ON BOOLEAN ALGEBRAS OF PROJECTIONS

by J. A. ERDOS

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In spectral theory on Banach spaces, certain more incisive results hold when the underlying space is weakly complete (that is, weakly *sequentially* complete). The standard proofs rely on the following deep theorem: any bounded linear map from the algebra of all complex continuous functions on a compact Hausdorff space to a weakly complete Banach space is weakly compact. The proof of this result depends in turn on a considerable amount of measure-theoretic machinery (see [4, Section VI.7]). We present here some alternative methods which avoid these technicalities. The results are then used to give an example of a set of projections, each having unit norm, which generate an unbounded Boolean algebra.

The results in question are as follows. Standard terminology (see, for example [4]) will be used.

THEOREM 1. *A strongly closed bounded Boolean algebra of projections in a weakly complete Banach space is complete.*

Proof. It is true in general ([4, Lemma XVIII.3.4]) that a strongly closed Boolean algebra $\mathscr{B}$ is complete if every increasing net of elements of $\mathscr{B}$ is strongly convergent.

Suppose now that $\mathscr{B}$ is a Boolean algebra of projections on the weakly complete space X and that $\|P\| \leqq K$, for all $P \in \mathscr{B}$. Let $\{P_\alpha\}$ be an increasing net of elements of $\mathscr{B}$. If $\{P_\alpha\}$ is not strongly convergent then for some $x \in X$, $\{P_\alpha x\}$ is not norm convergent and so is not a Cauchy net in $(X, \|\cdot\|)$. Hence, for some $k > 0$, we can find an increasing sequence $\{\alpha_i\}$ such that, (writing P_i for P_{α_i}),

$$\|(P_{i+1} - P_i)x\| > k. \tag{1}$$

We shall in due course obtain a contradiction to this statement.

First we show that $\{P_i x\}$ is a weak Cauchy sequence. Since every element of the dual X^* of X is a linear combination of two real-valued elements of X^*, it is sufficient to show that $\{\psi(P_i x)\}$ is a Cauchy sequence for every real-valued element ψ of X^*. Let

$$\mathscr{I}^+ = \{i : \psi[(P_{i+1} - P_i)x] \geqq 0\} \quad \text{and} \quad \mathscr{I}^- = \{i : \psi[(P_{i+1} - P_i)x] < 0\}.$$

Then for every integer n,

$$\begin{aligned}\sum_{i=1}^{n} |\psi[(P_{i+1} - P_i)x]| &= \sum_{i \in \mathscr{I}^+, i \leqq n} \psi[(P_{i+1} - P_i)x] - \sum_{i \in \mathscr{I}^-, i \leqq n} \psi[(P_{i+1} - P_i)x] \\ &= \psi\Big[\sum_{i \in \mathscr{I}^+, i \leqq n} (P_{i+1} - P_i)x\Big] - \psi\Big[\sum_{i \in \mathscr{I}^-, i \leqq n} (P_{i+1} - P_i)x\Big] \\ &\leqq 2K \|\psi\| \|x\|.\end{aligned}$$

Hence $\sum \psi[(P_{i+1} - P_i)x]$ is convergent and so $\{\psi(P_i x)\}$ is a convergent sequence. Since X is weakly complete, it follows that $\{P_i x\}$ converges weakly to some element y of X.

For any $\phi \in X^*$ and any integers i and j with $i > j$, since $P_i > P_j$ we have

$$|\phi[P_j(x-y)]| = |\phi(P_jP_ix - P_jy)| = |P_j^*\phi(P_ix - y)|.$$

The limit, as i tends to infinity, of the right-hand term is zero and so it follows that

$$P_jx = P_jy, \tag{2}$$

for any integer j. Let N_i be the range of P_i. Then $\bigcup_{i=1}^{\infty} N_i$ is a linear subspace of X and so its norm closure is the same as its weak closure. Since y is in the weak closure it follows that for any $\varepsilon > 0$, there exists an integer i and a vector $z \in N_i$ such that $\|y - z\| < \varepsilon$. Then, since $P_{i+1} > P_i$, $P_iz = P_{i+1}z$ and so, using (2)

$$P_{i+1}x - P_ix = P_{i+1}y - P_iy = (P_{i+1} - P_i)(y - z).$$

Hence

$$\|(P_{i+1} - P_i)x\| < K\varepsilon.$$

Since ε is arbitrary this contradicts (1) and so completes the proof.

The above proof uses ideas from Barry **[2]**. A shorter proof could be given by referring to the result of **[2]**. However, giving the proof in full serves to underline its elementary nature.

THEOREM 2. *Let $f \mapsto T_f$ be a continuous homomorphism from the algebra $C(\Lambda)$ of all continuous complex functions on the complex Hausdorff space Λ into the algebra of bounded linear operators on a weakly complete Banach space X. Then there exists a unique spectral measure $E(\cdot)$ on X defined on the Baire sets and countably additive in the strong operator topology such that, for all $f \in C(\Lambda)$,*

$$T_f = \int_\Lambda f(\lambda)E(d\lambda).$$

Proof. Without the hypothesis that X is weakly complete, an argument based on the Riesz representation theorem shows that there exists a unique spectral measure $F(\cdot)$ on the dual of X such that for all $f \in C(\Lambda)$

$$T_f^* = \int_\Lambda f(\lambda)F(d\lambda)$$

(see **[4**, Theorem XVII.2.4]). Thus it is only required to show that for every Baire set δ, there exists an operator $E(\delta)$ on X such that $E(\delta)^* = F(\delta)$.

We first assume that δ is compact. Then there exists a decreasing sequence (f_n) of continuous functions converging pointwise to the characteristic function of δ (see **[5**, p. 240, Theorem A]). For any $x \in X$, $\phi \in X^*$ we have

$$\phi(T_{f_n}x) = (T_{f_n}^*\phi)(x) = \int_\Lambda f_n(\lambda)F(d\lambda)\phi(x)$$

and so, by the monotone convergence theorem, the sequence $\{\phi(T_{f_n}x)\}$ converges to $(F(\delta)\phi)(x)$. Since X is weakly complete, this shows that $\{T_{f_n}\}$ converges in the weak operator topology to some operator $E(\delta)$. Clearly $E(\delta)^* = F(\delta)$.

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For an arbitrary Baire set δ, note that the set $\{E(\beta) : \beta \text{ compact}, \beta \subseteq \delta\}$ forms an increasing net of projections. The Boolean algebra generated by this set is bounded since for any element E we have $E^* = F(\gamma)$ for some Baire set γ. Hence by Theorem 1 this net converges to some projection which we call $E(\delta)$. Then for all $x \in X$, $\phi \in X^*$, since $(F(\cdot)\phi)(x)$ is a regular measure,

$$\phi(E(\delta)x) = \lim_{\beta} \phi(E(\beta)x) = \lim_{\beta} (F(\beta)\phi)(x) = (F(\delta)\phi)(x)$$

and so $E(\delta)^* = F(\delta)$. This completes the proof.

The above theorem is slightly weaker than the standard result (Theorem XVIII.2.5 of [4]) in that the measure is defined on the Baire sets rather than the Borel sets. However, in most applications Λ is metrisable and so the Baire and Borel sets coincide.

In the following example, the underlying Banach space is the trace class $\mathscr{C}_1$ of operators on the Hilbert space H. If $A \in \mathscr{C}_1$ then $\tau(A)$ denotes the trace of A. We refer to [4, Chapter XI] for properties of $\mathscr{C}_1$. In particular recall that every element of the dual of $\mathscr{C}_1$ is of the form $A \mapsto \tau(XA)$ where X is some bounded linear operator on H. Using this fact it is easy to prove that $\mathscr{C}_1$ is weakly complete; alternatively the weak completeness of $\mathscr{C}_1$ may be deduced from Corollary III.3 of [1] since $\mathscr{C}_1$ is the pre-dual of a W^*-algebra.

EXAMPLE. Let $\mathscr{E}$ be a set of commuting self-adjoint projections on the Hilbert space H which is totally ordered under the natural ordering of projections. Suppose that the commutant $\mathscr{E}'$ of $\mathscr{E}$ contains no non-zero compact operator. (For example, on $L^2[0, 1]$, if E_t is the operator of multiplication by the characteristic function of $[0, t]$, then $\mathscr{E} = \{E_t : 0 \leqq t \leqq 1\}$ is such a set.) Let $\mathscr{D}$ be the set of all finite subsets of $\mathscr{E}$ directed by inclusion. If $\Delta = \{E_1, E_2, \ldots, E_n\} \in \mathscr{D}$, we define $P_\Delta : \mathscr{B}(H) \to \mathscr{B}(H)$ by

$$P_\Delta(T) = \sum_{i=0}^{n} (E_{i+1} - E_i)T(E_{i+1} - E_i)$$

where (for notational convenience) $E_0 = 0$ and $E_{n+1} = I$. Since the trace class is an ideal of $\mathscr{B}(H)$, if $T \in \mathscr{C}_1$ then $P_\Delta(T) \in \mathscr{C}_1$. We shall prove that the set $\{P_\Delta : \Delta \in \mathscr{D}\}$, regarded as operators on $\mathscr{C}_1$, form a set of projections of norm 1 but the Boolean algebra they generate is not uniformly bounded.

It is clear that P_Δ is a projection. Also, for all $x \in H$,

$$\begin{aligned} \| P_\Delta(T)x \|^2 &= \sum_{i=0}^{n} \| (E_{i+1} - E_i)T(E_{i+1} - E_i)x \|^2 \\ &\leqq \| T \|^2 \sum_{i=0}^{n} \| (E_{i+1} - E_i)x \|^2 \\ &= \| T \|^2 \| x \|^2, \end{aligned}$$

where $\| \cdot \|$ is the norm in $\mathscr{B}(H)$. Hence $\| P_\Delta(T) \| \leqq \| T \|$. Now if $A \in \mathscr{C}_1$, for all $T \in \mathscr{B}(H)$,

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since τ is linear and satisfies $\tau(XY) = \tau(YX)$,

$$\tau[TP_\Delta(A)] = \sum_{i=0}^{n} \tau[T(E_{i+1}-E_i)A(E_{i+1}-E_i)]$$
$$= \sum_{i=0}^{n} \tau[(E_{i+1}-E_i)T(E_{i+1}-E_i)A]$$
$$= \tau[P_\Delta(T)A].$$

Therefore, denoting the norm in $\mathscr{C}_1$ by $\|\cdot\|_1$,

$$\|P_\Delta(A)\|_1 = \sup_{\|T\|\leqq 1} \tau[TP_\Delta(A)] = \sup_{\|T\|\leqq 1} \tau[P_\Delta(T)A] \leqq \sup_{\|T\|\leqq 1} \|P_\Delta(T)\|\,\|A\|_1 \leqq \|A\|_1$$

and so P_Δ is a projection of norm 1 on $\mathscr{C}_1$.

We now prove that the Boolean algebra generated by $\{P_\Delta : \Delta\in\mathscr{D}\}$ is not uniformly bounded. To do this we suppose the contrary and obtain a contradiction. Uniform boundedness would imply by the proof of Theorem 1, that the net $\{P_\Delta : \Delta\in\mathscr{D}\}$ converges strongly to a projection P. Now for each $A\in\mathscr{C}$, $P_\Delta(A)$ commutes with every element E belonging to Δ. Therefore $P(A)$ is in the commutant $\mathscr{E}'$ of $\mathscr{E}$. Since also $P(A)\in\mathscr{C}_1$ and $\mathscr{E}'$ contains no non-zero compact operator, it follows that $P = 0$.

However, if x is any unit vector and $\Delta = \{E_1, E_2, \ldots, E_n\}\in\mathscr{D}$, then

$$\tau[P_\Delta(x\otimes x)] = \tau[\sum_{i=0}^{n} (E_{i+1}-E_i)(x\otimes x)(E_{i+1}-E_i)]$$
$$= \sum_{i=0}^{n} \tau[(E_{i+1}-E_i)x\otimes(E_{i+1}-E_i)x]$$
$$= \sum_{i=0} \|(E_{i+1}-E_i)x\|^2 = 1.$$

This is true for all $\Delta\in\mathscr{D}$. Since τ is continuous on $\mathscr{C}_1$ this shows that the net $\{P_\Delta(x\otimes x)\}$ does not converge to zero and so contradicts the fact that $P = 0$. Therefore the Boolean algebra generated by $\{P_\Delta : \Delta\in\mathscr{D}\}$ is not uniformly bounded.

Another example of this phenomenon arises in a different context in [3]. However in this case the set of projections which generate an unbounded Boolean algebra are only uniformly bounded in norm but are not of norm 1.

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