THE EXISTENCE OF A DISTRIBUTION FUNCTION FOR AN ERROR TERM RELATED TO THE EULER FUNCTION

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1. Introduction. The average order of the Euler function $\phi(n)$, the number of integers less than n which are relatively prime to n, raises many difficult and still unanswered questions. Thus, for

$$R(x) = \sum_{n \leqslant x} \phi(n) - \frac{3}{\pi^2} x^2, \tag{1.1}$$

and

$$H(x) = \sum_{n \leqslant x} \frac{\phi(n)}{n} - \frac{6}{\pi^2} x, \tag{1.2}$$

it is known that $R(x) = O(x \log x)$ and $H(x) = O(\log x)$. However, though these results are quite old, they were not improved until recently. Walfisz **(1)** has given the outline of a proof of

$$R(x) = O(x(\log x)^{3/4}(\log \log x)^2).$$

On the other hand it is known **(3)** that

$$R(x) \neq O(x \log \log \log x). \tag{1.3}$$

and

$$H(x) \neq O(\log \log \log x). \tag{1.4}$$

In this direction it was proved in **(4)** that each of the following inequalities holds for infinitely many integral x (c a certain positive constant):

$$R(x) > cx \log \log \log \log x, \tag{1.5}$$

$$R(x) < - cx \log \log \log \log x, \tag{1.6}$$

$$H(x) > c \log \log \log \log x, \tag{1.7}$$

$$H(x) < - c \log \log \log \log x. \tag{1.8}$$

In this paper we propose to continue the study of the error function $H(x)$, and will prove that $H(x)$ possesses a continuous distribution function. By this we mean that for $N(n, u)$ = the number of $m \leqslant n$ such that $H(m) \geqslant u$, we have for each u, $-\infty < u < \infty$, that the limit

$$\lim_{n \to \infty} \frac{N(n, u)}{n} = F(u) \tag{1.9}$$

exists; and the non-increasing function $F(u)$ is continuous for all u.

In the case of additive arithmetic functions, necessary and sufficient conditions for the existence of a distribution function are known **(5; 6)**. The methods used in **(5)** to establish the sufficient conditions seem to apply in a fairly general way for establishing the existence of a continuous distribu-

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tion function even for a function which is not additive (7). This method serves also as the basic framework of the proof given here for the existence of a continuous distribution function for $H(x)$.

There are essentially three steps. *First*, we introduce for each integer $k \geqslant 1$, the function

$$H_k(x) = \sum_{n \leqslant x} \frac{\phi((n, A_k))}{(n, A_k)} - x \prod_{p \leqslant p_k} \left(1 - \frac{1}{p^2}\right), \tag{1.10}$$

where

$$A_k = \prod_{p \leqslant p_k} p;$$

where p_k is the kth prime. It is then shown that for each u, with fixed k, if $N_k(n, u)$ is the number of $m \leqslant n$ such that $H_k(x) \geqslant u$, the limit

$$\lim_{n \to \infty} \frac{N_k(n, u)}{n} = F_k(u) \tag{1.11}$$

exists. We then see that (1.9) follows if we can show that, for a given u and any $\epsilon > 0$, the inequality

$$|N(n, u) - N_k(n, u)| < \epsilon n \tag{1.12}$$

holds for each $k \geqslant k_0 = k_0(\epsilon)$ and all $n \geqslant n_0 = n_0(k)$. For from (1.12) we have

$$\left| \sup \frac{N(n, u)}{n} - \lim_{n \to \infty} \frac{N_k(n, u)}{n} \right| \leqslant \epsilon, \qquad \left| \inf \frac{N(n, u)}{n} - \lim_{n \to \infty} \frac{N_k(n, u)}{n} \right| \leqslant \epsilon,$$

for $k \geqslant k_0$. This in turn gives

$$\left| \sup \frac{N(n, u)}{n} - \inf \frac{N(n, u)}{n} \right| \leqslant 2\epsilon,$$

and the existence of the limit (1.9) follows.

The next two steps of the proof are devoted to establishing (1.12). This asserts that the number of $m \leqslant n$ such that either

$$(\alpha) \quad H(m) < u \text{ and } H_k(m) \geqslant u$$

or

$$(\beta) \quad H(m) \geqslant u \text{ and } H_k(m) < u$$

is less than ϵn for each $k \geqslant k_0$, and sufficiently large n. It suffices (since the argument is the same for the other case) to consider only the case (α). At this point the *second* step of the proof comes in. It is proved that, given any $\delta > 0$, $\epsilon > 0$, for k fixed sufficiently large, and n sufficiently large,

$$|H(m) - H_k(m)| < \delta \tag{1.13}$$

except for at most $\frac{1}{2}\epsilon n$ integers $m \leqslant n$. Thus in case (α),

$$H(m) < u - \delta, \quad H_k(m) \geqslant u$$

can hold for at most $\frac{1}{2}\epsilon n$ integers $m \leqslant n$. Hence we need consider only those m for which

$$u - \delta < H(m) < u. \tag{1.14}$$

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This then brings us to the *third* step of the proof. It is shown that given $\epsilon > 0$ there exists a $\delta > 0$ ($\delta = \delta(\epsilon)$, independent of u), such that for sufficiently large n, the number of $m \leqslant n$ such that (1.14) holds is less than $\frac{1}{2}\epsilon n$. This clearly completes the proof of the existence of $F(u)$. Furthermore, the result of this third step implies that for a fixed u, given any $\epsilon > 0$, there is a $\delta = \delta(\epsilon)$ such that $0 \leqslant F(u - \delta) - F(u) < \epsilon$, which yields the continuity of $F(u)$.

The main component of the argument used to carry out this last step is the result that, for any fixed integer l, the function

$$\Phi_l(x) = \frac{\phi(x)}{x} + \frac{\phi(x+1)}{x+1} + \ldots + \frac{\phi(x+l)}{x+l}$$

has a continuous distribution function. Though we shall not bother to delineate the proof of this, it is contained in the arguments given. The idea in the proof of the result desired in the third step is that its negation would for some l imply the existence of a discontinuity in the distribution function of $\Phi_l(x)$.

2. First step: The existence of $F_k(u)$. We have

$$\begin{aligned}\sum_{n \leqslant x} \frac{\phi((n, A_k))}{(n, A_k)} &= \sum_{n \leqslant x} \sum_{d \mid (n, A_k)} \frac{\mu(d)}{d} \\ &= \sum_{d \mid A_k} \frac{\mu(d)}{d}\left[\frac{x}{d}\right] \\ &= x \sum_{d \mid A_k} \frac{\mu(d)}{d^2} - \sum_{d \mid A_k} \frac{\mu(d)}{d}\left\{\frac{x}{d}\right\},\end{aligned}$$

where $\{z\} = z - [z]$ denotes the fractional part of z. This in turn yields, from (1.10),

$$H_k(x) = -\sum_{d \mid A_k} \frac{\mu(d)}{d}\left\{\frac{x}{d}\right\}, \tag{2.1}$$

Since $\{x/d\}$ is, for fixed d, a periodic function of x with period d, we see from (2.1) that $H_k(x)$ is a periodic function of x with A_k as a period. Thus we have

$$N_k(n, u) = \sum_{\substack{m \leqslant n \\ H_k(m) \geqslant u}} 1 = \frac{n}{A_k} \sum_{\substack{m \leqslant A_k \\ H_k(m) \geqslant u}} 1 + O(1),$$

so that

$$\lim_{\to\infty} \frac{N_k(n, u)}{n} = \frac{1}{A_k} N_k(A_k, u) = F_k(u)$$

exists.

3. Second step. We will prove in this section that, given any $\eta > 0$, for each $k \geqslant k_0 = k_0(\eta)$, and all $x \geqslant x_0 = x_0(k)$, we have

$$\sum_{n=1}^{x} (H(n) - H_k(n))^2 < \eta x. \tag{3.1}$$

From this it follows that if $M(x)$ is the number of $n \leqslant x$ such that

$$|H(n) - H_k(n)| > \delta,$$

$M(x) < \eta x/\delta^2$, which yields the statement concerning (1.13).

(3.1) is established in a rather straightforward fashion in the following sequence of lemmas.

LEMMA 3.1.

$$\sum_{n \leqslant x} H^2(n) \sim \left(\frac{1}{2\pi^2} + \frac{6}{\pi^4}\right) x. \tag{3.2}$$

Proof. This is essentially Lemma 12 of **(8)**, which asserts that

$$\int_1^x H^2(u)\, du \sim \frac{x}{2\pi^2}. \tag{3.3}$$

The passage from (3.3) to (3.2) is simple and we omit it. In passing it is perhaps of some interest to note that (3.3) is proved by means of a method of Walfisz **(2)**, and seems to be slightly "deeper" than the rest of our estimates which require only elementary methods together with a strong form of the prime number theorem.

LEMMA 3.2.

$$\sum_{n \leqslant x} H_k^2(n) \sim \alpha_k x, \tag{3.4}$$

where

$$\begin{aligned}\alpha_k = {} & \frac{1}{12} \sum_{\substack{d_1 | A_k \\ d_2 | A_k}} \frac{\mu(d_1)\,\mu(d_2)}{d_1^2 d_2^2} (d_1, d_2)^2 \\ & + \frac{1}{6} \prod_{p \leqslant p_k} \left(1 - \frac{1}{p^2}\right)^2 + \frac{1}{4} \prod_{p \leqslant p_k} \left(1 - \frac{1}{p}\right)^2 \\ & - \frac{1}{2} \prod_{p \leqslant p_k} \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p^2}\right).\end{aligned} \tag{3.5}$$

Furthermore,

$$\lim_{k \to \infty} \alpha_k = \frac{1}{2\pi^2} + \frac{6}{\pi^4}. \tag{3.6}$$

Proof. From (2.1) we have

$$\begin{aligned}\sum_{n \leqslant x} H_k^2(n) &= \sum_{n \leqslant x} \sum_{\substack{d_1 | A_k \\ d_2 | A_k}} \frac{\mu(d_1)}{d_1} \frac{\mu(d_2)}{d_2} \left\{\frac{n}{d_1}\right\}\left\{\frac{n}{d_2}\right\} \\ &= \sum_{\substack{d_1 | A_k \\ d_2 | A_k}} \frac{\mu(d_1)\,\mu(d_2)}{d_1 d_2} \sum_{n \leqslant x} \left\{\frac{n}{d_1}\right\}\left\{\frac{n}{d_2}\right\}.\end{aligned} \tag{3.7}$$

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(3.8)
$$\sum_{n\leqslant x}\left\{\frac{n}{d_1}\right\}\left\{\frac{n}{d_2}\right\} = \sum_{\substack{1\leqslant i\leqslant d_1-1\\ 1\leqslant j\leqslant d_2-1\\ (d_1,d_2)\mid(i-j)}} \frac{ij}{d_1d_2}\sum_{\substack{n\leqslant x\\ n\equiv i(d_1)\\ n\equiv j(d_2)}} 1$$
$$= \frac{1}{d_1d_2}\sum_{\substack{1\leqslant i\leqslant d_1-1\\ 1\leqslant j\leqslant d_2-1\\ (d_1,d_2)\mid i-j}} ij\left(\frac{x}{\{d_1,d_2\}}+O(1)\right)$$
$$= \frac{\lambda}{d_1^2d_2^2}x\sum_{\substack{1\leqslant i\leqslant d_1-1\\ 1\leqslant j\leqslant d_2-1\\ \lambda\mid(i-j)}} ij + O(1),$$

where $\lambda = (d_1, d_2)$ is greatest common divisor of d_1, d_2, and $\{d_1, d_2\}$ is the least common multiple of d_1, d_2.

A simple calculation gives that

$$\sum_{\substack{1\leqslant i\leqslant d_1-1\\ 1\leqslant j\leqslant d_2-1\\ \lambda\mid i-j}} ij = \sum_{l=0}^{\lambda-1}\left(\sum_{\substack{i=1\\ i\equiv l(\lambda)}}^{d_1-1} i\right)\left(\sum_{\substack{j=1\\ j\equiv l(\lambda)}}^{d_2-1} j\right)$$

(3.9)
$$= \frac{d_1d_2}{\lambda}\left(\frac{\lambda^2}{12}+\frac{1}{6}+\frac{d_1d_2}{4}-\frac{(d_1+d_2)}{4}\right).$$

Combining (3.7), (3.8) and (3.9) yields

$$\sum_{n\leqslant x} H_k^2(n) = x\sum_{\substack{d_1\mid A_k\\ d_2\mid A_k}}\frac{\mu(d_1)}{d_1^2}\frac{\mu(d_2)}{d_2^2}\left\{\frac{\lambda^2}{12}+\frac{1}{6}+\frac{d_1d_2}{4}-\frac{d_1}{2}\right\}+O(1)$$
$$\sim x\left(\frac{1}{12}\sum_{\substack{d_1\mid A_k\\ d_2\mid A_k}}\frac{\mu(d_1)}{d_1^2}\frac{\mu(d_2)}{d_2^2}(d_1,d_2)^2\right.$$
$$+\frac{1}{6}\prod_{p\leqslant p_k}\left(1-\frac{1}{p^2}\right)^2+\frac{1}{4}\prod_{p\leqslant p_k}\left(1-\frac{1}{p}\right)^2$$
$$\left.-\frac{1}{2}\prod_{p\leqslant p_k}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right)\right)$$

which is precisely (3.4) and (3.5). Since

$$\prod_{p\leqslant p_k}\left(1-\frac{1}{p}\right)\to 0 \text{ as } k\to\infty; \text{ and } \prod_p\left(1-\frac{1}{p^2}\right)=\frac{6}{\pi^2},$$

it follows from (3.5) that

$$\lim_{k\to\infty}\alpha_k = \frac{1}{12}\sum_{d_1,d_2}\frac{\mu(d_1)\,\mu(d_2)\,(d_1,d_2)^2}{d_1^2d_2^2}+\frac{6}{\pi^4}.$$

(3.6) then follows from

$$\sum_{d_1,d_2}\frac{\mu(d_1)\,\mu(d_2)\,(d_1,d_2)^2}{d_1^2d_2^2}=\frac{6}{\pi^2}.$$

LEMMA 3.3.

(3.10)
$$\sum_{n\leqslant x} H(n)\,H_k(n)\sim\beta_k x$$

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where

(3.11)
$$\beta_k = \frac{1}{12}\sum_{\substack{d_1|A_k\\ d_2|A_k}} \frac{\mu(d_1)}{d_1^2}\frac{\mu(d_2)}{d_2^2}(d_1,d_2)^2 - \frac{3}{2\pi^2}\prod_{p\leqslant p_k}\left(1-\frac{1}{p}\right)+\frac{1}{\pi^2}\prod_{p\leqslant p_k}\left(1-\frac{1}{p^2}\right).$$

Furthermore,

(3.12)
$$\lim_{k\to\infty}\beta_k = \frac{1}{2\pi^2}+\frac{6}{\pi^4}.$$

Proof. Setting

$$M(u) = \sum_{d\leqslant u}\frac{\mu(d)}{d},$$

since, by the prime number theorem, $M(u) = O(\log^{-c}u)$ for any fixed $c > 0$, we have for $uv = x$

$$\begin{aligned}\sum_{n\leqslant x}\frac{\phi(n)}{n} &= \sum_{dd'\leqslant x}\frac{\mu(d)}{d}\\ &= \sum_{d\leqslant u}\frac{\mu(d)}{d}\left[\frac{x}{d}\right]+\sum_{d'\leqslant v}M\left(\frac{x}{d'}\right)-M(u)[v]\\ &= \frac{6}{\pi^2}x-\sum_{d\leqslant u}\frac{\mu(d)}{d}\left\{\frac{x}{d}\right\}+O(v\log^{-2c}u).\end{aligned}$$

Taking $u = x\log^{-c}x$, we get

(3.13)
$$H(x) = -\sum_{d\leqslant x\log^{-c}x}\frac{\mu(d)}{d}\left\{\frac{x}{d}\right\}+O(\log^{-c}x).$$

(This is essentially Lemma 2 of **(8)**.)

From (2.1) and (3.13) we obtain

$$\sum_{n\leqslant x}H_k(n)\,H(n) = \sum_{\substack{d_1|A_k\\ d_2\leqslant x\log^{-c}x}}\frac{\mu(d_1)}{d_1}\frac{\mu(d_2)}{d_2}\sum_{n\leqslant x}\left\{\frac{n}{d_1}\right\}\left\{\frac{n}{d_2}\right\}+O(\log^{-c}x).$$

Using a slight modification of (3.8) and (3.9) we get

$$\begin{aligned}\sum_{n\leqslant x}H_k(n)\,H(n) &= x\sum_{\substack{d_1|A_k\\ d_2\leqslant x\log^{-c}x}}\frac{\mu(d_1)}{d_1^2}\frac{\mu(d_2)}{d_2^2}\left\{\frac{\lambda^2}{12}+\frac{1}{6}+\frac{d_1d_2}{4}-\frac{(d_1+d_2)}{4}\right\}+O(x\log^{-c}x)\\ &\sim x\left(\frac{1}{12}\sum_{\substack{d_1|A_k\\ d_2|A_k}}\frac{\mu(d_1)}{d_1^2}\frac{\mu(d_2)}{d_2^2}(d_1,d_2)^2-\frac{3}{2\pi^2}\prod_{p\leqslant p_k}\left(1-\frac{1}{p_k}\right)+\frac{1}{\pi^2}\prod_{p\leqslant p_k}\left(1-\frac{1}{p^2}\right)\right)\end{aligned}$$

which gives (3.10) and (3.11). From this it follows easily that

$$\lim_{k\to\infty}\beta_k = \frac{1}{2\pi^2}+\frac{6}{\pi^4}.$$

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Applying (3.2), (3.4), and (3.10) we have

$$\sum_{n\leqslant x} (H(n) - H_k(n))^2 = \sum_{n\leqslant x} H^2(n) - 2\sum_{n\leqslant x} H_k(n)\,H(n) + \sum_{n\leqslant x} H_k^2(n)$$
$$\sim x\left\{\frac{1}{2\pi^2} + \frac{6}{\pi^4} - 2\beta_k + \alpha_k\right\}.$$

From (3.6) and (3.12) we see that

$$\lim_{k\to\infty}\left(\frac{1}{2\pi^2} + \frac{6}{\pi^4} - 2\beta_k + \alpha_k\right) = 0,$$

and the assertion concerning (3.1) made at the beginning of this section follows immediately.

4. The third step. In this section we propose to prove that, given any $\epsilon > 0$, there exists a $\delta > 0$ such that the number of $m \leqslant x$ such that

(4.1) $$u < H(m) < u + \delta \qquad \text{for some } u,$$

is (for sufficiently large x) less than ϵx.

We shall suppose that the above statement is false and derive a contradiction. Negating the above assertion yields that for *some* constant $A > 0$ and *each* $\delta > 0$, there exist infinitely many positive integers x (depending possibly on δ) such that for some u (depending possibly on x as well as on u) the number of $m \leqslant x$ such that (4.1) is satisfied is at least Ax.

Since from (3.2) we have

(4.2) $$\sum_{m=1}^{x} H^2(m) < c_1 x,$$

it follows that for these $u = u(x, \delta)$ (we restrict ourselves to $0 < \delta < 1$), we have that either $-2 \leqslant u \leqslant 0$, or from (4.2)

$$\frac{u^2}{4} Ax < c_1 x,$$

so that in any event the possible values of $u = u(x, \delta)$ are bounded. Thus for each δ $(0 < \delta < 1)$ we can find an infinite sequence of positive integers $\{x_i(\delta)\}$ such that

(4.3) $$\lim_{i\to\infty} u(x_i(\delta), \delta) = u^*(\delta),$$

where the set of $u^*(\delta)$ is also clearly bounded. Thus again we can choose a sequence $\delta_j \to 0$ such that the limit

(4.4) $$\lim_{j\to\infty} u^*(\delta_j) = \bar{u}$$

exists.

Given any $\delta > 0$ we can find a $\delta_j < \frac{1}{3}\delta$ such that

$$|\bar{u} - u^*(\delta_j)| < \tfrac{1}{3}\delta.$$

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Since from (4.3) we know that, for all sufficiently large i,

$$|u^*(\delta_j) - u(x_i(\delta_j),\ \delta_j)| < \tfrac{1}{3}\delta,$$

it follows that for this sequence $\{x_i(\delta_j)\}$ we have

$$|\bar{u} - u(x_i(\delta_j),\ \delta_j)| < \tfrac{2}{3}\delta. \tag{4.5}$$

For $m \leqslant x_i(\delta_j)$ there are more than Ax_i integers $m \leqslant x_i$ such that

$$u(x_i(\delta_j),\ \delta_j) < H(m) < u(x_i(\delta_j),\ \delta_j) + \delta_j. \tag{4.6}$$

But since (4.5) implies that

$$\bar{u} - \delta < u(x_i(\delta_j),\ \delta_j) < u(x_i(\delta_j),\ \delta_j) + \delta_j < \bar{u} + \delta,$$

it follows that for at least $\tfrac{1}{2}Ax_i$ of the $m \leqslant x_i$ we have one of the inequalities

$$(\alpha)\quad \bar{u} \leqslant H(m) < \bar{u} + \delta,$$
$$(\beta)\quad \bar{u} - \delta < H(m) \leqslant \bar{u}.$$

Since at least one of (α) or (β) must occur for a sequence of δ's approaching 0, at least one of these is the case for all $\delta > 0$. Since the treatment of the other case is exactly the same, we assume (α). Thus we have that, for any $\delta > 0$, there exist infinitely many positive integers n such that the number of integers $m \leqslant n$ for which

$$\bar{u} \leqslant \sum_{r=1}^{m} \frac{\phi(r)}{r} - \frac{6}{\pi^2}m < \bar{u} + \delta \tag{4.7}$$

is greater than $\tfrac{1}{2}An$.

Let $m_1 < m_2 < \ldots < m_t \leqslant n$ $(t > \tfrac{1}{2}An)$ be the integers $\leqslant n$ which satisfy (4.7). Clearly $m_{i+1} - m_i < 4/A$ has at least $\tfrac{1}{4}An$ solutions. Thus there exists an integer $l < 4/A$ such that $m_{i+1} - m_i = l$ has at least $A^2n/16$ solutions. Furthermore, by extracting a suitable subsequence from our infinite sequence of n, we may assume that l is independent of n.

The above in turn implies that for any $\delta > 0$ there exists an infinite sequence of n such that

$$\left|\sum_{r=m}^{m+l-1} \frac{\phi(r)}{r} - \frac{6}{\pi^2}l\right| \leqslant \delta \tag{4.8}$$

has at least $A^2n/16$ solutions $m \leqslant n$. In deriving a contradiction from this, the underlying idea is that this implies that the distribution function (it exists, though we forego a proof of this) of

$$\frac{\phi(x)}{x} + \ldots + \frac{\phi(x+l-1)}{x+l-1}$$

would have to have a discontinuity at $6l/\pi^2$, and this in turn would lead to the existence of a discontinuity in the distribution function of $\phi(x)/x$ (which is known to exist and be continuous) **(5)**.

We set

$$\frac{\phi_D(x)}{x} = \prod_{\substack{p|x \\ p \leqslant D}} \left(1 - \frac{1}{p}\right),$$

$$\mu_D(n) = \begin{cases} \mu(n) \text{ if } n \text{ is divisible only by primes } p \leqslant D, \\ 0 \quad \text{otherwise;} \end{cases}$$

so that

$$\frac{\phi_D(x)}{x} \geqslant \frac{\phi(x)}{x},$$

and

$$0 \leqslant \sum_{n \leqslant x} \left\{\frac{\phi_D(n)}{n} - \frac{\phi(n)}{n}\right\} = \sum_{d \leqslant x} \left\{\frac{\mu_D(d)}{d} - \frac{\mu(d)}{d}\right\}\left[\frac{x}{d}\right]$$
$$= x \sum_{d \leqslant x} \frac{\mu_D(d) - \mu(d)}{d^2} + O(\log x)$$
$$\sim x\left(\prod_{p \leqslant D}\left(1 - \frac{1}{p^2}\right) - \frac{6}{\pi^2}\right).$$

From this it follows that, given $\eta_1, \eta_2 > 0$, we can choose $D \geqslant D(\eta_1, \eta_2)$ sufficiently large so that for all but $\eta_1 n$ integers $x \leqslant n$ we have

$$0 \leqslant \frac{\phi_D(x)}{x} - \frac{\phi(x)}{x} < \eta_2.$$

Thus, taking $\eta_2 = \delta/l$ and $\eta_1 = A^2/32$, we obtain from (4.8) that for each sufficiently large D, there exist infinitely many positive integers n such that the inequalities

$$\left|\sum_{r=m}^{m+l-1} \frac{\phi_D(r)}{r} - \frac{6}{\pi^2} l\right| \leqslant 2\delta \tag{4.9}$$

and

$$\left|\frac{\phi(m)}{m} + \sum_{r=m+1}^{m+l-1} \frac{\phi_D(r)}{r} - \frac{6}{\pi^2} l\right| \leqslant 2\delta \tag{4.10}$$

are satisfied simultaneously by at least $A^2n/32$ integers $m \leqslant n$.

LEMMA 4.1. *There exist absolute constants $\rho > 0$, and $\delta_0 > 0$ (independent of D) such that for at least $A^2n/64$ of the solutions $m \leqslant n$ of (4.9) and (4.10) we have for $\delta < \delta_0$*

$$\frac{6}{\pi^2} l - \sum_{r=m+1}^{m+l-1} \frac{\phi_D(r)}{r} \geqslant \rho. \tag{4.11}$$

Proof. For if (4.11) is false, (4.10) implies

$$0 \leqslant \frac{\phi(m)}{m} \leqslant \rho + 2\delta. \tag{4.12}$$

Since the distribution function of $\phi(m)/m$ exists and is continuous, for ρ and δ sufficiently small, (4.12) can have at most $A^2n/64$ solutions $m \leqslant n$.

Thus we may restrict ourselves to solutions m of (4.9) for which (4.11) holds. Also there is no loss of generality in assuming $\delta < \frac{1}{3}\rho$, as we shall do henceforth.

Next, we discard a certain "small" set of integers. Since

$$\sum_{m=1}^{n} \sum_{p \mid m} \frac{1}{p} = \sum_{p \leqslant n} \frac{1}{p}\left[\frac{n}{p}\right] < n \sum_{p} \frac{1}{p^2} = c_2 n \tag{4.13}$$

it follows that the number of $m \leqslant n$ such that

$$\sum_{p \mid m+i} \frac{1}{p} < E, \qquad 0 \leqslant i \leqslant l-1, \tag{4.14}$$

fails to hold is less than $lc_2 n/E$, which for $E > 128lc_2/A^2$ is in turn less than $A^2n/128$. Thus for such an E we have an infinite sequence of n such that (4.9), (4.11) and (4.14) hold simultaneously for more than $A^2n/128$ integers $m \leqslant n$.

We now attempt to show that the set of integers m which satisfy (4.9), (4.11) and (4.14) has small density, thereby obtaining a contradiction. For a given integer m define

$$\lambda(m) = \prod_{\substack{p \mid m \\ p \leqslant D}} p.$$

We then associate with each integer m an $(l-1)$-dimensional vector $\vec{\lambda}(m)$ as follows:

$$\vec{\lambda}(m) = (\lambda(m+1),\ \lambda(m+2), \ldots, \lambda(m+l-1)).$$

Next, for a given vector $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{l-1})$, wherein each λ_i is an integer which is a product of distinct primes $\leqslant D$, and

$$\sum_{p \mid \lambda_i} \frac{1}{p} < E, \qquad i = 1, \ldots, l-1, \tag{4.15}$$

and

$$\frac{6}{\pi^2} l - \sum_{i=1}^{l-1} \frac{\phi(\lambda_i)}{\lambda_i} \geqslant \rho, \tag{4.16}$$

we estimate the number of $m \leqslant n$ satisfying (4.9) such that $\vec{\lambda}(m) = \vec{\lambda}$ (possibly none). For such m we have

$$m + i \equiv 0 \pmod{\lambda_i}, \qquad i = 1, \ldots, l-1, \tag{4.17}$$

so that if there are any solutions they belong to a single arithmetic progression modulo $[\vec{\lambda}] = \{\lambda_1, \ldots, \lambda_{l-1}\}$, the least common multiple of the λ_i, $i = 1, \ldots, l-1$. Furthermore, in order that such solutions exist we must have

$$(\lambda_i, \lambda_j) \mid i - j \qquad i \neq j,\ 1 \leqslant i, j \leqslant l-1. \tag{4.18}$$

Suppose then that the aforementioned progression is $m \equiv \alpha \pmod{[\vec{\lambda}]}$. For those m such that $\vec{\lambda}(m) = \vec{\lambda}$ which satisfy (4.9) it follows that

$$\frac{6}{\pi^2} l - 2\delta - \sum_{i=1}^{l-1} \frac{\phi_D(\lambda_i)}{\lambda_i} \leqslant \frac{\phi_D(m)}{m} \leqslant \frac{6}{\pi^2} l + 2\delta - \sum_{i=1}^{l-1} \frac{\phi_D(\lambda_i)}{\lambda_i}, \tag{4.19}$$

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so that for these m, $\phi_D(m)/m$ lies in a fixed interval of length 4δ which we shall denote by $I_\delta = I_\delta(\vec{\lambda})$. Thus the number of $m \leqslant n$ such that $\vec{\lambda}(m) = \vec{\lambda}$ and which satisfy (4.9) equals the number of $m \leqslant n$ which satisfy

(a) $m \equiv \alpha \pmod{[\vec{\lambda}]}$

(b) $\left(\dfrac{m+i}{\lambda_i}, \dfrac{\Delta}{\lambda_i}\right) = 1, \qquad i = 1, \ldots, l-1; \Delta = \prod_{p \leqslant D} p,$

(c) $\dfrac{\phi_D(m)}{m} \in I_\delta.$

LEMMA 4.2. *Given any $\eta > 0$, for D fixed sufficiently large, and δ sufficiently small* (*these requirements are however independent of* $\vec{\lambda}$), *the number of $m \leqslant n$ such that* (a), (b) *and* (c) *hold is less than*

$$(4.20) \qquad (\eta n/[\vec{\lambda}]) \prod_{p \leqslant D} \left(1 - \frac{1}{p}\right)^{l-1}.$$

Proof. Suppose that the above statement concerning the estimate (4.20) is false, so that for infinitely many n, the number of $m \leqslant n$ satisfying (a), (b) and (c) is more than

$$(4.21) \qquad (c_3 n/[\vec{\lambda}]) \prod_{p \leqslant D} \left(1 - \frac{1}{p}\right)^{l-1}.$$

Let $z_1, z_2, \ldots$ be those integers, composed of primes $\leqslant D$, which can occur as divisors of an integer $m \equiv \alpha \pmod{[\vec{\lambda}]}$ and such that (we denote the z_k generically by z)

$$(4.22) \qquad \frac{\phi_D(z)}{z} = \frac{\phi(z)}{z} \in I_\delta.$$

From (4.16), (4.19) and our assumption $\delta < \rho/3$, (4.22) yields

$$(4.23) \qquad \frac{\phi(z)}{z} \geqslant \frac{\rho}{3}.$$

Consider the number of $m \leqslant n$ such that (a), (b) above hold, and in addition for a fixed z,

$$\text{(d)} \qquad m \equiv 0 \pmod{z}, \left(\frac{m}{z}, \Delta\right) = 1.$$

Clearly (d) implies (c).

Delete from $\Delta/[\vec{\lambda}]$ all prime factors $\leqslant l$ and any other prime factors of z; denoting the resulting integer by ψ. Then the number of $m \leqslant n$ which satisfy (a), (b), (d) is less than or equal to the number which satisfy (a) and

$$\text{(b}'\text{)} \qquad (m+i, \psi) = 1, \qquad i = 1, \ldots, l-1$$

and

$$\text{(d}'\text{)} \qquad m \equiv 0(z), \left(\frac{m}{z}, \psi\right) = 1.$$

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Setting $m = vz$ we have that the number of such m equals

$$(4.24) \qquad \sum_{\substack{v \leqslant n/z \\ (v,\psi)=1 \\ (vz+i)\equiv 0(\lambda_i) \\ (vz+i,\psi)=1 \\ 1\leqslant i\leqslant l-1}} 1 = \sum_{\substack{d_i \mid \psi \\ i=0,\ldots,l-1}} \mu(d_0)\,\mu(d_1)\ldots\mu(d_{l-1})\,. \sum_{\substack{v\leqslant n/z \\ v\equiv 0(d_0) \\ vz+i\equiv 0(d_i\lambda_i) \\ 1\leqslant i\leqslant l-1}} 1$$

Since $(d_i, z) = 1$, $(\lambda_i, \lambda_j)|i - j$, and the primes which divide ψ are $>l$, we see that the system of congruences

$$v \equiv 0(d_0), \quad vz + i \equiv 0(d_i\lambda_i), \qquad 1 \leqslant i \leqslant l-1,$$

has solutions if and only if

$$(4.25) \qquad (d_i, d_j) = 1, \text{ and } (z, \lambda_i)|i; \qquad i \neq j, \quad 0 \leqslant i, \quad j \leqslant l-1.$$

Furthermore, if (4.25) holds we have, since $(d_i, \lambda_j) = 1$,

$$\sum_{\substack{v\leqslant n/z \\ v\equiv 0(d_0) \\ vz+i\equiv 0(d_i\lambda_i) \\ 1\leqslant i\leqslant l-1}} 1 = \frac{n}{z}\,\frac{1}{\left\{d_0, \dfrac{d_1\lambda_1}{(z,\lambda_1)}, \ldots, \dfrac{d_{l-1}, \lambda_{l-1}}{(z,\lambda_{l-1})}\right\}} + O(1)$$

$$= \frac{n}{zd_0d_1\ldots d_{l-1}}\,\frac{1}{\left\{\dfrac{\lambda_1}{(z,\lambda_1)}, \ldots, \dfrac{\lambda_{l-1}}{(z,\lambda_{l-1})}\right\}} + O(1).$$

Inserting this in (4.24) we get

$$(4.26) \qquad M = \frac{n}{z}\cdot\frac{1}{\left\{\ldots, \dfrac{\lambda_i}{(z,\lambda_i)}, \ldots\right\}} \sum_{\substack{d_i \mid \psi \\ (d_i,d_j)=1 \\ 0\leqslant i,\, j\leqslant l-1}} \frac{\mu(d_0)\ldots\mu(d_{l-1})}{d_0\ldots d_{l-1}} + O(1).$$

Since

$$\sum_{\substack{d_i \mid \psi \\ (d_i,d_j)=1 \\ 0\leqslant i,\, j\leqslant l-1}} \frac{\mu(d_0)\ldots\mu(d_{l-1})}{d_0\ldots d_{l-1}} = \sum_{c\mid\psi}\frac{\mu(c)}{c}\,l^{\nu(c)} = \prod_{p\mid\psi}\left(1-\frac{l}{p}\right)$$

$$< c_4\prod_{p\mid\psi}\left(1-\frac{1}{p}\right)^l$$

$$< c_5\left(\frac{z}{\phi(z)}\right)^l \prod_{p\mid[\vec{\lambda}]}\left(1+\frac{1}{p}\right)^l \prod_{p\leqslant D}\left(1-\frac{1}{p}\right)^l$$

and from (4.15), (4.23)

$$\prod_{p\mid[\vec{\lambda}]}\left(1+\frac{1}{p}\right)^l \leqslant e^{El^2},$$

$$\left(\frac{z}{\phi(z)}\right)^l \leqslant \left(\frac{3}{\rho}\right)^l,$$

(4.26) yields (since $(z, \lambda_i)|i$)

$$M < c_6\left(\frac{3}{\rho}\right)^l e^{El^2} l!\,(n/z[\vec{\lambda}]) \prod_{p\leqslant D}\left(1-\frac{1}{p}\right)^l$$

or

$$(4.27) \qquad M < c_7(n/z[\vec{\lambda}]) \prod_{p\leqslant D}\left(1-\frac{1}{p}\right)^l,$$

where $c_7 > 0$ is independent of D.

(4.27) together with (4.21) implies

$$\sum_k \frac{1}{z_k} \prod_{p \leqslant D} \left(1 - \frac{1}{p}\right) > \frac{c_3}{c_7}. \tag{4.28}$$

On the other hand, the number of $m \leqslant n$ such that (d) holds for some z_k is, for large n, greater than or equal to

$$\frac{n}{2} \sum_k \frac{1}{z_k} \prod_{p \leqslant D} \left(1 - \frac{1}{p}\right) > \frac{c_3}{2c_7} n,$$

using (4.28). Since for these m, $\phi_D(m)/m$ lies in a fixed interval I_δ of length 4δ, we see that for at least $c_3 n/4c_7$ of these m, $\phi(m)/m$ lies in a fixed interval of length 8δ (if D is large enough). Since $c_3/4c_7$ is independent of δ (and of D), this would contradict the continuity of the distribution function of $\phi(m)/m$. Thus the lemma is proved.

Finally then, letting T denote the number of $m \leqslant n$ which satisfy (4.9), (4.11) and (4.14), we have

$$T < \eta n \prod_{p \leqslant D} \left(1 - \frac{1}{p}\right)^{l-1} \sum_{\vec{\lambda}} [\vec{\lambda}]^{-1}.$$

Since

$$\sum_{\vec{\lambda}} [\vec{\lambda}]^{-1} \leqslant c_8 \left(\sum_{\lambda | \Delta} \frac{1}{\lambda} \right)^{l-1}$$
$$\leqslant c_8 \prod_{p \leqslant D} \left(1 + \frac{1}{p}\right)^{l-1},$$

we have

$$T < c_9 \eta n.$$

But for η sufficiently small, $c_9 \eta < A^2/128$, so that we obtain a contradiction, and the proof is completed.

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