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# Attracting Cantor set of positive measure for a $C^\infty$ map of an interval

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I dedicate this paper to the memory of V. M. Alexeyev

*Abstract.* We give an example of a smooth map of an interval into itself, conjugate to the Feigenbaum map, for which the attracting Cantor set has positive Lebesgue measure.

## 0. *Introduction*

Let us consider a one-parameter family of smooth unimodal (i.e. with 'one hump') maps of an interval into itself. As an example one can take $f_\mu(x) = \mu x(1-x)$. If a map depends on a parameter continuously and if the family contains maps with both zero and positive topological entropy, it also contains a map $f$ with periodic points of periods $1, 2, 2^2, 2^3, \ldots$, and no other periods. Suppose that $f$ has no homtervals (i.e. open intervals, on which all iterates of $f$ are homeomorphisms). Denote by $I_n$ the interval between the periodic point of period $2^n$, closest to the critical point, and the second point with the same image under $f$. Assume also that one of the endpoints of the whole interval is a fixed point and the second endpoint is mapped to the first one. Then $f^{2^n}|_{I_n}$ is topologically conjugate to $f$. Feigenbaum [**3**] conjectured that for a 'good' map $f$, the sequence $(f^{2^n}|_{I_n})_{n=0}^{\infty}$, after rescaling (i.e. an affine change of a coordinate) converges to a certain map $F$. We shall call this limit map the Feigenbaum map. The detailed description of this and other connected problems can be found in [**2**].

For the Feigenbaum map, $F^{2^n}|_{I_n}$, after rescaling, is equal to $F$. The existence of this (real analytic) map was proved by Campanino and Epstein [**1**] and Lanford [**5**].

From the kneading theory we know that if a map $f$ has the same kneading invariant as $F$ (i.e. the images of the critical point lie to the left or right of the critical point for the same iterates of both $f$ and $F$) and $f$ has no homtervals, then $f$ is topologically conjugate to $F$ (see [**2**]).

The set of non-wandering points for a map $f$, topologically conjugate to $F$, consists of a Cantor set (more exactly, a set homeomorphic to the Cantor set) $C = \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{2^n-1} f^k(I_n)$, and periodic points, lying in the gaps of $C$. This Cantor set attracts all points which are not eventually periodic ([**6**]).

In connection with the general question about the Lebesgue measure of attractors, one can ask, what the measure of $C$ is. If $f$ satisfies the Feigenbaum conjecture, then the answer is zero. Since the rescaling constant for $F$ is $\delta < \frac{1}{2}$, the set $\bigcup_{k=0}^{2^n-1} f^k(I_n)$ consists of $2^n$ intervals, the longest of which has a length of approximately $\alpha\delta^m$ (for some constant $\alpha$). Since $2^n\alpha\delta^n \to 0$ as $n \to \infty$, $C$ has Lebesgue measure zero.

The results for diffeomorphisms would suggest that the answer should be zero for all $C^{1+\varepsilon}$ maps. However, here the situation is different. Namely, we prove the following theorem:

THEOREM. *There exists a $C^\infty$ concave map f, conjugate to the Feigenbaum map, with the attracting Cantor set C of positive Lebesgue measure.*

This result does not give the complete solution to the problem. One can ask, whether there is an example of such a map with some additional properties. The desired properties would be, for instance:

($a$) polynomial behaviour in a neighbourhood of a critical point (i.e. a critical point not 'flat');

($b$) absolute continuity of the unique invariant probabilistic measure on $C$ with respect to the Lebesgue measure.

Notice that ($a$) implies that the map is 'almost symmetric' (for $x$, $y$ with $f(x)=f(y)$, the ratio of distances of $x$ and $y$ from the critical point is bounded). It can be shown that even this 'almost symmetry' cannot be obtained by the technique used in this paper. Lemma 4 shows that our example does not have property ($b$).

## 1. *Construction*

We start by defining two sequences of points of the interval [0, 1]: $0=a_2<b_2<a_4<b_4<a_6<b_6<\cdots<b_7<a_7<b_5<a_5<b_3<a_3<b_1<a_1=1$, by setting:

$$|a_n-b_n|=\frac{1}{(n+1)^2}, \qquad |a_{n+2}-b_n|=\frac{1}{n(n+1)^2}.$$

Since

$$\sum_{n=1}^{\infty}\left(\frac{1}{(n+1)^2}+\frac{1}{n(n+1)^2}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1,$$

we see that all points with even indices lie to the left of all points with odd indices and there exists a common limit lying between even and odd points

$$c=\lim_n a_n=\lim_n b_n.$$

Now we begin to define $f$:

$$f(c)=1, \qquad f(a_n)=1-\frac{1}{n\cdot n!}, \qquad f(b_n)=1-\frac{1}{n\cdot(n+1)!}.$$

Notice that

$$0=f(a_1)<f(b_1)<f(a_2)<f(b_2)<f(a_3)<f(b_3)<\cdots$$

and

$$\lim_n f(a_n)=\lim_n f(b_n)=f(c)=1.$$

We define $f$ as a linear (we use this word instead of the more precise 'affine') map on each interval:

$$L_n = \begin{cases} [a_n, b_n] & \text{if } n \text{ is even,} \\ [b_n, a_n] & \text{if } n \text{ is odd.} \end{cases}$$

We also set

$$M_n = \begin{cases} [b_n, a_{n+2}] & \text{if } n \text{ is even,} \\ [a_{n+2}, b_n] & \text{if } n \text{ is odd.} \end{cases}$$

Since

$$[0, 1] = \{c\} \cup \bigcup_{n\geq 1} L_n \cup \bigcup_{n\geq 1} M_n,$$

it remains to define $f$ on intervals ('gaps') $M_n$. Before doing it, we compute the slope (i.e. the absolute value of the derivative) of $f$ on $L_n$. Denote this slope by $\lambda_n$. Then

$$\lambda_n = \frac{|f(b_n) - f(a_n)|}{|b_n - a_n|} = \frac{n+1}{n!}.$$

Set

$$\mu_n = \lambda_{n-1}^{2^0} \cdot \lambda_{n-2}^{2^1} \cdot \lambda_{n-3}^{2^2} \cdot \ldots \cdot \lambda_1^{2^{n-2}}.$$

We have $\mu_1 = 1$ and $\mu_{n+1} = \lambda_n \cdot \mu_n^2$. From this it is easy to check by induction that $\mu_n = n!$ (this result may be surprising at a first glance – the numbers $\lambda_k$ are mainly very small, but their product $\mu_n$ is large).

Consider the intervals $L_n$ and $L_{n+2}$ and the gap $M_n$ between them. We already can draw the graphs of $f$ on $L_n$ and $L_{n+2}$; they are segments of straight lines. Let us see where these lines intersect each other. Denote this point by

$$p_n = \begin{cases} (a_{n+2} - x_n, f(a_{n+2}) - y_n) & \text{if } n \text{ is even,} \\ (a_{n+2} + x_n, f(a_{n+2}) - y_n) & \text{if } n \text{ is odd.} \end{cases}$$

We have then:

$$\begin{cases} y_n/x_n = \lambda_{n+2} \\ \dfrac{f(a_{n+2}) - f(b_n) - y_n}{1/(n(n+1)^2) - x_n} = \lambda_n. \end{cases}$$

Solving this system of equations we get

$$x_n = 1/(n+2)[(n+1)^2(n+2) - (n+3)]. \tag{1}$$

For every $n \geq 1$ we have $0 < x_n < \frac{1}{2}|M_n|$. To check it, it is enough to notice that the first inequality is equivalent to

$$(n+1)^2 > 1 + 1/(n+2),$$

and the second one to

$$(n^2 + 2n + 1)(n^2 + 2n + 4) > (n+2)(n+3).$$

We consider an auxiliary function $\varphi:\mathbb{R}\to\mathbb{R}$ given by

$$\varphi(x)=\begin{cases} x & \text{if } x\le 0\\ \int_0^x (1-\psi(y))\,dy & \text{if } 0<x<1\\ \frac{1}{2} & \text{if } x\ge 1\end{cases}$$

where

$$\psi(y)=\int_0^y \eta(t)\,dt\Big/\int_0^1 \eta(t)\,dt \quad\text{and}\quad \eta(t)=\exp\frac{1}{t(t-1)}.$$

It is easy to check that $\varphi$ is of class $C^\infty$ and concave.

We show how to use $\varphi$ for filling the gaps. If a function $g$ defined on $(a-\varepsilon, a]\cup[b, b+\varepsilon)$ is such that

$$g(x)=\begin{cases} g(a)+\alpha(x-a) & \text{for } x\in(a-\varepsilon, a]\\ g(b)+\beta(x-b) & \text{for } x\in[b, b+\varepsilon)\end{cases} \tag{2}$$

where

$$\frac{g(a)-g(b)}{a-b}=\frac{\alpha+\beta}{2} \tag{3}$$

and $a<b$, $\alpha>\beta$, then we can extend it to a concave function of class $C^\infty$ on $(a-\varepsilon, b+\varepsilon)$ by setting for $x\in(a,b)$

$$g(x)=g(a)+\beta(x-a)+(\alpha-\beta)(b-a)\varphi\left(\frac{x-a}{b-a}\right). \tag{4}$$

To prove this, it is enough to show that the formulae (2) and (4) coincide on $(a-\varepsilon, a]\cup[b, b+\varepsilon)$. This is a simple computation and we omit it. Concavity of $g$ follows from the fact that

$$g''(x)=\frac{\alpha-\beta}{b-a}\varphi''\left(\frac{x-a}{b-a}\right)$$

and the concavity of $\varphi$.

We estimate the derivative of $g$

$$|g'(x)|\le\max(|\alpha|, |\beta|) \tag{5}$$

by the concavity of $g$. For $k>1$, we have

$$g^{(k)}(x)=(\alpha-\beta)(b-a)^{1-k}\varphi^{(k)}\left(\frac{x-a}{b-a}\right).$$

Thus,

$$\sup_{[a,b]}|g^{(k)}|=(\alpha-\beta)(b-a)^{1-k}\cdot\sup_{[0,1]}|\varphi^{(k)}| \quad\text{for } k>1. \tag{6}$$

Notice that condition (3) is equivalent to the fact that the point of intersection of the lines defined by

$$y=g(a)+\alpha(x-a) \quad\text{and}\quad y=g(b)+\beta(x-b)$$

has the first coordinate $(a+b)/2$.

Now we are ready to define $f$ on $M_n$. On the interval $(b_n, a_{n+2}-2x_n]$ (if $n$ is even) or $[a_{n+2}+2x_n, b_n)$ (if $n$ is odd), we define it as the same linear function as on $L_n$. Then the gap which remains is such that the procedure described above may be used. We do it and get $f$ defined on the whole interval $[0, 1]$.

## 2. *Properties*

From the construction it follows that $f$ is continuous and concave on $[0, 1]$ and of class $C^\infty$ on $[0, c)\cup(c, 1]$. To see that it is of class $C^\infty$ on the whole interval $[0, 1]$, we use (5) and (6). Since $\lim_n \lambda_n = 0$, we get by (5),

$$\lim_{x\to c} |f'(x)| = 0.$$

For $k>1$, we have

$$\lim_n (\lambda_n - \lambda_{n+2})(2x_n)^{1-k} \sup_{[0,1]} |\varphi^{(k)}| = 0,$$

because $(2x_n)^{1-k}$ is a polynomial function of $n$ and

$$\lambda_n - \lambda_{n+2} = \frac{1}{n!}\left(n+1-\frac{n+3}{(n+1)(n+2)}\right).$$

Hence, by (6),

$$\lim_{x\to c} |f^{(k)}(x)| = 0.$$

Now the smoothness of $f$ follows from the inductive use of the following fact: if $\psi$ is continuous on $[0, 1]$ and of class $C^1$ on $[0, c)\cup(c, 1]$, and the limit $\lim_{x\to c}\psi'(x)$ exists, then $\psi$ is of class $C^1$ on $[0, 1]$.

Hence, we have proved the following properties of $f$:

$(A)$ $f$ is concave and of class $C^\infty$ on $[0, 1]$.

*Remark.* Our function is defined on $[f^2(c), f(c)]$. If we want it to be defined on some $[a, b]$ such that $f(a)=f(b)=a$ then we can take $a=-\frac{3}{2}$, $b=\frac{7}{4}$, and set

$$f(x)=\begin{cases}\frac{3}{2}x+\frac{3}{4} & \text{for } x\in[-\frac{3}{2}, 0),\\ -2x+2 & \text{for } x\in(1, \frac{7}{4}],\end{cases}$$

and $f$ remains $C^\infty$ and concave. Notice also that outside $[0, 1]$ the slope of $f$ defined in this way is larger than 1.

We continue investigating the properties of $f$. For $n\ge 1$ we set $g_n = f^{2^{n-1}-1}|_{[f(a_n),1]}$ and $h_n = f^{2^{n-1}}$.

LEMMA 1. *For every $n\ge 1$ we have:*

(*a*) $h_n(a_n)=a_{n+1}$;
(*b*) $h_n(b_n)=a_{n+2}$;
(*c*) $h_n(a_{n+1})=b_n$;
(*d*) $h_n(c)=a_n$;
(*e*) *$g_n$ is linear and has slope $\mu_n$;*
(*f*) *$g_n$ is orientation-reversing if $n$ is even and orientation-preserving if $n$ is odd;*
(*g*) *$f$ is linear on $f^i([f(a_n), 1])$, $i=0, 1, 2, \ldots, 2^{n-1}-2$.*

*Proof.* Notice first that since

$$f(a_n) < f(b_n) < f(a_{n+1}) < 1 = f(c),$$

$(e)$ implies:

$$|h_n(a_n) - h_n(b_n)| = |g_n(f(a_n)) - g_n(f(b_n))| = \frac{1}{n+1},$$

$$|h_n(b_n) - h_n(a_{n+1})| = |g_n(f(b_n)) - g_n(f(a_{n+1}))| = \frac{1}{n(n+1)^2},$$

and

$$|h_n(a_n) - h_n(c)| = |g_n(f(a_n)) - g_n(1)| = \frac{1}{n}.$$

Consequently, $(a)$, $(e)$ and $(f)$ imply $(b)$, $(c)$ and $(d)$.

Now we shall prove $(a)$, $(e)$, $(f)$ and $(g)$ by induction. For $n = 1$, we have $2^{n-1} = 1$ and $g_n = f^0 = \text{id}$. Hence, $(e)$, $(f)$ and $(g)$ hold for $n = 1$. We have $f(a_1) = 0 = a_2$ and therefore also $(a)$ holds for $n = 1$.

We assume that $(a)$, $(e)$, $(f)$ and $(g)$ hold for $n = k$ and shall prove them for $n = k+1$. We have shown already that $(b)$, $(c)$ and $(d)$ hold for $n = k$. By $(b)$ and $(c)$,

$$f^{2^k}(a_{k+1}) = f^{2^{k-1}}(b_k) = a_{k+2},$$

and thus $(a)$ holds for $n = k+1$. By $(e)$ (for $n = k$), $g_k$ is monotone, and hence by $(c)$ and $(d)$ (also for $n = k$), we have $g_k([f(a_{k+1}), 1]) = L_k$. Therefore

$$g_{k+1} = g_k \circ f \circ g_k|_{[f(a_{k+1}), 1]}$$

is a composition of three linear maps, and consequently is linear itself. Its slope is equal to

$$\lambda_k \cdot \mu_k^2 = \mu_{k+1},$$

and this proves $(e)$ for $n = k+1$. It is affecting the orientation in the same way as $f|_{L_k}$, and this proves $(f)$ for $n = k+1$. To prove $(g)$ (for $n = k+1$), notice that for $i = 0, 1, 2, \ldots, 2^{k-1} - 2$ it follows immediately from $(g)$ for $n = k$ that $f$ is linear on $f^i([f(a_{k+1}), 1])$. For $i = 2^{k-1} - 1$ we know already that $f^i([f(a_{k+1}), 1]) = L_k$. For $i = 2^{k-1}, 2^{k-1} + 1, \ldots, 2^k - 2$ we also have to use the fact that

$$f^i|_{[f(a_{k+1}), 1]} = f^{i-2^{k-1}} \circ f \circ g_k|_{[f(a_{k+1}), 1]}$$

and

$$f \circ g_k([f(a_{k+1}), 1]) = f(L_k) \subset [f(a_k), 1],$$

and $(g)$ for $n = k+1$ follows. □

Set

$$K_n = \begin{cases} [a_n, a_{n+1}] & \text{if } n \text{ is even,} \\ [a_{n+1}, a_n] & \text{if } n \text{ is odd,} \end{cases}$$

$n = 1, 2, \ldots$.

LEMMA 2. *The sets $K_n$ have the following properties:*
(*a*) *The sequence $(K_n)_{n=1}^{\infty}$ is descending;*
(*b*) $h_n(K_{n+1})=L_n$;
(*c*) $h_n(K_n)=K_n$;
(*d*) *The sets $f^i(K_n)$, $i=1,2,3,\ldots,2^{n-1}$ (for $n$ fixed) are disjoint;*
(*e*) $f|_{f^i(K_n)}$ *is linear for $i=1,2,3,\ldots,2^{n-1}-1$;*
(*f*) $f^{2^{n-1}+i}(K_{n+1})\cap f^i(K_{n+1})=\varnothing$ *for $i=1,2,3,\ldots,2^{n-1}$;*
(*g*) $\dfrac{|\bigcup_{i=1}^{2^n} f^i(K_{n+1})|}{|\bigcup_{i=1}^{2^{n-1}} f^i(K_n)|}=1-\left(\dfrac{1}{n+1}\right)^2$

(*where $|\cdot|$ denotes the Lebesgue measure of a set*).

*Proof.* (*a*) follows from the definition of the points $a_n$. (*b*) follows from lemma 1(*c*), (*d*) and (*e*).

By the definition of the points $a_n$, $b_n$ and their images, we have $f(L_n)\cap f(K_{n+1})=\varnothing$. Now, (*f*) follows from this, (*b*) and lemma 1 (*g*). Since $L_k\cup K_{k+1}\subset K_k$, (*d*) for $n=k+1$ follows from (*f*) and (*d*) for $n=k$ (for $n=1$ it is obvious). (*e*) is equivalent to lemma 1(*g*). (*c*) follows from lemma 1 (*a*), (*d*), (*e*). To prove (*g*), we have to make computations, using (*b*)–(*f*):

$$\begin{aligned}\left|\bigcup_{i=1}^{2^n} f^i(K_{n+1})\right| &= \sum_{i=1}^{2^n} |f^i(K_{n+1})| = \sum_{i=1}^{2^{n-1}} (|f^i(K_{n+1})|+|f^i(L_n)|)\\ &= \frac{|f(K_{n+1})|+|f(L_n)|}{|f(K_n)|}\cdot\sum_{i=1}^{2^{n-1}}|f^i(K_n)|\\ &= \frac{|f(K_{n+1})|+|f(L_n)|}{|f(K_n)|}\cdot\left|\bigcup_{i=1}^{2^{n-1}} f^i(K_n)\right|.\end{aligned}$$

But we have

$$\frac{|f(K_{n+1})|+|f(L_n)|}{|f(K_n)|}=\frac{(1-a_{n+1})+(f(b_n)-f(a_n))}{1-a_n}=1-\left(\frac{1}{n+1}\right)^2. \qquad \square$$

We claim that

(*B*) $f$ has the same kneading invariant as the Feigenbaum map.

To prove this, we have to know the trajectory of $c$. By lemma 1, $h_n(c)=a_n$ and $h_n$ is monotone on $[a_n, c]$ (or $[c, a_n]$). Besides, $h_n(a_n)=a_{n+1}$, and hence $c$ belongs to $h_n((a_n, c))$ (or $h_n((c, a_n))$). Consequently, if we set

$$\xi_n=\begin{cases}+1 & \text{if } f^n(c)<c,\\ -1 & \text{if } f^n(c)>c,\end{cases}$$

then

$$\xi_{2^{n-1}+i}=\xi_i \quad \text{for } i=1,2,3,\ldots,2^{n-1}-1,$$

and

$$\xi_{2^n}=-\xi_{2^{n-1}}.$$

Hence, $\xi_1\cdot\xi_2\cdot\ldots\cdot\xi_{2^n}=-1$ and

$$\xi_1\cdot\xi_2\cdot\ldots\cdot\xi_{2^n+i}=-\xi_1\cdot\xi_2\cdot\ldots\cdot\xi_i \quad \text{for } i=1,2,3,\ldots,2^n-1.$$

By the results of [4], this is equivalent to the existence of periodic points of all periods being powers of 2, and of no other periods. This proves (*B*).

Define

$$S=\bigcap_{n=1}^{\infty}\bigcup_{i=1}^{2^{n-1}} f^i(K_n).$$

By lemma 2, and since

$$\bigcup_{i=1}^{2^{1-1}} f^i(K_1)=K_1=[0,1],$$

we have

$$|S|=\prod_{n=1}^{\infty}\left(1-\left(\frac{1}{n+1}\right)^2\right)>0,$$

i.e. the set $S$ has positive Lebesgue measure.

LEMMA 3. *The measures of the sets* $S\cap f^i(K_n)$, $i=1,2,3,\dots,2^{n-1}$, are

$$|S|\cdot\prod_{k=1}^{n-1}\frac{k\cdot\varepsilon_k+1}{k+2},$$

*where* $(\varepsilon_1,\dots,\varepsilon_k)$ *runs over all* 0–1 *sequences of length* $n-1$.

*Proof.* We use induction with respect to $n$. For $n=1$, the conclusion of the lemma obviously holds. Suppose now that it holds for $n=m$; we shall prove it for $n=m+1$.

Every set of form $S\cap f^i(K_m)$, $i=1,2,3,\dots,2^{m-1}$, is a union of two disjoint sets: $S\cap f^i(K_{m+1})$ and $S\cap f^i(L_m)$. Since

$$S\cap f^i(K_m)=f^{i-1}(S\cap f(K_m))$$

and all $f^{i-1}$, $i=1,2,3,\dots,2^{m-1}$, are linear on $f(K_m)$, we have

$$\frac{|S\cap f^i(K_{m+1})|}{|S\cap f^i(L_m)|}=\frac{|f(K_{m+1})|}{|f(L_m)|}=\frac{1}{m+1}.$$

Hence, the measures of $S\cap f^i(K_{m+1})$ and $S\cap f^{2^{m-1}+i}(K_{m+1})$ are

$$\frac{1}{m+2}|S\cap f^i(K_m)|\quad\text{and}\quad\frac{m+1}{m+2}|S\cap f^i(K_m)|.$$

But if

$$|S\cap f^i(K_m)|=|S|\cdot\prod_{k=1}^{m-1}\frac{k\cdot\varepsilon_k+1}{k+1},$$

then these measures are

$$|S|\cdot\prod_{k=1}^{m}\frac{k\cdot\varepsilon_k+1}{k+2},\quad\varepsilon_m=0,1. \qquad\square$$

LEMMA 4. (*a*) *S is a Cantor set.*

(*b*) *The measure* $\nu$ *on S, defined by* $\nu(f^i(K_n))=1/2^{n-1}$ *for* $i=1,2,3,\dots,2^{n-1}$, *is not absolutely continuous with respect to the Lebesgue measure.*

*Proof.* (*a*) By the definition of $S$, it is enough to show that $\max_{1\le i\le 2^{n-1}} |S\cap f^i(K_m)|\to 0$ as $n\to\infty$. But indeed,

$$\max_{1\le i\le 2^{n-1}} |S\cap f^i(K_n)| = \prod_{k=1}^{n-1}\frac{k+1}{k+2} = \prod_{j=3}^{n+1}\left(1-\frac{1}{j}\right)\to 0$$

as $n\to\infty$. This proves (*a*).

(*b*) follows from the fact that for every $n$, one can find a set of $\nu$-measure $\frac{1}{2}$ and Lebesgue measure $|S|/(n+1)$. This set is the union of these intervals $f^i(K_n)$ (intersected with $S$), for which the corresponding $\varepsilon_{n-1}$ is equal to 1. □

Now we shall prove that

(*C*) $f$ is topologically conjugate to the Feigenbaum map.

For this we have to prove that $f$ has no homtervals. Consider again the gap $M_m$ between $L_n$ and $L_{n+2}$. It is mapped by $f$ monotonically and then by $f^{2^{n-1}-1}$ linearly (by lemma 1(*e*) and since $f(M_n)\subset[f(a_n), 1]$). Since we have $f(a_n)<f(b_n)<f(a_{n+1})<f(a_{n+2})$, $h_n(b_n)=a_{m+2}$ and $h_n(a_{n+1})=b_n$, our interval $M_n$ is mapped by $h_n$ homeomorphically onto some interval containing $M_n$. By lemma 1(*f*), $h_n|_{M_n}$ reverses orientation. Hence, $h_n|_{M_n}$ has a unique fixed point. Call this point $u_n$.

We know that $f$ has slope $\lambda_n$ on $L_n$, $\lambda_{n+2}$ on $L_{n+2}$, and $f^{2^{n-1}-1}$ has slope $\mu_n$ on $[f(a_n), 1]$. Therefore (taking into account the orientation), we get:

$$h'_n|_{L_n} = -\lambda_n\mu_n = -(n+1) \tag{7}$$

$$h'_n|_{L_{n+2}} = -\lambda_{n+2}\mu_n = -(n+3)/(n+1)(n+2). \tag{8}$$

By the construction, $f$ (and consequently, also $h_n$) is linear on the interval $[a_{n+2}+2x_n, a_n]$ (if $n$ is odd) or $[a_n, a_{n+2}-2x_n]$ (if $n$ is even), where $x_n$ is given by (1). If $h_n$ has a fixed point on this interval, this fixed point has to be equal to $u_n$. To find it we have to solve the equation (where $u_n=b_n+t_n$):

$$b_n+t_n = a_{n+2}-t_n(n+1)$$

(remember that the derivative is given by (7)). We get for $t_n$:

$$t_n = (a_{n+2}-b_n)/n+2.$$

This implies that the sign of $t_n$ is the same as of $a_{n+2}-b_n$. Hence, to prove that $u_n$ lies on the considered interval, it is enough to show that $|t_n|+2x_n\le|a_{n+2}-b_n|$ (cf. figure 1). This inequality is equivalent to $(n+1)(n^2+n+2)\ge n+3$, which holds because $n+1>1$ and $n^2+2\ge 3$. Hence, we have shown that

$$h'_n(u_n) = -(n+1). \tag{9}$$

Since $h_n$ is monotone on $[c, b_n]$ (or $[b_n, c]$) and

$$h_n(b_n)=a_{n+2}, \qquad h_n(c)=a_n,$$

the interval $M_n$ is mapped by $h_n$ homeomorphically onto some sub-interval of $M_n\cup L_n$. The interval $M_n\cup L_n$ is divided by $u_n$ into two parts. By (7), (8), (9) and the facts that $f$ is concave and $h_n$ is linear on $f(M_n\cup L_n)$, the slope of $h_n$ on one of these parts is constant and equal to $n+1$ and on the other one is at least $(n+3)/(n+1)(n+2)$. Since

$$(n+1)\cdot\frac{n+3}{(n+1)(n+2)}>1$$

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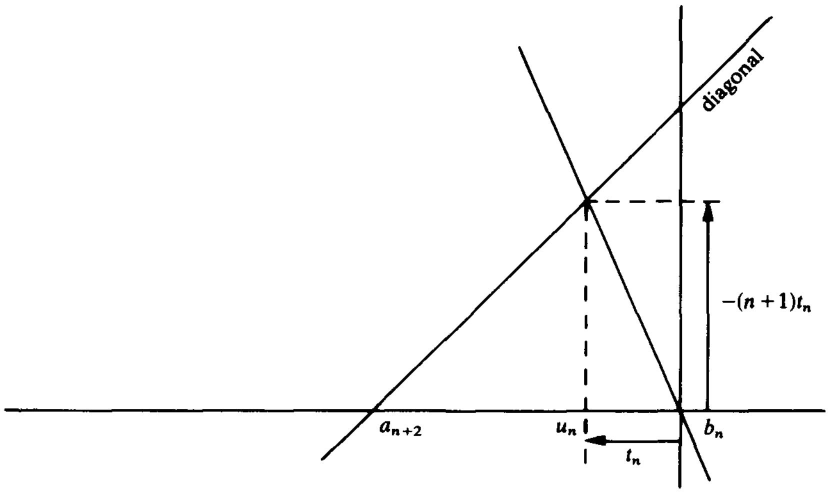


FIGURE 1

and $h_n$ reverses orientation on $M_n \cup L_n$, we have

$$|h'_n|_{M_n}| > 1. \tag{10}$$

Suppose now that $J$ is a homterval. Since $S$ is a Cantor set, $J$ has to intersect some of its gaps (or an 'outer gap'). If it also intersects $S$, then it has to contain some $f^i(K_n)$ (remember that $J$ is open). But some image of $f^i(K_n)$ contains $c$, and we get a contradiction. Hence $J$ (and all its images) is disjoint from $S$.

Since, by lemma 1, all points $a_n$ and $b_n$ are images of $c$, they all belong to $S$ ($S$ is invariant). Consequently, none of these points belong to $J$, or its images. We have

$$[0, 1] = \{c\} \cup \bigcup_{n\geq 0} (M_n \cup L_n).$$

We take the subsequence of images of $J$, $(f^{k(n)}(J))_{n=1}^{\infty}$, defined by induction: first, $k(0) = 0$, second if $f^{k(n)}(J) \subset M_n$, then $k(n+1) = k(n) + 2^m$; if $f^{k(n)}(J) \subset L_m$, then $k(n+1) = k(n) + 2^{m-1}$. By (7) and (10), we get

$$|(f^{k(n+1)-k(n)})'|_{f^{k(n)}(J)}| > 1. \tag{11}$$

Therefore, for all $n$, $|f^{k(n)}(J)| \geq |J|$. This is possible only if $J$ is contained in a basin of a sink. But this contradicts (11).

Hence, $f$ has no homtervals, and consequently, $(C)$ is proved.

To complete the proof of the theorem, we need to prove only that $S = C$ where the set $C$ was defined in the introduction. But this follows from the straightforward remark that for every descending sequence $(Q_n)_{n=1}^{\infty}$ of neighbourhoods of $c$ with $\bigcap_{n=1}^{\infty} Q_n = \{c\}$, the set $\bigcap_{n=1}^{\infty} \bigcup_{k=0}^{\infty} f^k(Q_n)$ is the same.

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