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ON THE COERCIVITY OF ELLIPTIC SYSTEMS IN TWO DIMENSIONAL SPACES

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We establish necessary conditions for quadratic forms corresponding to strongly elliptic systems in divergence form to have various coercivity properties in a smooth domain in $\mathbb{R}^2$. We prove that if the quadratic form has some coercivity property, then certain types of BMO seminorms of the coefficients of the system cannot be very large. We use the connection between Jacobians and Hardy spaces and the special structures of elliptic quadratic forms defined on 2×2 matrices.

In this note, we study the coercivity of elliptic systems with measurable coefficients satisfying a strong ellipticity condition — the Legendre-Hadamard condition. In two dimensions, we find some interesting necessary conditions for coercivity which provide new and important tools for the study of homogenisation and spectra of these systems.

In [**2**], among other results, the following were established:

(A) If $u \in W^{1,n}(\mathbb{R}^n,\mathbb{R}^n)$, then $\det Du \in \mathcal{H}^1(\mathbb{R}^n)$ ($\mathcal{H}^1$ is the Hardy space) and

$$\|\det Du\|_{\mathcal{H}^1(\mathbb{R}^n)} \leqslant C(n)\,\|Du\|^n_{L^n(\mathbb{R}^n)}.$$

(B) There exists $c(n) > 0$ such that

$$c(n)\,\|b\|_{BMO(\mathbb{R}^n)} \leqslant \sup\Big\{\int_{\mathbb{R}^n} b \det Du\, dx;\quad u = (u_1,\dots,u_n) \in W^{1,n}(\mathbb{R}^n,\mathbb{R}^n),\ \|Du_i\|_{L^2(\mathbb{R}^n)} \leqslant 1\Big\}.$$

We apply these results to the study of coercivity of strongly elliptic quadratic forms with measurable coefficients, defined in a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary,

$$a(u,\Omega) = \int_\Omega A^{ij}_{\alpha,\beta}(x) D_\alpha u^i D_\beta u^j\, dx, \tag{1}$$

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where the summation convention is understood and $u \in W_0^{1,2}(\Omega, \mathbb{R}^2)$. The coefficients $A^{ij}_{\alpha,\beta}(x)$ belong to $L^\infty(\Omega)$ and satisfy the Legendre-Hadamard condition

$$A^{ij}_{\alpha\beta}(x)\xi_\alpha\xi_\beta\eta^i\eta^j \geqslant c\,|\xi|^2\,|\eta|^2, \tag{2}$$

for some constant $c > 0$. It is known [9, 7] that $A^{ij}_{\alpha,\beta}(x)P^i_\alpha P^j_\beta$ can be written in the form

$$B^{ij}_{\alpha,\beta}(x)P^i_\alpha P^j_\beta + b(x)\det P \tag{3}$$

for $P \in M^{2\times 2}$, the set of real-valued 2×2 matrices, and $B^{ij}_{\alpha,\beta}(x) \in L^\infty(\Omega)$ satisfying

$$c\,|P|^2 \leqslant B^{ij}_{\alpha,\beta}(x)P^i_\alpha P^j_\beta \leqslant C\,|P|^2, \tag{4}$$

where c, $C > 0$ are constants. Therefore $A^{ij}_{\alpha,\beta}(x)P^i_\alpha P^j_\beta$ is strongly polyconvex (see [1]).

In the two-dimensional case, the above quadratic form comes naturally from the linearisation of polyconvex variational integrals studied in nonlinear elasticity by Ball [1]. In [5], a quantity Λ is defined which gives a criterion for determining whether an elliptic system satisfying the Legendre-Hadamard condition can be homogenised. It is defined as

$$\Lambda = \inf\left\{\frac{\int_{\mathbb{R}^n} A^{ij}_{\alpha,\beta}(x)D_\alpha u^i D_\beta u^j\,dx}{\int_{\mathbb{R}^n} |Du|^2\,dx};\ u \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)\right\}, \tag{5}$$

where $A^{ij}_{\alpha,\beta}(x)$ is a periodic and measurable function and $1 \leqslant i,j \leqslant n$, $1 \leqslant \alpha,\beta \leqslant n$. It was establised in [5] that if $\Lambda \geqslant 0$, some homogenisation results can be obtained for the system

$$\begin{cases} \mathrm{Div}_\alpha A^{ij}_{\alpha,\beta}\left(\frac{x}{\varepsilon}\right)D_\beta u^j = f & \text{in } \Omega \\ u|_{x\in\partial\Omega} = 0. \end{cases} \tag{6}$$

If $\Lambda < 0$, the system cannot be homogenised. A natural question arises as to which conditions on the coefficients of the system imply $\Lambda \geqslant 0$. We answer this question for $n = 2$.

In [10, 11], counterexamples were given showing that Gårding's inequality may not hold in general for systems with L^∞ coefficients which satisfy the Legendre-Hadamard condition. In [3], examples were exhibited showing that system (6) cannot be homogenised even when the coefficients are continuous.

In this note, we establish necessary conditions such that (i) $a(u,\Omega) \geqslant 0$, or equivalently $\Lambda \geqslant 0$ if $\Omega = \mathbb{R}^n$; (ii) Gårding's inequality holds for $a(u,\Omega)$; (iii) the first eigenvalue through homogenisation is bounded (Theorem 3). The conditions are that certain types of BMO norms on b obtained from (3) cannot be too large. Before we state the main results, let us give some basic definitions and facts.

Let $\Omega \subset \mathbb{R}^n$ ($n \geqslant 2$) be a connected open set. A function $b : \Omega \to \mathbb{R}$ is in $\mathrm{BMO}(\Omega)$ if b is integrable in Ω and

$$\sup_Q \frac{1}{|Q|} \int_Q |b - b_Q|\, dx = \|u\|_{BMO(\Omega)} < \infty. \tag{7}$$

The above supremum is taken over all cubes Q with sides parallel to the axes and $b_Q = 1/(|Q|) \int_Q b\, dx$.

An extension theorem due to Jones [**6**] states that under certain conditions on Ω (which include the case that Ω has Lipschitz boundary) there exists a continuous extension of $\mathrm{BMO}(\Omega)$ to $\mathrm{BMO}(\mathbb{R}^n)$. If we denote by

$$\|b\|_{BMO(\Omega)} = \sup\left\{ \left(\frac{1}{|Q|} \int_Q |b - b_Q|^2\, dx \right)^{1/2} ; Q \subset \Omega \right\}, \tag{7'}$$

where the supremum is taken over all cubes with sides parallel to the axes, the seminorms given by (7) and (7′) are equivalent (see [**6**] for example).

If we consider Ω as a space of homogeneous type, we have another type of $\mathrm{BMO}(\Omega)$ which we denote by $\mathrm{BMO}_H(\Omega)$ with its BMO seminorm given by taking cubes with side length $l(Q) \leqslant \mathrm{dist}\,(Q, \Omega^c)$, and

$$\|b\|_{BMO_H(\Omega)} = \sup\left\{ \left(\frac{1}{|Q|} \int_Q |b - b_Q|^2\, dx \right)^{1/2} ; Q \subset \Omega,\, l(Q) \leqslant \mathrm{dist}\,(Q, \Omega^c) \right\}. \tag{7''}$$

We have

$$\|b\|_{BMO_H(\Omega)} \leqslant \|b\|_{BMO(\Omega)}.$$

After an extensive search of the literature in harmonic analysis, the author was not able to find a reference to confirm that under suitable conditions, the two seminorms given by (7') and (7") are equivalent.

The following are the main results of this note.

Theorem 1. *Suppose $\Omega \subset \mathbb{R}^2$ is open with Lipschitz boundary, $A^{ij}_{\alpha,\beta} : \Omega \to \mathbb{R}^2$ is measurable for $1 \leqslant i, j, \alpha, \beta \leqslant 2$, such that*

$$A^{ij}_{\alpha,\beta}(x) P^i_\alpha P^j_\beta = B^{ij}_{\alpha,\beta}(x) P^i_\alpha P^j_\beta + b(x) \det P, \tag{8}$$

where $b \in BMO(\Omega)$ and $B^{ij}_{\alpha,\beta}$ are measurable functions satisfying

$$c_0 |P|^2 \leqslant B^{ij}_{\alpha,\beta}(x) P^i_\alpha P^j \leqslant C_0 |P|^2, \tag{9}$$

for some constants $0 < c_0 \leqslant C_0$. Then there exists a constant $C_1 > 0$ depending only on C_0 such that $a(u, \Omega) \geqslant 0$ for all $u \in W^{1,2}_0(\Omega, \mathbb{R}^2)$ implies that $\||b\||_{BMO_H(\Omega)} \leqslant C_1$.

REMARK 1. If $\|b\|_{BMO(\Omega)}$ is sufficiently small, from (A) and the extension theorem in [6], we see that $a(u,\Omega) \geqslant 0$ in Theorem 1.

DEFINITION 1. *(See [8] for example.) For $b \in BMO(\Omega)$, the oscillation norm of b is defined by*

$$\|b\|_{*,\Omega} = \limsup_{d\to 0+}\Big(\sup\Big\{\Big(\frac{1}{|Q|}\int_Q |b-b_Q|^2\,dx\Big)^{1/2};\; Q\subset\Omega,\, l(Q)\leqslant d,\, \operatorname{dist}(Q,\Omega^c)\geqslant l(Q)\Big\}\Big),$$

where $\operatorname{dist}(\cdot,\Omega^c)$ *is the distance function. Obviously,* $\|b\|_{*,\Omega} \leqslant \|b\|_{BMO_H(\Omega)}$.

It is easy to see that $\|b\|_{*,\Omega} = 0$ if b is uniformly continuous in Ω. The following simple example shows that if b has points of jump discontinuity, $\|b\|_{*,\Omega} \neq 0$

EXAMPLE 1. Let us first look at the Heaviside function in $\mathbb{R}^1$,

$$H_k(x) = \begin{cases} 0 & \text{if } x<0,\\ k & \text{if } x>0,\\ \text{undefined} & \text{if } x=0.\end{cases}$$

It is easy to check that

$$\|b\|_{*,\mathbb{R}^1} = \|b\|_{BMO_H(\mathbb{R}^1)} = \|b\|_{BMO(\mathbb{R}^1)} = k/2.$$

We can generalise this example to a square $Q_1 = (-1,1)^2$ in $\mathbb{R}^2$. Let

$$f(x,y) = \begin{cases} 0 & \text{if } -1<x<0,\, -1<y<1\\ k & \text{if } 0<x<1,\, -1<y<1\\ \text{undefined} & \text{if } x=0.\end{cases}$$

We have

$$\|b\|_{*,Q_1} = \|b\|_{BMO_H(Q_1)} = \|b\|_{BMO(Q_1)} = k/2.$$

THEOREM 2. *Suppose the assumptions in Theorem 1 are satisfied. If Gårding's inequality holds for $a(u,\Omega)$, that is, there exist $\lambda_0>0$, $\lambda_1\geqslant 0$ such that*

$$a(u,\Omega) \geqslant \lambda_0\int_\Omega |Du|^2\,dx - \lambda_1\int_\Omega |u|^2\,dx \tag{10}$$

for all $u\in W_0^{1,2}(\Omega,\mathbb{R}^2)$, then

$$\|b\|_{*,\Omega} \leqslant C_1$$

where $C_1>0$ is given by Theorem 1.

REMARK 2. Define the oscillation norm of b on $\overline{\Omega}$ by

$$\|b\|_{*,\overline{\Omega}} = \limsup_{d\to 0+}\left(\sup\left\{\frac{1}{|Q|}\int_Q |b-b_Q|\,dx;\ Q\cap\Omega\neq\emptyset,\ l(Q)\leqslant d\right\}\right),$$

where we extend b to be a BMO $(\mathbb{R}^2)$ function (see [6]). If $\|b\|_{*,\overline{\Omega}}$ is small enough, we have, by using a classical partition of unity method used, for example, in [4, Chapter 1] and inequality (A), that Gårding's inequality holds for $a(u,\Omega)$ in Theorem 2.

THEOREM 3. *Suppose b and $B^{ij}_{\alpha\beta}$ given by (3) are periodic and continuous. Let*

$$\lambda_\varepsilon = \inf\left\{\lambda,\ a_\varepsilon(u,\Omega)+\lambda\int_\Omega |u|^2\,dx\geqslant 0;\ u\in W_0^{1,2}(\Omega,\mathbb{R}^2)\right\},$$

where

$$a_\varepsilon(u,\Omega)=\int_\Omega A^{ij}_{\alpha,\beta}\left(\frac{x}{\varepsilon}\right)D_\alpha u^i D_\beta u^j\,dx.$$

If λ_ε is bounded above when $\varepsilon\to 0$, then $\|b\|_{BMO(D)}\leqslant C_1$, where D is the period of b and $C_1>0$ is given by Theorem 1.

REMARK 3. If $\|b\|_{BMO(D)}$ is sufficiently small, λ_ε defined in Theorem 3 is nonnegative if we simply apply (A) and the partition of unity.

The following lemma is a simple consequence of the proof of (B), Theorem III.2 in [2].

LEMMA 1. *Let $\Omega\subset\mathbb{R}^2$ be an open set. For $b\in BMO(\Omega)$, there exists a constant $C>0$ independent of Ω and b, such that*

$$\|b\|_{BMO_H(\Omega)}\leqslant C\sup\left\{\int_\Omega b\det Du\,dx;\right.$$
$$\left. u=(u_1,u_2)\in W_0^{1,2}(\Omega,\mathbb{R}^2),\ \|Du_1\|_{L^2(\Omega)}\leqslant 1,\ \|Du_2\|_{L^2(\Omega)}\leqslant 1\right\}.$$

PROOF OF THEOREM 1: For any $\varepsilon>0$, we have from Lemma 1 that there exists $u^{(\varepsilon)}=\left(u_1^{(\varepsilon)},u_2^{(\varepsilon)}\right)\in W_0^{1,2}(\Omega,\mathbb{R}^2)$, with $\left\|Du_1^{(\varepsilon)}\right\|\leqslant 1$, $\left\|Du_2^{(\varepsilon)}\right\|\leqslant 1$, such that

$$\|b\|_{BMO_H(\Omega)}-\varepsilon\leqslant C_2\int_\Omega b\det Du^{(\varepsilon)}dx.$$

On replacing $u^{(\varepsilon)}$ by $v^{(\varepsilon)}=\left(u_1^{(\varepsilon)},-u_2^{(\varepsilon)}\right)$, we see that

$$C_2\int_\Omega b\det Dv^{(\varepsilon)}dx\leqslant -\|b\|_{BMO_H(\Omega)}+\varepsilon.$$

Suppose $a(u,\Omega) \geqslant 0$ for all $u \in W_0^{1,2}(\Omega,\mathbb{R}^2)$. Then we have

$$\begin{aligned} 0 \leqslant a\left(v^{(\varepsilon)},\Omega\right) &= \int_\Omega [B^{ij}_{\alpha,\beta}(x) D_\alpha v_i^{(\varepsilon)} D_\beta v_j^{(\varepsilon)} + b(x) \det Dv^{(\varepsilon)}]\, dx \\ &\leqslant C_0 \left|Dv^{(\varepsilon)}\right|^2 - \frac{1}{C_2}[\|b\|_{BMO_H(\Omega)} - \varepsilon] \\ &\leqslant 2C_0 - \frac{1}{C_2}[\|b\|_{BMO_H(\Omega)} - \varepsilon]. \end{aligned}$$

Therefore

$$\|b\|_{BMO_H(\Omega)} \leqslant 2C_0C_2 + C_2\varepsilon.$$

Let $C_1 = 2C_0C_2$. Since $\varepsilon > 0$ is arbitrary, $|||b|||_{BMO(\Omega)} \leqslant C_1$. The proof is finished. □

PROOF OF THEOREM 2: Let $Q_{d_k} \subset \Omega$ be a sequence of cubes with side length $2d_k$ such that $\operatorname{dist}(Q_{d_k},\Omega^c) \geqslant 2d_k$, $d_k \to 0$ and

$$\left(\frac{1}{|Q_{d_k}|}\int_{Q_{d_k}} \left|b - b_{Q_{d_k}}\right|^2 dx\right)^{1/2} \to \|b\|_{*,\Omega}.$$

Let $\widetilde{Q}_k$ be a cube with the same centre as Q_{d_k} and with side length $4d_k$. Let $v \in W_0^{1,2}(\Omega,\mathbb{R}^2)$ be such that v is supported in the closure of $\widetilde{Q}_k$. Since Gårding's inequality holds for $a(\cdot,\Omega)$, there exist $\lambda_0 > 0$ and $\lambda_1 \geqslant 0$ such that (10) holds for all $u \in W_0^{1,2}(\Omega)$. In particular, it holds for v. Let x_k be the centre of $\widetilde{Q}_k$. Change variables $x - x_k = 2d_k y$ in (10) and let $u(y) = v(x_k + 2d_k y)$, and $b(x_k + 2d_k y) = b_k(y)$. We have

$$\int_{Q_1} [B^{ij}_{\alpha\beta}(x_k + 2d_k y) D_\alpha u^i D_\beta u^j + b_k(y) \det Du(y)\, dy$$
$$\geqslant \lambda_0 \int_{Q_1} |Du|^2\, dy - (2d_k)^2 \lambda_1 \int_{Q_1} |u|^2\, dy,$$

where Q_1 is the cube with side length 2 centred at 0. Since this inequality is true for all $u \in W_0^{1,2}(Q_1,\mathbb{R}^2)$, we have, from the Sobolev-Poincaré inequality and by taking $d_k > 0$ small enough,

$$\int_{Q_1} [B^{ij}_{\alpha\beta}(x_k + 2d_k y) D_\alpha u^i D_\beta u^j + b_k(y) \det Du(y)\, dy \geqslant 0,$$

where $Q_{1/2}$ is the cube centred at 0 with side length 1. Hence from Theorem 1,

$$\left(\frac{1}{|Q_{1/2}|}\int_{Q_{1/2}} \left|b_k - (b_k)_{Q_{1/2}}\right|^2 dy\right)^{1/2} \leqslant C_1.$$

$$\left(\frac{1}{|Q_{1/2}|}\int_{Q_{1/2}} \left|b_k - (b_k)_{Q_{1/2}}\right|^2 dy\right)^{1/2} = \left(\frac{1}{|Q_{d_k}|}\int_{Q_{d_k}} \left|b - b_{Q_{d_k}}\right|^2 dx\right)^{1/2}.$$

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Therefore

$$\|b\|_{*,\Omega} \leqslant C_1.$$

The proof is complete. ∎

PROOF OF THEOREM 3: Let us assume that b is of period 2, that is, $b(x+z) = b(x)$ for all $z = (2j, 2k)$ where j, k are integers. Suppose that $\lambda_\varepsilon \leqslant C$ for some constant $C > 0$. Then we have

$$a_\varepsilon(u,\Omega) + \lambda_\varepsilon \int_\Omega |u|^2\, dx \geqslant 0,$$

for all $u \in W_0^{1,2}(\Omega, \mathbb{R}^2)$. Let us take a cube Q_ε with side length 2ε such that $Q_{2\varepsilon} \subset \Omega$ with side length 4ε has the same centre as Q_ε. Let $v \in W_0^{1,2}(\Omega, \mathbb{R}^2)$ be such that v is supported in the closure of $Q_{2\varepsilon}$. We have

$$\begin{aligned} 0 &\leqslant a_\varepsilon(v, Q_{2\varepsilon}) + \lambda_\varepsilon \int_{Q_{2\varepsilon}} |v|^2\, dx \\ &= \int_{Q_{2\varepsilon}} [B^{ij}_{\alpha\beta}\Big(\frac{x}{\varepsilon}\Big) D_\alpha v^i D_\beta v^j + b\Big(\frac{x}{\varepsilon}\Big) \det Dv(x) + \lambda_\varepsilon |v|^2]\, dx. \end{aligned}$$

Let x_0 be the centre of $Q_{2\varepsilon}$. Change variables $x - x_0 = \varepsilon y$, and let $u(y) = v(x_0 + \varepsilon y)$. We have from the Sobolev-Poincaré inequality, and the bound of $B^{ij}_{\alpha\beta}$, that

$$\begin{aligned} 0 &\leqslant \int_{Q_2} [B^{ij}_{\alpha\beta}(y) D_\alpha u^i D_\beta u^j + b(y) \det Du(y) + \varepsilon^2 \lambda_\varepsilon |u|^2]\, dy \\ &\leqslant \int_{Q_2} [(C_0 + CC(Q_2)\varepsilon^2)\, |Du|^2 + b \det Du]\, dy, \end{aligned}$$

for all $u \in W_0^{1,2}(Q_2, \mathbb{R}^2)$. Therefore, $\|b\|_{BMO(Q_1)} \leqslant C_1 + O(\varepsilon)$, and hence $\|b\|_{BMO(Q_1)} \leqslant C_1$. The proof is complete. ∎

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