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ON VARIATION OF EQUICONTINUITY IN DYNAMICAL SYSTEMS

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In this paper we investigate the relationships among the notions of minimality, characteristic 0, equicontinuity, Lipschitz stability and isometry in dynamical systems. Examples are provided to show that the results obtained are sharp.

1. INTRODUCTION

We consider a dynamical system (M, π) (see [1, 8]) where M is a locally compact metric space with a specified metric d and the action group G is either the additive group of integers Z or the additive group of reals R. For each $t \in G$, the transition map $\pi^t\colon M \to M$ is defined by $\pi^t(x) = \pi(x, t) = xt$. The orbit $O(x)$ of $x \in M$ is defined to be the set $\{xt \mid t \in G\}$. The prolongation set $D(x)$ is the set $\{y \mid$ there exist sequences $\{x_i\}$ in M and $\{t_i\}$ in G such that $x_i \to x$, $x_i t_i \to y\}$. The positive limit set $L^+(x)$ of x is the set $\{y \mid xt_i \to y$, for some sequence $\{t_i\}$ in G with $t_i \to \infty\}$. We also need the following definitions.

DEFINITION 1.1: [6]. A closed nonempty subset A of M is said to be minimal if it is invariant ($At \subset A$ for all $t \in G$) and contains no proper closed invariant subset. The dynamical system (M, π) is said to be minimal if the orbit closure $\overline{O(x)}$ of each $x \in M$ is minimal.

DEFINITION 1.2: [5]. A point $x \in M$ is said to be of characteristic 0 if $D(x) = \overline{O(x)}$. The dynamical system (M, π) is said to be of characteristic 0 if every point $x \in M$ is of characteristic 0.

DEFINITION 1.3: [6]. The dynamical system (M, π) is said to be equicontinuous at $x \in M$ if given $\varepsilon > 0$, there exists $\delta = \delta(x, \varepsilon)$ such that $d(xt, yt) < \varepsilon$ whenever $d(x, y) < \delta$ and $y \in M$. The dynamical system (M, π) is said to be pointwise equicontinuous if it is equicontinuous at every $x \in M$. Furthermore, (M, π) is uniformly equicontinuous if it is pointwise equicontinuous and δ can be chosen independently of $x \in M$.

DEFINITION 1.4: [3, 4]. The dynamical system (M, π) is said to be Lipschitz stable if there exist $k \geqslant 1$ and $\delta > 0$ such that $d(xt, yt) \leqslant k\, d(x, y)$, for all $t \in G$ and $x, y \in M$, whenever $d(x, y) < \delta$.

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DEFINITION 1.5: The dynamical system (M, π) is said to be isometric if for each $t \in G$, $d(xt, yt) = d(x, y)$ for all $x, y \in M$. In otherwords, π^t is an isometry for every $t \in G$.

The following diagram summarises the known connection among the preceding notions.

Isometric $\Longrightarrow$ Lipschitz stable $\overset{[3][4]}{\Longrightarrow}$ uniform equicontinuous $\Longrightarrow$ pointwise equicontinuous $\overset{[2]}{\Longrightarrow}$ characteristic 0 $\Longrightarrow$ minimal.

In the sequel we show that the arrows in the above diagram may not be reversed even in compact spaces (the only exception is that uniform equicontinuity and pointwise equicontinuity are equivalent in compact spaces). However, we will give conditions under which the above implications can be reversed.

2. Minimality and characteristic 0

There are many examples of dynamical systems which are minimal but not of characteristic 0 (see Example 5.4 in [5]). Another simple example is the saddle in the plane generated by the differential system $x_1' = -x_1$, $x_2' = x_2$.

We give below conditions under which the converse is true. But before doing so we need to give the following definition.

DEFINITION 2.1: [1]. Let X be a subset of M. Then the set $A_\omega(X) = \{y \in M \mid L^+(y) \cap X \neq \emptyset\}$ is called the region of weak attraction of X. If $A_\omega(X)$ is a neighbourhood of X, then X is said to be a weak attractor.

THEOREM 2.2. *Let (M, π) be a minimal dynamical system on a connected space M in which every orbit closure is a weak attractor. Then (M, π) is of characteristic* 0.

PROOF: Let $x \in M$ and $A_\omega\left(\overline{O(x)}\right)$ be the region of weak attraction of $\overline{O(x)}$. It follows from [1] that $A_\omega\left(\overline{O(x)}\right)$ is an open neighbourhood of $\overline{O(x)}$. Let $y \in A_\omega\left(\overline{O(x)}\right)$. Since $L^+(y) \cap \overline{O(x)} \neq \emptyset$, we have $\overline{0(y)} \cap \overline{O(x)} \neq \emptyset$. By the minimality assumption, it follows that $\overline{O(x)} = \overline{0(y)}$ and thus $y \in \overline{O(x)}$. Hence $A_\omega\left(\overline{O(x)}\right) = \overline{O(x)}$. This implies that $\overline{O(x)}$ is both open and closed. Since M is connected, $\overline{O(x)} = M$ and consequently, (M, π) is of characteristic 0. ∎

3. Characteristic 0 and equicontinuity

We first remark that the discrete dynamical system given in [5, 5.3] is of characteristic 0 but not pointwise equicontinuous. We now give an example of a continuous dynamical system which is of characteristic 0 but not pointwise equicontinuous.

EXAMPLE 3.1. Consider the continuous dynamical system (R^2, π) defined on the plane by

$$xt = \begin{bmatrix} \cos\left(\|x\|^2 t\right) & \sin\left(\|x\|^2 t\right) \\ -\sin\left(\|x\|^2 t\right) & \cos\left(\|x\|^2 t\right) \end{bmatrix} x$$

for each $x \in R^2$ and $t \in R$, where $\|x\|$ is the Euclidean norm of x. Then π^t rotates each $x \in R^2$ through an angle $\|x\|^2 t$ counterclockwise about the origin. Every point x in the plane is periodic with period $2\pi/\|x\|^2$, while the origin is a rest point. Then (R^2, π) is of characteristic 0. If a point x is closer to the origin than a point y, then the period of y is smaller than that of x. Hence xt, $t > 0$, lags increasingly behind yt and no matter how close is x to y, after a sufficient time $s > 0$, xs will be as much as half a cycle behind ys. Hence the dynamical system is not pointwise equicontinuous.

One may argue similarly to show that the dynamical system generated by the differential system $x' = y$, $y' = \sin x$, is of characteristic 0 but not pointwise equicontinuous.

It is well known [6] that if a system (M, π) is pointwise equicontinuous then so is the squared system $(M \times M, \pi \times \pi)$. This implies [2] that $(M \times M \pi \times \pi)$ is of characteristic 0. The following theorem is a partial converse.

THEOREM 3.2. *Let (M, π) be a dynamical system such that the orbit closure of every $x \in M$ is compact. If $(M \times M, \pi \times \pi)$ is of characteristic 0 then (M, π) is pointwise equicontinuous.*

PROOF: Suppose that (M, π) is not equicontinuous at some $x \in M$. Then there are sequences $\{x_i\}$, $\{y_i\}$ in M, $\{t_i\}$ in G with $x_i \to x$, $y_i \to x$, $t_i \to \infty$ (or $-\infty$) and $\varepsilon > 0$ such that $d(x_i t_i, y_i t_i) > \varepsilon$ for all i. If $\{x_i t_i\}$ diverges, then there exists an open neighbourhood U with compact closure $\overline{U}$, $\overline{O(x)} \subset U$ and $x_i t_i \notin \overline{U}$ for all $i \geqslant i_0$, for some i_0. One may find a sequence $\{s_i\}$ in G with $x_i s_i \in U$ and $x_i(s_i + 1) \notin U$. Since $\{x_i s_i\} \subset \overline{U}$, there exists a subsequence $\{r_i\}$ of $\{s_i\}$ such that $x_i r_i \to y \in \overline{U}$. Thus $x_i(r_i + 1) \to y(1) \notin U$. But $y(1) \in D(x) = \overline{O(x)} \subset U$ and hence we have a contradiction. Hence we may assume that $x_i t_i \to a$ and $y_i t_i \to b$. This implies that $(a, b) \in D(x, x) = 0(x, x)$. Therefore $a = b$ which contradicts the assumption that $d(x_i t_i, y_i t_i) > \varepsilon$ for all i. Thus (M, π) is pointwise equicontinuous and the proof of the theorem is now complete. ∎

As remarked earlier, pointwise equicontinuity implies uniform equicontinuity if the space is compact. This is false however if the space is not compact as it is the case for the discrete system defined on the open interval $(0, 1)$ by $\pi^n(x) = x^{2^n}$.

4. EQUICONTINUITY AND LIPSCHITZ STABILITY

It has been believed for quite some time that uniform equicontinuity is equivalent

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to Lipschitz stability. The following example shows that this is not true.

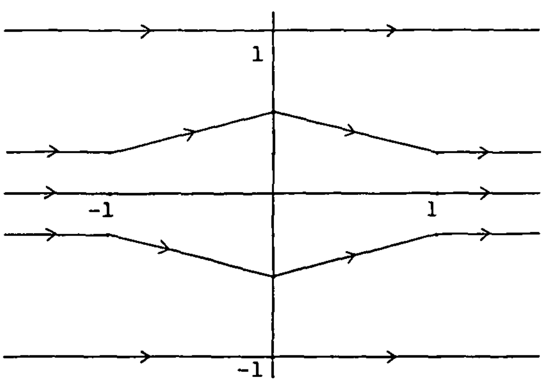
EXAMPLE 4.1. Let $M = \{(x, y) \in R^2 \mid -1 \leqslant y \leqslant 1\}$ with the usual metric. To define a continuous dynamical system on M, we start by describing the curves that will constitute the orbits. These will be the graphs of the functions $y = f_c(x)$, $c \in [-1, 1]$ defined by:

(i) if $c \neq 0$ then

$$f_c(x) = \begin{cases} c[\left(1 - \frac{1}{\sqrt{|c|}}\right)|x| + \frac{1}{\sqrt{|c|}}] & \text{if } |x| \leqslant 1 \\ c & \text{if } |x| \geqslant 1 \end{cases}$$

(ii) $f_O(x) = 0$.

The dynamical system π on M can be defined via the transition map: $\pi^t(a, b) = (a + t, f_b(a + t))$. (See the figure). It is trivial to check that this system is uniformly equicontinuous. To see that it is not Lipschitz stable take, for example, the two points $(-5, 0)$, $(-5, \varepsilon)$, $\varepsilon > 0$. $\pi^5(-5, 0) = (0, 0)$ and $\pi^5(-5, \varepsilon) = (0, \sqrt{\varepsilon})$.



Hence

$$\frac{d(\pi^5(-5, 0), \pi^5(-5, \varepsilon))}{d((-5, 0), (-5, \varepsilon))} = \frac{1}{\sqrt{\varepsilon}}.$$

Now it is clear that the above ratio can be made arbitrarily large by making ε arbitrarily small. This shows that this sytem is not Lipschitz stable.

For the rest of the paper, all the dynamical systems considered will be smooth. That is to say, the space M will be smooth connected complete Riemannian manifold and, $\forall t \in G$, $\pi^t \colon M \to M$ will be a diffeomorphism.

It is natural to ask whether uniform equicontinuity and Lipschitz stability are equivalent in the nice situation of smooth dynamical systems. The answer is again in the negative as can be seen from the following example which is, in fact, a modification of Example 4.1.

EXAMPLE 4.2. The orbits in Example 4.1 can be smoothed at $x = -1, 0, 1$. After doing so we take the restriction of the smooth system obtained to the subset $[-a, a] \times [-1, 1]$ for a sufficiently large $a > 0$. By identifying each point $(-a, y)$ with (a, y) and each $(x, -1)$ with $(x, 1)$ we obtain a differentiable dynamical system on an anchor ring. This system is again uniformly equicontinuous but not Lipschitz stable.

Before we give our result on the conditions under which uniform equicontinuity implies Lipschitz stability we need the following lemma.

LEMMA 4.3. *Let (M, π) be a uniformly equicontinuous dynamical system. Then for any pair $x, y \in M$, we have $S_{xy} = \sup\{d(xt, yt) \mid t \in G\} < \infty$.*

PROOF: For a small $\varepsilon > 0$, there exists $\delta > 0$ such that $d(at, bt) < \varepsilon$, for all $t \in G$, whenever $d(a, b) < \delta$. Now if for some $x, y \in M$, S_{xy} is not finite, then the orbits $O(x)$ and $O(y)$ will be diverging from each other. Let x_1 be midpoint between x and y on a minimal geodesic connecting x and y [9]. Then the orbit $O(x_1)$ is diverging either from $O(x)$ or from $O(y)$. Assume that $O(x_1)$ is diverging from $O(x)$. Now, in a similar way, take x_2 the midpoint between x and x_1, then $O(x_2)$ either diverges from $O(x)$ or $O(x_1)$, say from $O(x_1)$. By repeating this process one obtains two points x_i, x_j with $d(x_i, x_j) < \delta$ and $d(x_i s, x_j s) > \varepsilon$ for some $s \in G$, which is a contradiction. This completes the proof of the lemma. ∎

THEOREM 4.4. *Let (M, π) be a uniformly equicontinuous dynamical system in which M is compact. If for each $a \in M$, $\lim_{x \to a} S_{xa}/d(x, a)$ exists, then (M, π) is Lipschitz stable.*

PROOF: For each $a \in M$, let $\varphi_a = \lim_{x \to a} S_{xa}/d(x, a)$. Define a function $h: M \times M \to R$ by

$$h(a, b) = \begin{cases} \frac{S_{xa}}{d(x, a)} & \text{if } a \neq b \\ \varphi_a & \text{if } a = b. \end{cases}$$

Then h is continuous. Since $M \times M$ is compact, it follows that for all $x, y \in M$, $h(x, y) \leqslant k$ for some constant $k > 0$. Hence $S_{xy}/d(x, y) \leqslant k$ for all $x, y \in M$. This implies that $d(xt, yt)/d(x, y) \leqslant k$ and thus (M, π) is Lipschitz stable. ∎

5. LIPSCHITZ STABILITY AND ISOMETRIES

Recall that we are working on smooth dynamical systems (M, π), where M is a complete Riemannian manifold. If g denotes the Riemannian metric on M then g induces a norm $\|\cdot\|$ on the tangent bundle TM and a distance function d on M. For each π^t, $T\pi^t$ denotes the differential of π^t.

It was shown in [4] that Lipschitz stability is independent of the choice of the metric g or the distance function d provided that the manifold is compact. This suggests one way to construct Lipschitz stable non-isometric systems on compact manifolds. To construct such a system we simply start with an isometric (and hence Lipschitz stable) dynamical system (M, π) where (M, g) is a compact Riemannian manifold. Then one can find another Riemannian metric g' on M under which (M, π) is not isometric. Since the system is Lipschitz stable on (M, g) then it is also Lipschitz stable on (M, g'). This shows that we have plenty of examples of non-isometric Lipschitz stable dynamical systems. However, all these systems "came from" isometric ones. This raises the fundamental question whether every Lipschitz stable system must be isometric with respect to some metric. Before we answer this question we need the following definitions.

DEFINITION 5.1: The dynamical system (M, π) is said to be isometric in variation if $\|T\pi^t(v)\| = \|v\|$ for all $v \in M$.

In [9], Kobayashi and Nomizu showed that the notions of isometry and isometry in variation are equivalent. In [4] the authors introduced the notion of Lipschitz stability in variation for smooth dynamical systems. This is defined as follows.

DEFINITION 5.2: A smooth dynamical system (M, π) is said to be Lipschitz stable in variation if there exist $k \geqslant 1$, $\delta > 0$ such that $\|T\pi^t(v)\| \leqslant k\|v\|$ for all $v \in TM$ with $\|v\| < \delta$ and all $t \in G$.

The authors then generalised the result of Kobayashi and Nomizu mentioned above by showing that the notion of Lipschitz stability, Lipschitz stability in variation and global Libschitz stability (that is, when $\delta = \infty$) are all equivalent [4].

Now we use the above information to give a positive answer to the question raised earlier in the section.

THEOREM 5.3. *Let (M, π) be a Lipschitz stable dynamical system on a manifold M with a Riemannian metric g. Then M admits a Riemannian metric $\widehat{g}$ under which (M, π) is isometric.*

PROOF: For $v \in TM$, let

$$|||v||| = \sup\{\|T\pi^t(v)\| \mid t \in G\}.$$

Since $\|T\pi^t(v)\| \leqslant k\|v\|$ for all $t \in G$, then $|||.|||$ is a norm on TM. It is straightforward to verify that $|||.|||$ satisfies the parallelogram law $|||v+w|||^2 + |||v-w|||^2 = 2[|||v|||^2 + |||w|||^2]$ for any $v, w \in TM$. This implies [7] that $|||.|||$ is generated by an inner product or a Riemannian metric $\widehat{g}$ on M in the sense that $[\widehat{g}(v, v)]^{1/2} = |||v|||$. It is clear that (M, π) is isometric with respect to $\widehat{g}$. ∎

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