BULL. AUSTRAL. MATH. SOC. 05A19, 10A35, 11B39
VOL. 50 (1994) [117–121]

A SIMPLE PROOF OF AN IDENTITY OF PETHE AND HORADAM

KUO-JYE CHEN

An identity for two generalised Tribonacci sequences is obtained. Earlier identities by Pethe and Horadam follow as special cases.

1. INTRODUCTION

In 1986, Pethe and Horadam [2] extended Harman's idea [1] and proved

$$(1)\quad p_2\sum_{j=1}^{k}(q_1q_2)^{k-j}U_{m+2j+s}V_{n+2j+s} - p_1q_2\sum_{j=1}^{k}(q_1q_2)^{k-j}U_{m+2j-1+s}V_{n+2j-1+s} = U_{m+2k+s}V_{n+2k+1+s} - (q_1q_2)^kU_{m+s}V_{n+1+s},$$

where $U_{n+2} = p_1U_{n+1} - q_1U_n$, $V_{n+2} = p_2V_{n+1} - q_2V_n$, p_1, p_2, q_1 and q_2 are fixed real numbers, $s = 0$ if k is even, and $s = 1$ if k is odd.

Using (1), they "obtained a wealth of significant summation identities involving the products of combinations of Fibonacci numbers and polynomials, Pell numbers and polynomials, Chebyshev polynomials and sine functions".

The object of this paper is to show that (1) can be proved without using Harman's idea and that (1) is indeed an easy consequence of a very simple reduction formula (see Lemma 1).

We shall use Lemma 1 to obtain an identity (see Theorem 1) for generalised Tribonacci sequences, of which (1) is a special case. Also from Theorem 1 we obtain as a bonus the main result in Pethe's paper [3], that is,

$$(2)\quad P\sum_{j=1}^{k}Q^{k-j}J_{m+j}J_{n+j} + R\left[\sum_{j=1}^{[k/2]+s}Q^{k-2j+s}J_{m+2j-s}J_{n+2j-2-s} + \sum_{j=1}^{[k/2]}Q^{k-2j+1-s}J_{m+2j-3+s}J_{n+2j-1+s}\right] = J_{m+k}J_{n+k+1} - Q^kJ_{m+s}J_{n+1-s},$$

where $J_{n+3} = PJ_{n+2} + QJ_{n+1} + RJ_n$, and P, Q, R are fixed real numbers.

Received 5th October, 1993.

2. The Main Result

Lemma 1. *Let $(U_n)_{n=0}^{\infty}$ and $(V_n)_{n=0}^{\infty}$ be two sequences satisfying*

$$U_{n+3} = p_1 U_{n+2} + q_1 U_{n+1} + r_1 U_n$$

and

$$V_{n+3} = p_2 V_{n+2} + q_2 V_{n+1} + r_2 V_n,$$

where $p_i, q_i, r_i, i = 1, 2$, are fixed real numbers (or polynomials). Then

$$(3) \quad U_{m+k}V_{n+k+1} = (p_2 U_{m+k}V_{n+k} + p_1 q_2 U_{m+k-1}V_{n+k-1} + r_2 U_{m+k}V_{n+k-2} + r_1 q_2 U_{m+k-3}V_{n+k-1}) + q_1 q_2 U_{m+k-2}V_{n+k-1}.$$

Proof: By the recurrences satisfied by U_n and V_n, we have

$$\begin{aligned} U_{m+k}V_{n+k+1} &= U_{m+k}(p_2 V_{n+k} + q_2 V_{n+k-1} + r_2 V_{n+k-2}) \\ &= p_2 U_{m+k}V_{n+k} + r_2 U_{m+k}V_{n+k-2} + q_2 U_{m+k}V_{n+k-1} \\ &= p_2 U_{m+k}V_{n+k} + r_2 U_{m+k}V_{n+k-2} \\ &\quad + q_2(p_1 U_{m+k-1} + q_1 U_{m+k-2} + r_1 U_{m+k-3})V_{n+k-1} \\ &= (p_2 U_{m+k}V_{n+k} + p_1 q_2 U_{m+k-1}V_{n+k-1} + r_2 U_{m+k}V_{n+k-2} \\ &\quad + r_1 q_2 U_{m+k-3}V_{n+k-1}) + q_1 q_2 U_{m+k-2}V_{n+k-1}. \end{aligned}$$

□

Theorem 1. *Let U_n and V_n be as defined in Lemma 1. Then*

(4)

$$\begin{aligned} p_2 \sum_{j=1}^{\lfloor k/2 \rfloor} (q_1 q_2)^{\lfloor k/2 \rfloor - j} U_{m+2j+s}V_{n+2j+s} &+ p_1 q_2 \sum_{j=1}^{\lfloor k/2 \rfloor} (q_1 q_2)^{\lfloor k/2 \rfloor - j} U_{m+2j-1+s}V_{n+2j-1+s} \\ &+ r_2 \sum_{j=1}^{\lfloor k/2 \rfloor} (q_1 q_2)^{\lfloor k/2 \rfloor - j} U_{m+2j+s}V_{n+2j-2+s} \\ &+ r_1 q_2 \sum_{j=1}^{\lfloor k/2 \rfloor} (q_1 q_2)^{\lfloor k/2 \rfloor - j} U_{m+2j-3+s}V_{n+2j-1+s} \\ &= U_{m+k}V_{n+k+1} - (q_1 q_2)^{\lfloor k/2 \rfloor} U_{m+s}V_{n+1+s}, \end{aligned}$$

or equivalently

(5)
$$p_2 \sum_{j=1}^{\lfloor k/2\rfloor+s} (q_1q_2)^{\lfloor k/2\rfloor-j+s} U_{m+2j-s}V_{n+2j-s} + p_1q_2 \sum_{j=1}^{\lfloor k/2\rfloor} (q_1q_2)^{\lfloor k/2\rfloor-j} U_{m+2j-1+s}V_{n+2j-1+s}$$
$$+ r_2 \sum_{j=1}^{\lfloor k/2\rfloor+s} (q_1q_2)^{\lfloor k/2\rfloor-j+s} U_{m+2j-s}V_{n+2j-2-s}$$
$$+ r_1q_2 \sum_{j=1}^{\lfloor k/2\rfloor} (q_1q_2)^{\lfloor k/2\rfloor-j} U_{m+2j-3+s}V_{n+2j-1+s}$$
$$= U_{m+k}V_{n+k+1} - q_2^s(q_1q_2)^{\lfloor k/2\rfloor} U_{m+s}V_{n+1-s}$$

where $s = 0$ if k is even and $s = 1$ if k is odd.

PROOF: When k is even, repeated use of (3) yields

$$\begin{aligned} U_{m+k}V_{n+k+1} = {} & (p_2U_{m+k}V_{n+k} + p_1q_2U_{m+k-1}V_{n+k-1} \\ & +r_2U_{m+k}V_{n+k-2} + r_1q_2U_{m+k-3}V_{n+k-1}) \\ & + q_1q_2\,(p_2U_{m+k-2}V_{n+k-2} + p_1q_2U_{m+k-3}V_{n+k-3} \\ & \quad +r_2U_{m+k-2}V_{n+k-4} + r_1q_2U_{m+k-5}V_{n+k-3}) \\ & + \cdots \\ & + (q_1q_2)^{(k-4)/2}\,(p_2U_{m+4}V_{n+4} + p_1q_2U_{m+3}V_{n+3} \\ & \quad +r_2U_{m+4}V_{n+2} + r_1q_2U_{m+1}V_{n+3}) \\ & + (q_1q_2)^{(k-2)/2}\,(p_2U_{m+2}V_{n+2} + p_1q_2U_{m+1}V_{n+1} \\ & \quad +r_2U_{m+2}V_{n} + r_1q_2U_{m-1}V_{n+1}) \\ & \quad + (q_1q_2)^{k/2}U_mV_{n+1}, \end{aligned}$$

whence

(6)
$$p_2\sum_{j=1}^{k/2}(q_1q_2)^{(k/2)-j}U_{m+2j}V_{n+2j} + p_1q_2\sum_{j=1}^{k/2}(q_1q_2)^{(k/2)-j}U_{m+2j-1}V_{n+2j-1}$$
$$+r_2\sum_{j=1}^{k/2}(q_1q_2)^{(k/2)-j}U_{m+2j}V_{n+2j-2} + r_1q_2\sum_{j=1}^{k/2}(q_1q_2)^{(k/2)-j}U_{m+2j-3}V_{n+2j-1}$$
$$= U_{m+k}V_{n+k+1} - (q_1q_2)^{k/2}U_mV_{n+1}, \qquad k \text{ even}.$$

https://doi.org/10.1017/S000497270000962X Published online by Cambridge University Press

Similarly, when k is odd, we have

$$(7)\qquad \begin{aligned} U_{m+k}V_{n+k+1} = {} & (p_2U_{m+k}V_{n+k} + p_1q_2U_{m+k-1}V_{n+k-1} \\ & +r_2U_{m+k}V_{n+k-2} + r_1q_2U_{m+k-3}V_{n+k-1}) \\ & + q_1q_2\,(p_2U_{m+k-2}V_{n+k-2} + p_1q_2U_{m+k-3}V_{n+k-3} \\ & \quad +r_2U_{m+k-2}V_{n+k-4} + r_1q_2U_{m+k-5}V_{n+k-3}) \\ & + \cdots \\ & + (q_1q_2)^{(k-5)/2}\,(p_2U_{m+5}V_{n+5} + p_1q_2U_{m+4}V_{n+4} \\ & \quad +r_2U_{m+5}V_{n+3} + r_1q_2U_{m+2}V_{n+4}) \\ & + (q_1q_2)^{(k-3)/2}\,(p_2U_{m+3}V_{n+3} + p_1q_2U_{m+2}V_{n+2} \\ & \quad +r_2U_{m+3}V_{n+1} + r_1q_2U_mV_{n+2}) \\ & + (q_1q_2)^{(k-1)/2}U_{m+1}V_{n+2}, \end{aligned}$$

whence

$$(8)\qquad \begin{aligned} & p_2\sum_{j=1}^{(k-1)/2}(q_1q_2)^{(k-1)/2-j}U_{m+2j+1}V_{n+2j+1} + p_1q_2\sum_{j=1}^{(k-1)/2}(q_1q_2)^{(k-1)/2-j}U_{m+2j}V_{n+2j} \\ & +r_2\sum_{j=1}^{(k-1)/2}(q_1q_2)^{(k-1)/2-j}U_{m+2j+1}V_{n+2j-1} + r_1q_2\sum_{j=1}^{(k-1)/2}(q_1q_2)^{(k-1)/2-j}U_{m+2j-2}V_{n+2j} \\ & \qquad = U_{m+k}V_{n+k+1} - (q_1q_2)^{(k-1)/2}U_{m+1}V_{n+2}, \qquad k \text{ odd.} \end{aligned}$$

Since

$$\begin{aligned} U_{m+1}V_{n+2} &= U_{m+1}(p_2V_{n+1} + q_2V_n + r_2V_{n-1}) \\ &= p_2U_{m+1}V_{n+1} + q_2U_{m+1}V_n + r_2U_{m+1}V_{n-1}, \end{aligned}$$

(8) can be rewritten as

$$(9)\qquad \begin{aligned} & p_2\sum_{j=1}^{(k+1)/2}(q_1q_2)^{(k+1)/2-j}U_{m+2j-1}V_{n+2j-1} + p_1q_2\sum_{j=1}^{(k-1)/2}(q_1q_2)^{(k-1)/2-j}U_{m+2j}V_{n+2j} \\ & +r_2\sum_{j=1}^{(k+1)/2}(q_1q_2)^{(k+1)/2-j}U_{m+2j-1}V_{n+2j-3} + r_1q_2\sum_{j=1}^{(k-1)/2}(q_1q_2)^{(k-1)/2-j}U_{m+2j-2}V_{n+2j} \\ & \qquad = U_{m+k}V_{n+k+1} - q_2(q_1q_2)^{(k-1)/2}U_{m+1}V_n, \qquad k \text{ odd.} \end{aligned}$$

Now, it is readily seen that (4) follows from (6) and (8), and (5) follows from (6) and (9). ∎

https://doi.org/10.1017/S000497270000962X Published online by Cambridge University Press

3. Special Cases

In this section, it will be shown, as we proposed in Introduction, that (1) and (2) are special cases of (4) and (5), respectively.

Case 1. Replacing k first by $2k$ and then by $2k+1$ in (4) gives two identities which can be combined in one identity, that is,

(10)
$$\begin{aligned} p_2 \sum_{j=1}^{k} (q_1 q_2)^{k-j} U_{m+2j+s} V_{n+2j+s} + p_1 q_2 \sum_{j=1}^{k} (q_1 q_2)^{k-j} U_{m+2j-1+s} V_{n+2j-1+s} & \\ + r_2 \sum_{j=1}^{k} (q_1 q_2)^{k-j} U_{m+2j+s} V_{n+2j-2+s} & \\ + r_1 q_2 \sum_{j=1}^{k} (q_1 q_2)^{k-j} U_{m+2j-3+s} V_{n+2j-1+s} & \\ = U_{m+2k+s} V_{n+2k+1+s} - (q_1 q_2)^k U_{m+s} V_{n+1+s} & \end{aligned}$$

Let $q_1 \to -q_1$, $q_2 \to -q_2$, and $r_1 = r_2 = 0$. Then (10) reduces to (1).

Case 2. Let $U_n = V_n = J_n$, $p_1 = p_2 = P$, $q_1 = q_2 = Q$, and $r_1 = r_2 = R$. Then we obtain (2) from (5) by combining even and odd cases of k.

References

[1] C.J. Harman, 'Complex Fibonacci numbers', *Fibonacci Quart.* **19** (1981), 82–86.

[2] S. Pethe and A.F. Horadam, 'Generalized Gaussian Fibonacci numbers', *Bull. Austral. Math. Soc.* **33** (1986), 37–48.

[3] S. Pethe, 'Some identities for Tribonacci sequences', *The Fibonacci quarterly* **26** (1988), 144–151.

Department of Mathematics
Changhua University of Education
Changhua 500
Taiwan