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Weighted Carleson Measure Spaces Associated with Different Homogeneities

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Abstract. In this paper, we introduce weighted Carleson measure spaces associated with different homogeneities and prove that these spaces are the dual spaces of weighted Hardy spaces studied in a forthcoming paper. As an application, we establish the boundedness of composition of two Calderón–Zygmund operators with different homogeneities on the weighted Carleson measure spaces; this, in particular, provides the weighted endpoint estimates for the operators studied by Phong–Stein.

1 Introduction and Statement of Main Results

The purpose of this paper is to develop a new theory of weighted Carleson measure spaces associated with different homogeneities, identify the dual of the weighted Hardy spaces studied in [Wu] with these new spaces, and prove that the composition of two Calderón–Zygmund operators with different homogeneities studied in [PS] is bounded on these spaces. This is a continuation of the paper [Wu] studying the questions of the composition of operators that cannot be answered by using the properties of each operator separately. To be more precise, let $e(\xi)$ and $h(\xi)$ be functions on $\mathbb{R}^N$ homogeneous of degree 0 in the isotropic sense and the anisotropic sense, and smooth away from the origin. Then it is well known that the Fourier multipliers T_1 defined by $\widehat{T_1(f)}(\xi) = e(\xi)\widehat{f}(\xi)$ and T_2 given by $\widehat{T_2(f)}(\xi) = h(\xi)\widehat{f}(\xi)$ are both bounded on L^p for $1 < p < \infty$, and satisfy various other regularity properties such as being of weak-type $(1, 1)$ and bounded on the classical isotropic and non-isotropic Hardy spaces, respectively. Rivieré in [WW] asked the question: is the composition $T_1 \circ T_2$ still of weak-type $(1, 1)$? Phong and Stein in [PS] answered this question and gave a necessary and sufficient condition for which $T_1 \circ T_2$ is of weak-type $(1, 1)$. The operators Phong and Stein studied are in fact compositions with different kinds of homogeneities that arise naturally in the $\overline{\partial}$-Neumann problem.

There are some other questions of this kind about the composition operator $T_1 \circ T_2$ that cannot be answered by using the properties of T_1 and T_2 separately. Recently, Han et al. [HLLRS] developed a theory of multiparameter Hardy spaces and proved that the composition $T_1 \circ T_2$ is bounded on these Hardy spaces. More recently, the author [Wu] introduced and studied a new class of Muckenhoupt weights $A_p^{\mathscr{C}}$. In terms of these weights, a theory of weighted Hardy space $H^p_{\mathscr{C},w}(\mathbb{R}^N)$ was established, and weighted norm inequalities for $T_1 \circ T_2$ in $H^p_{\mathscr{C},w}(\mathbb{R}^N)$ were derived. Such questions

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also arise in the context of BMO and Lipschitz spaces due to the examples constructed in [MR]. These questions motivate this paper.

In order to describe more precisely the questions and results studied in this paper, we begin by considering all functions and operators on $\mathbb{R}^N = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$. For $x = (x_1, \dots, x_m) \in \mathbb{R}^N$ and $\delta > 0$, we consider the *isotropic homogeneity* on $\mathbb{R}^N$:

$$\delta \circ (x_1, \dots, x_m) = (\delta x_1, \dots, \delta x_m),$$

and two kinds of *anisotropic homogeneities* on $\mathbb{R}^N$:

$$\delta \circ_1 (x_1, \dots, x_m) = (\delta^{a_1} x_1, \dots, \delta^{a_m} x_m),$$

$$\delta \circ_2 (x_1, \dots, x_m) = (\delta^{b_1} x_1, \dots, \delta^{b_m} x_m),$$

with $1 \le a_1 \le \cdots \le a_m < \infty$ and $1 \le b_1 \le \cdots \le b_m < \infty$. Note that the additive group $(\mathbb{R}^N, +)$ equipped with either of the dilations $\circ_1$ and $\circ_2$ is a homogeneous Lie group (see [FS2]). Let $N_1 = a_1 n_1 + \cdots + a_m n_m$ and $N_2 = b_1 n_1 + \cdots + b_m n_m$ denote the homogeneous dimensions and let $|x|_1 = \sup_{1 \le i \le m} |x_i|^{\frac{1}{a_i}}$ and $|x|_2 = \sup_{1 \le i \le m} |x_i|^{\frac{1}{b_i}}$ be the homogeneous norms. For $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$, set

$$\|\alpha\| = |\alpha_1| + \cdots + |\alpha_m|,$$

$$\|\alpha\|_1 = |\alpha_1| a_1 + \cdots + |\alpha_m| a_m,$$

$$\|\alpha\|_2 = |\alpha_1| b_1 + \cdots + |\alpha_m| b_m.$$

For $j, k \in \mathbb{Z}$, we will frequently use the discrete dilations

$$2^j \circ_1 (x_1, \dots, x_m) = (2^{j_1} x_1, \dots, 2^{j_m} x_m),$$

$$2^k \circ_2 (x_1, \dots, x_m) = (2^{k_1} x_1, \dots, 2^{k_m} x_m),$$

where $j_i = a_i j$ and $k_i = b_i k$ for $i = 1, \dots, m$.

As pointed out in [Wu], the weighted theory of function spaces is closely related to the family of *acceptable rectangles*, which nicely reflects the geometry structure of $\mathbb{R}^N$ with mixed homogeneities. Throughout this paper, all rectangles are assumed to have edges parallel to coordinate axes. We say that a rectangle R is *acceptable* if $R = I_1 \times \cdots \times I_m$, where each I_i is a Euclidean cube in $\mathbb{R}^{n_i}$ with side-length $\ell(I_i) = 2^{j_i \vee k_i} = 2^{(a_i j) \vee (b_i k)}, 1 \le i \le m$ for some $j, k \in \mathbb{Z}$. Denote by $\mathcal{R}_{\mathscr{C}}$ the set of all acceptable rectangles and by $\mathcal{R}_{\mathscr{C}}^d$ the set of all dyadic acceptable rectangles. Let $\mathcal{R}_{\mathscr{C}}^{j,k}$ be the subset of $\mathcal{R}_{\mathscr{C}}^d$ that consists of all *dyadic* acceptable rectangles $R = I_1 \times \cdots \times I_m$ with side-length $\ell(I_i) = 2^{j_i \vee k_i}, 1 \le i \le m$.

The maximal function and Muckenhoupt weights associated with different homogeneities were introduced in [Wu] as follows.

Definition 1.1 The maximal function associated with different homogeneities is defined by

$$\mathcal{M}_{\mathscr{C}}(f)(x) = \sup_{\substack{R \in \mathcal{R}_{\mathscr{C}} \\ R \ni x}} \frac{1}{|R|} \int_R |f(y)| dy.$$

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Definition 1.2 Let w be a nonnegative locally integrable function on $\mathbb{R}^N$. For $1 < p < \infty$, we say that w is in $A_p^{\mathscr{C}}(\mathbb{R}^N)$ if there is a constant $0 < C < \infty$ such that

$$\sup_{R\in\mathcal{R}_{\mathscr{C}}} \Big(\frac{1}{|R|}\int_R w(x)dx\Big)\Big(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}dx\Big)^{p-1} < C.$$

We say that $w \in A_1^{\mathscr{C}}(\mathbb{R}^N)$ if there is a constant $C > 0$ such that

$$\mathcal{M}_{\mathscr{C}}(w)(x) \le Cw(x), \quad \text{for almost every } x \in \mathbb{R}^N.$$

The weight class $A_\infty^{\mathscr{C}}(\mathbb{R}^N)$ is defined by $A_\infty^{\mathscr{C}}(\mathbb{R}^N) = \bigcup_{1\le p<\infty} A_p^{\mathscr{C}}(\mathbb{R}^N)$. We use $q_w \equiv \inf\{q : w \in A_q^{\mathscr{C}}(\mathbb{R}^N)\}$ to denote the *critical index* of w. For any subset $A \subseteq \mathbb{R}^N$, denote $w(A) = \int_A w(x)dx$.

Let $\mathcal{R}_{(1)}$ denote the set of all "cubes" associated with $\circ_1$ (*i.e.*, rectangles with side-length $(2^{j_1},\dots,2^{j_m})$ for some $j_i \in \mathbb{Z}$) and similarly for $\mathcal{R}_{(2)}$. Associated with $\circ_i$, $i = 1, 2$, the anisotropic Hardy–Littlewood maximal function $\mathcal{M}_{(i)}$ and the Muckenhoupt weight class $A_p^{(i)}$ can be defined by replacing $\mathcal{R}_{\mathscr{C}}$ with $\mathcal{R}_{(i)}$ in the definitions above.

Let w be a weight function (*i.e.*, a nonnegative measurable function) on $\mathbb{R}^N$. The characterizations of the Muckenhoupt weight class $A_p^{\mathscr{C}}(\mathbb{R}^N)$ are given by the following theorem.

Theorem 1.3 ([Wu]) *Suppose $1 < p < \infty$. Then the following four statements are equivalent:*

(i) $w \in A_p^{\mathscr{C}}(\mathbb{R}^N)$;

(ii) $w \in A_p^{(1)} \cap A_p^{(2)}(\mathbb{R}^N)$;

(iii) $\mathcal{M}_{(1)} \circ \mathcal{M}_{(2)}$ *is bounded on* $L_w^p(\mathbb{R}^N)$ *or on* $L_w^p(\ell^q;\mathbb{R}^N)$;

(iv) $\mathcal{M}_{\mathscr{C}}$ *is bounded on* $L_w^p(\mathbb{R}^N)$ *or on* $L_w^p(\ell^q;\mathbb{R}^N)$.

The singular integral operators considered in this paper are defined as follows.

Definition 1.4 A locally integrable function $\mathcal{K}_i$, $i = 1, 2$ on $\mathbb{R}^N\backslash\{0\}$ is said to be a Calderón–Zygmund kernel associated with $\circ_i$ if for any multi-index $\alpha = (\alpha_1,\dots,\alpha_m)$,

$$|\partial^\alpha \mathcal{K}_i(x)| \le A|x|_i^{-N_i-\|\alpha\|_i}, \quad \text{for } x \in \mathbb{R}^N\backslash\{0\}, \tag{1.1}$$

and

$$\Big|\int_{\delta<|x|_i<r} \mathcal{K}_i(x)dx\Big| \le C, \quad \text{uniformly for all } r > \delta > 0. \tag{1.2}$$

The Calderón–Zygmund singular integral operator associated with $\circ_i$ is defined by $T_i(f)(x) = p.v.(\mathcal{K}_i * f)(x)$, where $\mathcal{K}_i$ satisfies conditions of (1.1) and (1.2).

When $w \in A_p^{\mathscr{C}}$, the composition $T_1 \circ T_2$ is bounded on L_w^p, $1 < p < \infty$, but in general, it is bounded neither on $H_{(1),w}^p$ nor on $H_{(2),w}^p$. Recently, in [Wu], new weighted Hardy spaces associated with different homogeneities were developed and weighted norm inequality in $H_{\mathscr{C},w}^p$ was established. The main goal of this paper is to characterize the dual of the weighted Hardy spaces $H_{\mathscr{C},w}^p$ and prove the boundedness of composition operator on the dual spaces. We would like to mention that characterizations of product BMO spaces have been established earlier by many authors (Chang and Fefferman [CF1, CF2], Krug and Torchinsky [KT], Ferguson and Lacey [FL], Lacey, Petermichl, Pipher, and Wick [LPPW], etc.). We will provide further details regarding these earlier works in what follows.

We now introduce the weighted Carleson measure spaces associated with different homogeneities. Our crucial idea is to use the family of acceptable rectangles to define the weighted multiparameter Carleson measure. More precisely, let $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^N)$ satisfy

$$\text{supp}\ \widehat{\psi^{(1)}}(\xi) \subseteq \{\xi : 1/2 < |\xi| \leq 2\}, \tag{1.3}$$

and

$$\sum_{j\in\mathbb{Z}} \widehat{\psi^{(1)}}(2^j \circ_1 \xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^N \backslash \{0\}, \tag{1.4}$$

and let $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^N)$ satisfy

$$\text{supp}\ \widehat{\psi^{(2)}}(\xi) \subseteq \{\xi : 1/2 < |\xi| \leq 2\}, \tag{1.5}$$

and

$$\sum_{k\in\mathbb{Z}} \widehat{\psi^{(2)}}(2^k \circ_2 \xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^N \backslash \{0\}. \tag{1.6}$$

Let $\psi_{j,k} = \psi_j^{(1)} * \psi_k^{(2)}$, where

$$\psi_j^{(1)}(x) = 2^{-jN_1}\psi^{(1)}(2^{-j} \circ_1 x) \quad \text{and} \quad \psi_k^{(2)}(x) = 2^{-kN_2}\psi^{(2)}(2^{-k} \circ_2 x).$$

We now formally define the weighted Carleson measure spaces $\text{CMO}_{\mathscr{C},w}^p(\mathbb{R}^N)$ as follows.

Definition 1.5 Suppose $0 < p \leq 1$ and $w \in A_\infty^{\mathscr{C}}(\mathbb{R}^N)$. Let $\psi_{j,k}$ be defined as above and $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^N)$ (the space of tempered distributions modulo polynomials). We say that f belongs to $\text{CMO}_{\mathscr{C},w}^p(\mathbb{R}^N)$, if

$$\|f\|_{CMO_{\mathscr{C},w}^{p,\psi}(\mathbb{R}^N)} \equiv \sup_{\Omega} \left\{ \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}_{\mathscr{C}}^{j,k} \\ R\subseteq\Omega}} \frac{|R|^2}{w(R)} |\psi_{j,k} * f(x_R)|^2 \right\}^{\frac{1}{2}} < \infty$$

for all open sets Ω in $\mathbb{R}^N$ with $w(\Omega) < \infty$, where x_R denotes the minimal corner of R, *i.e.*, the corner of R with each coordinate component attaining the minimal value. When $p = 1$, we denote by $\text{BMO}_w^{\mathscr{C}}$ the space $\text{CMO}_{\mathscr{C},w}^1$.

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Note that multiparameter structures are involved in the definition of Carleson measure spaces. These new multiparameter structures are described via the family of dyadic acceptable rectangles, which nicely reflects the mixed homogeneities of the underlying space $\mathbb{R}^N$.

To see that the weighted Carleson measure spaces are well defined, we need to show that the definition of weighted Carleson measure spaces is independent of the choice of $\psi_{j,k}$. This will follow from the next result.

Theorem 1.6 *Let $0 < p \le 1$ and $w \in A_\infty^{\mathscr{C}}(\mathbb{R}^N)$. Suppose that $\psi_{j,k} = \psi_j^{(1)} * \psi_k^{(2)}$ are defined as above and $\varphi_{j,k} = \varphi_j^{(1)} * \varphi_k^{(2)}$ satisfy the same conditions as $\psi_{j,k}$. Then for $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^N)$,*

$$\|f\|_{\mathrm{CMO}^{p,\psi}_{\mathscr{C},w}(\mathbb{R}^N)} \approx \|f\|_{\mathrm{CMO}^{p,\varphi}_{\mathscr{C},w}(\mathbb{R}^N)}.$$

Before stating the main results of this paper, let us first recall the definition of weighted Hardy spaces $H^p_{\mathscr{C},w}$ introduced in [Wu]. For $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^N)$, the Littlewood–Paley–Stein square function $g_{\mathscr{C}}(f)$ of f was defined by

$$g_{\mathscr{C}}(f)(x) = \Big\{ \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}^{j,k}_{\mathscr{C}}} |\psi_{j,k} * f(x_R)|^2 \chi_R(x) \Big\}^{\frac{1}{2}},$$

where x_R denotes the minimal corner of R. Let $0 < p < \infty$ and $w \in A_\infty^{\mathscr{C}}(\mathbb{R}^N)$. The weighted Hardy space $H^p_{\mathscr{C},w}(\mathbb{R}^N)$ was introduced by

$$H^p_{\mathscr{C},w}(\mathbb{R}^N) \equiv \big\{ f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^N) : g_{\mathscr{C}}(f) \in L^p_w(\mathbb{R}^N) \big\}.$$

The $H^p_{\mathscr{C},w}(\mathbb{R}^N)$ quasi-norm of f was given by $\|f\|_{H^p_{\mathscr{C},w}(\mathbb{R}^N)} \equiv \|g_{\mathscr{C}}(f)\|_{L^p_w(\mathbb{R}^N)}$.

The main results of this paper are as follows.

Theorem 1.7 *Let $0 < p \le 1$ and $w \in A_\infty^{\mathscr{C}}(\mathbb{R}^N)$. Then*

$$\big(H^p_{\mathscr{C},w}(\mathbb{R}^N)\big)^* = \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N).$$

More precisely, if $g \in \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$, the mapping ℓ_g given by $\ell_g(f) = \langle f, g\rangle$, defined initially for $f \in \mathcal{S}_\infty(\mathbb{R}^N)$, extends to a continuous linear functional on $H^p_{\mathscr{C},w}(\mathbb{R}^N)$ with $\|\ell_g\| \approx \|g\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)}$. Conversely, for every $\ell \in (H^p_{\mathscr{C},w}(\mathbb{R}^N))^*$, there exists some $g \in \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$ so that $\ell = \ell_g$. In particular, $(H^1_{\mathscr{C},w}(\mathbb{R}^N))^* = \mathrm{BMO}^{\mathscr{C}}_w(\mathbb{R}^N)$.

Theorem 1.8 *Let $0 < p \le 1$ and $w \in A_\infty^{\mathscr{C}}(\mathbb{R}^N)$. Suppose that T_1 and T_2 are Calderón–Zygmund singular integral operators as defined in Definition 1.4. Then the composition operator $T_1 \circ T_2$ is bounded on $\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$, in particular, on $\mathrm{BMO}_{\mathscr{C},w}(\mathbb{R}^N)$. Moreover, there exists a constant C such that*

$$\|T_1 \circ T_2(f)\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)} \le C\|f\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)}.$$

Remark 1.9 It is worthwhile to point out that the homogeneities considered in this paper are more general than the ones considered in [PS, HLLRS]. The weighted endpoint estimates for the operators studied in [PS] are thus given by a special case of Theorem 1.8. Moreover, if $\circ_1 = \circ_2 = \circ$, our results also cover the classical ones in [Ga, ST, LLL]. We also point out that if the regularity condition (1.1) is weakened, then the result in Theorem 1.8 continues to hold for certain range of p.

Finally, we make the following remarks.

To prove Theorem 1.6, our strategy is to use the discrete Calderón reproducing formula (see Lemma 2.3) and the geometric argument involving certain annuli decomposition of the set of acceptable rectangles (See Sections 3 for more details). These ideas will also be used in the proofs of the other main results.

To establish the dual of $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ with $\mathrm{BMO}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, Chang and Fefferman [CF1] invoked the bi-Hilbert transform characterization of product Hardy spaces. Krug and Torchinsky [Kr, KT] described the dual of weighted product Hardy spaces $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ in a quite different way, and the method employed there relied on atomic decomposition characterizations of $H^p_w(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ and Journé's covering lemma. Journé's proof in [Jo] that a class of product singular integrals maps $L^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ boundedly to $\mathrm{BMO}(\mathbb{R}^n \times \mathbb{R}^m)$ also invokes the covering lemma of fundamental importance. However, these methods cannot be applied to our case, since these characterizations for $H^p_{\mathscr{C},w}$ and Journé's covering lemma in our setting are still absent.

We shall use techniques of weighted sequence spaces to prove Theorem 1.7. To be more specific, we first introduce weighted sequence spaces s^p_w, c^p_w, the lifting operator $\mathcal{L}$ and the projection operator $\mathcal{T}$. We then show in Theorem 4.2 that c^p_w is the dual space of s^p_w. The $H^p_{\mathscr{C},w} - s^p_w$ boundedness of $\mathcal{L}$ and $\mathrm{CMO}^p_{\mathscr{C},w} - c^p_w$ boundedness of $\mathcal{T}$ are then established in Theorem 4.3. The proof of Theorem 1.7 then follows from Theorems 4.2 and 4.3.

To prove Theorem 1.8, we first note that $\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N) \subseteq \mathcal{S}'/\mathcal{P}(\mathbb{R}^N)$. Thus the composition operator $T_1 \circ T_2$ may not be well defined on $\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$. Therefore, to prove Theorem 1.8, we first have to define $T_1 \circ T_2$ on $\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$. Recall that the key method used in [Wu, DHLW] to derive the boundedness of $T_1 \circ T_2$ on weighted Hardy spaces is based on the denseness of L^2 in weighted Hardy spaces. Unfortunately this method is not directly applicable to the current setting, since $L^2 \cap \mathrm{CMO}^p_{\mathscr{C},w}$ is not dense in the $\mathrm{CMO}^p_{\mathscr{C},w}$ norm. However, a weaker version of the density result holds. Namely, $L^2 \cap \mathrm{CMO}^p_{\mathscr{C},w}$ is dense in $\mathrm{CMO}^p_{\mathscr{C},w}$ in the *weak topology* $\langle H^p_{\mathscr{C},w}, \mathrm{CMO}^p_{\mathscr{C},w}\rangle$ (see Lemma 5.1). This implies that $T_1 \circ T_2$ can first be defined on $L^2 \cap \mathrm{CMO}^p_{\mathscr{C},w}$, and then be extended to $\mathrm{CMO}^p_{\mathscr{C},w}$. Furthermore, to show the boundedness of $T_1 \circ T_2$ on $\mathrm{CMO}^p_{\mathscr{C},w}$, it suffices to establish the boundedness on $L^2 \cap \mathrm{CMO}^p_{\mathscr{C},w}$. The boundedness on $L^2 \cap \mathrm{CMO}^p_{\mathscr{C},w}$ can be achieved by applying the Calderón type formula and the geometric argument.

The paper is organized as follows. In Section 2, we give some lemmas. The proof of Theorem 1.6 is presented in Section 3. Section 4 is devoted to the proof of the duality of $H^p_{\mathscr{C},w}$ with $\mathrm{CMO}^p_{\mathscr{C},w}$. In Section 5, we establish the boundedness of composition operators on $\mathrm{CMO}^p_{\mathscr{C},w}$.

2 Some Lemmas

The following lemma can be proved as in the classical case; see [St, GR].

Lemma 2.1 *Suppose $w \in A_\infty^{\mathscr{C}}(\mathbb{R}^n)$ and $q > q_w$. Then there exist $0 < C_1, C_2, \delta < \infty$ such that for all acceptable rectangles R and all measurable subsets A of R,*

$$C_1\Big(\frac{|A|}{|R|}\Big)^q \le \frac{w(A)}{w(R)} \le C_2\Big(\frac{|A|}{|R|}\Big)^\delta.$$

In particular, the measure $w(x)dx$ is doubling with respect to acceptable rectangles.

Lemma 2.2 *Let $w \in A_\infty^{\mathscr{C}}(\mathbb{R}^N)$. Then for all acceptable rectangles R and R' and for $q > q_w$,*

$$\frac{w(R')}{w(R)} \lesssim \prod_{i=1}^m \Big[\frac{|I_i|}{|I_i'|} \vee \frac{|I_i'|}{|I_i|}\Big]^q \Big[1 + \frac{|x_{I_i} - x_{I_i'}|}{\ell(I_i) \vee \ell(I_i')}\Big]^{n_i q}.$$

Here and in what follows, x_{I_i} and $x_{I_i'}$ denote the minimal corners, and $\ell(I_i)$ and $\ell(I_i')$ denote the side-lengths of I_i and I_i', respectively.

Proof Note that for $i = 1, \dots, m$, $I_i' \subseteq A_i I_i$, where

$$A_i = C\frac{\ell(I_i) \vee \ell(I_i') + |x_{I_i} - x_{I_i'}|}{\ell(I_i)},$$

with C being a constant depending only on the dimensions. This implies $R' \subseteq \overline{R}$, where $\overline{R} = C[(A_1 I_1) \times \cdots \times (A_m I_m)]$. Then by Lemma 2.1, for any $q > q_w$,

$$\begin{aligned}\frac{w(R')}{w(R)} &\le \frac{w(\overline{R})}{w(R)} \le C\Big[\frac{|\overline{R}|}{|R|}\Big]^q \le C\prod_{i=1}^m \Big[\frac{\ell(I_i) \vee \ell(I_i') + |x_{I_i} - x_{I_i'}|}{\ell(I_i)}\Big]^{n_i q}\\ &\le C\prod_{i=1}^m \Big[\frac{|I_i|}{|I_i'|} \vee \frac{|I_i'|}{|I_i|}\Big]^q \Big[1 + \frac{|x_{I_i} - x_{I_i'}|}{\ell(I_i) \vee \ell(I_i')}\Big]^{n_i q}.\end{aligned}$$

Hence the proof of Lemma 2.2 is concluded. ∎

Let $S_\infty(\mathbb{R}^N)$ be the set of all $f \in \mathcal{S}(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} f(x)x^\alpha dx = 0, \text{ for all multi-index } \alpha.$$

One of the key tools in this paper is the following discrete Calderón reproducing formula. The proof is essentially the same as that of [HLLRS, Theorem 1.3] and thus will be omitted. For the classical case, see [Ha, FJ, FJW].

Lemma 2.3 *Let $\psi_{j,k} = \psi_j^{(1)} * \psi_k^{(2)}$ satisfy (1.3)–(1.6). Then*

$$f(x) = \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathscr{C}}^{j,k}} |R|\psi_{j,k} * f(x_R)\psi_{j,k}(x - x_R),$$

where $x_R = (x_{I_1}, \dots, x_{I_m})$ is the minimal corner of R and the series converges in $L^2(\mathbb{R}^N)$, $\mathcal{S}_\infty(\mathbb{R}^N)$, and $\mathcal{S}'/\mathcal{P}(\mathbb{R}^N)$.

Throughout this paper, for $j, k \in \mathbb{Z}$, let $\mathbf{j} = (j_1, \dots, j_m)$ and $\mathbf{k} = (k_1, \dots, k_m)$. The following almost orthogonality estimate will be frequently used in the sequel (see [Wu, Lemma 3.1]).

Lemma 2.4 *Let $\psi_{j,k} = \psi_j^{(1)} * \psi_k^{(2)}, \varphi_{j,k} = \varphi_j^{(1)} * \varphi_k^{(2)}$ satisfy the conditions (1.3)–(1.6). Given any positive integers L and M, there exists a constant $C = C(L, M) > 0$ such that*

$$|\psi_{j,k} * \varphi_{j',k'}(x)| \leq C2^{-\|\mathbf{j}-\mathbf{j}'\|L}2^{-\|\mathbf{k}-\mathbf{k}'\|L} \prod_{i=1}^{m} \frac{2^{(j_i \vee j_i' \vee k_i \vee k_i')M}}{(2^{j_k \vee j_k' \vee k_i \vee k_i'} + |x_i|)^{n_i+M}}.$$

Remark 2.5 The almost orthogonality estimate also holds if the functions $\psi^{(1)}$, $\psi^{(2)}$, $\varphi^{(1)}$, $\varphi^{(2)}$ only satisfy moment conditions up to order M_0,

$$\int_{\mathbb{R}^N} \psi^{(i)}(x)x^\alpha dx = 0 = \int_{\mathbb{R}^N} \varphi^{(i)}(y)y^\beta dy$$

for any multi-indices $|\alpha|, |\beta| \leq M_0$, $i = 1, 2$. In this case, the almost orthogonality estimate indeed holds for all $M > 0$ and all $0 < L \leq M_0 + 1$.

The following useful estimate is also needed; see [Wu, Lemma 3.2].

Lemma 2.6 *Let $R \in \mathcal{R}_{\mathscr{C}}^{j,k}$. Then for any $x \in R$, $x_R = (x_{I_1}, \dots, x_{I_m}) \in R$, $x_{R'} = (x_{I_1'}, \cdots, x_{I_m'}) \in R'$ and for any $M, \delta > 0$ satisfying $\frac{N}{N+M} < \delta \leq 1$,*

$$\sum_{R' \in \mathcal{R}_{\mathscr{C}}^{j',k'}} |R'| \left[\prod_{i=1}^{m} \frac{2^{M(j_i \vee j_i' \vee k_i \vee k_i')}}{(2^{j_i \vee j_i' \vee k_i \vee k_i'} + |x_{I_i} - x_{I_i'}|)^{n_i+M}} \right] |g(x_{R'})|$$
$$\leq C \Big\{ \prod_{i=1}^{m} \big[2^{n_i(j_i - j_i')} \vee 1 \big] \Big\}^{\frac{1}{\delta}-1} \Big\{ \mathcal{M}_{\mathscr{C}} \Big[\Big(\sum_{R' \in \mathcal{R}_{\mathscr{C}}^{j',k'}} |g(x_{R'})|^2 \chi_{R'} \Big)^{\frac{\delta}{2}} \Big] (x) \Big\}^{\frac{1}{\delta}},$$

where the constant C depends only on M and the dimensions $n_1, \dots, n_m$.

3 Proof of Theorem 1.6

For $R = I_1 \times \cdots \times I_m, R' = I_1' \times \cdots \times I_m' \in \mathcal{R}_{\mathscr{C}}^d$, set

$$r(R, R') = \prod_{i=1}^{m} \left[\frac{|I_i|}{|I_i'|} \wedge \frac{|I_i'|}{|I_i|} \right]^L,$$
$$P(R, R') = \prod_{i=1}^{m} \frac{1}{(1 + [\ell(I_i) \vee \ell(I_i')]^{-1}|x_{I_i} - x_{I_i'}|)^{n_i+M}},$$

where $x_R = (x_{I_1}, \dots, x_{I_m})$ is the minimal corner of R and $\ell(I_i)$ denotes the side-length of I_i and similarly for $x_{R'}$ and $\ell(I_i')$. For $R \in \mathcal{R}_{\mathscr{C}}^{j,k}$ and $R' \in \mathcal{R}_{\mathscr{C}}^{j',k'}$, denote

$$S_R = |\psi_{j,k} * f(x_R)|^2, \quad T_{R'} = |\varphi_{j',k'} * f(x_{R'})|^2.$$

For any $L, M > 0$, applying the discrete Calderón reproducing formula in Lemma 2.3 and the almost orthogonality estimate in Lemma 2.4 yields

$$\begin{aligned} S_R^{\frac{1}{2}} &= \Big| \sum_{j',k'\in\mathbb{Z}} \sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j',k'}} |R'|\varphi_{j',k'} * f(x_{R'})\psi_{j,k} * \varphi_{j',k'}(x_R - x_{R'})\Big| \\ &\lesssim \sum_{j',k'\in\mathbb{Z}} \sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j',k'}} 2^{-L(\|\mathbf{j}-\mathbf{j}'\|+\|\mathbf{k}-\mathbf{k}'\|)} \\ &\qquad \prod_{i=1}^{m} \frac{|R'|2^{(j_i\vee j_i'\vee k_i\vee k_i')M}}{(2^{j_i\vee j_i'\vee k_i\vee k_i'} + |x_{I_i} - x_{I_i'}|)^{n_i+M}} |\varphi_{j',k'} * f(x_{R'})| \\ &= \sum_{j',k'\in\mathbb{Z}} \sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j',k'}} r(R,R')P(R,R')T_{R'}^{\frac{1}{2}}. \end{aligned}$$

Squaring both sides first, then multiplying by $|R|^2[w(R)]^{-1}$, adding up all the terms over $j,k \in \mathbb{Z}$, $R \in \mathcal{R}_{\mathcal{C}}^{j,k}, R \subseteq \Omega$, and finally applying Hölder's inequality, we obtain

$$\begin{aligned} &\sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}_{\mathcal{C}}^{j,k}\\ R\subseteq\Omega}} |R|^2[w(R)]^{-1} S_R \\ &\lesssim \sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}_{\mathcal{C}}^{j,k}\\ R\subseteq\Omega}} |R|^2[w(R)]^{-1} \Big[\sum_{j',k'\in\mathbb{Z}} \sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j',k'}} r(R,R')P(R,R')T_{R'}^{\frac{1}{2}} \Big]^2 \\ &\lesssim \sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}_{\mathcal{C}}^{j,k}\\ R\subseteq\Omega}} |R|^2[w(R)]^{-1} \Big[\sum_{j',k'\in\mathbb{Z}} \sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j',k'}} r(R,R')P(R,R') \Big] \\ &\qquad \times \Big[\sum_{j',k'\in\mathbb{Z}} \sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j',k'}} r(R,R')P(R,R')T_{R'} \Big]. \end{aligned} \tag{3.1}$$

Note that for $y = (y_1, \dots, y_m) \in R'$, $\ell(I_i') + |x_{I_i} - x_{I_i'}| \approx \ell(I_i') + |x_{I_i} - y_i|$, $i = 1, \dots, m$. Consequently,

$$\begin{aligned} \sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j',k'}} P(R,R') &\approx \sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j',k'}} \prod_{i=1}^{m} \int_{I_i'} \frac{2^{(j_i\vee j_i'\vee k_i\vee k_i')M}}{(2^{j_i\vee j_i'\vee k_i\vee k_i'} + |x_{I_i} - y_i|)^{n_i+M}} dy_i \\ &= \int_{\mathbb{R}^N} \Big(\prod_{i=1}^{m} \frac{2^{(j_i\vee j_i'\vee k_i\vee k_i')M}}{(2^{j_i\vee j_i'\vee k_i\vee k_i'} + |x_{I_i} - y_i|)^{n_i+M}} \Big) dy \lesssim 1. \end{aligned}$$

It follows that

$$\sum_{j',k'\in\mathbb{Z}} \sum_{R'\in\mathcal{R}_{\mathcal{C}}^{j',k'}} r(R,R')P(R,R') \lesssim \sum_{j',k'\in\mathbb{Z}} 2^{-L[\|\mathbf{j}-\mathbf{j}'\|+\|\mathbf{k}-\mathbf{k}'\|]} \lesssim 1. \tag{3.2}$$

For $R, R' \in \mathcal{R}^d_{\mathscr{C}}$, by Lemma 2.2, we also have

$$(3.3)\quad |R|^2[w(R)]^{-1} \lesssim |R'|^2[w(R')]^{-1} \prod_{i=1}^{m}\Big[\frac{|I_i|}{|I_i'|} \vee \frac{|I_i'|}{|I_i|}\Big]^{q+2}\Big[1 + \frac{|x_{I_i} - x_{I_i'}|}{\ell(I_i) \vee \ell(I_i')}\Big]^{n_i q}.$$

Combining the estimates in (3.1), (3.2), and (3.3) yields

$$\sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}^{j,k}_{\mathscr{C}}\\ R\subseteq\Omega}} |R|^2[w(R)]^{-1}S_R \lesssim \sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}^{j,k}_{\mathscr{C}}\\ R\subseteq\Omega}} \sum_{j',k'\in\mathbb{Z}} \sum_{R'\in\mathcal{R}^{j',k'}_{\mathscr{C}}} |R'|^2[w(R')]^{-1}\widetilde{r}(R,R')\widetilde{P}(R,R')T_{R'},$$

where

$$\widetilde{r}(R,R') = \prod_{i=1}^{m}\Big[\frac{|I_i|}{|I_i'|} \wedge \frac{|I_i'|}{|I_i|}\Big]^{L-q-2},$$

$$\widetilde{P}(R,R') = \prod_{i=1}^{m} \frac{1}{(1 + [\ell(I_i) \vee \ell(I_i')]^{-1}|x_{I_i} - x_{I_i'}|)^{n_i+M-n_i q}}.$$

Note that in the above inequality, L and M can be chosen arbitrarily large. Consequently,

$$(3.4)\quad \sup_{\Omega}\Big\{\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}^{d}_{\mathscr{C}}\\ R\subseteq\Omega}} |R|^2[w(R)]^{-1}S_R\Big\} \lesssim \sup_{\Omega}\Big\{\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R\in\mathcal{R}^{d}_{\mathscr{C}}\\ R\subseteq\Omega}} \sum_{R'\in\mathcal{R}^{d}_{\mathscr{C}}} |R'|^2[w(R')]^{-1}r(R,R')P(R,R')T_{R'}\Big\}.$$

Here and in what follows, $\sum_{R\in\mathcal{R}^d_{\mathscr{C}}}$ means $\sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{R}^{j,k}_{\mathscr{C}}}$ and similarly for $\sum_{R'\in\mathcal{R}^d_{\mathscr{C}}}$.

Now to finish the proof, it suffices to show that the last term in (3.4) is majorized by

$$\sup_{\overline{\Omega}}\Big\{\frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{\substack{R'\in\mathcal{R}^{d}_{\mathscr{C}}\\ R'\subseteq\overline{\Omega}}} |R'|^2 w(R')^{-1}T_{R'}\Big\}.$$

We point out that $r(R,R')$ and $P(R,R')$ characterize the geometrical properties between two acceptable rectangles R and R'. Namely, when the difference of the sizes of R and R' grows bigger, $r(R,R')$ becomes smaller; when the distance between R and R' gets larger, $P(R,R')$ becomes smaller. The following argument is quite geometric. To be precise, we shall first decompose the set of dyadic acceptable rectangles $\{R'\}$

into annuli according to the distance of R and R'. Next, in each annuli, precise estimates are given by considering the difference of the sizes of R and R'. Finally, add up all the estimates in each annuli to finish the proof.

We now turn to details. For $j,k \in \mathbb{Z}$ and $R = I_1 \times \cdots \times I_m \in \mathcal{R}^d_{\mathscr{C}}$, let

$$R_{j,k} = (2^{j_1 \vee k_1} I_1) \times \cdots \times (2^{j_m \vee k_m} I_m).$$

Denote $\Omega^{j,k} = \bigcup_{R \subseteq \Omega} 3R_{j,k}$. For any dyadic acceptable rectangle $R \subseteq \Omega$ and $j,k \in \mathbb{Z}_+$, let

$$\begin{aligned}
&\mathscr{A}_{0,0}(R) = \{R' \in \mathcal{R}^d_{\mathscr{C}} : 3R' \cap 3R \neq \varnothing\},\\
&\mathscr{A}_{j,0}(R) = \{R' \in \mathcal{R}^d_{\mathscr{C}} : 3R'_{j,0} \cap 3R \neq \varnothing, 3R'_{j-1,0} \cap 3R = \varnothing\},\\
&\mathscr{A}_{0,k}(R) = \{R' \in \mathcal{R}^d_{\mathscr{C}} : 3R'_{0,k} \cap 3R \neq \varnothing, 3R'_{0,k-1} \cap 3R = \varnothing\},\\
&\mathscr{A}_{j,k}(R) = \{R' \in \mathcal{R}^d_{\mathscr{C}} : 3R'_{j,k} \cap 3R \neq \varnothing, 3R'_{j-1,k} \cap 3R = \varnothing, 3R'_{j,k-1} \cap 3R = \varnothing\}.
\end{aligned}$$

Note that for any $R' \in \mathcal{R}^d_{\mathscr{C}}$ and for any $R \in \mathcal{R}^d_{\mathscr{C}}$ contained in Ω, there exist $j,k \in \mathbb{N}$ such that $R' \in \mathscr{A}_{j,k}(R)$. Therefore $\{R'\} \subseteq \cup_{j,k \in \mathbb{N}} \mathscr{A}_{j,k}(R)$. Hence,

$$\begin{aligned}
&\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R \in \mathcal{R}^d_{\mathscr{C}} \\ R \subseteq \Omega}} \sum_{R' \in \mathcal{R}^d_{\mathscr{C}}} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}\\
&\quad \le \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R \in \mathcal{R}^d_{\mathscr{C}} \\ R \subseteq \Omega}} \sum_{R' \in \mathscr{A}_{0,0}(R)} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}\\
&\qquad + \sum_{j \in \mathbb{Z}_+} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R \in \mathcal{R}^d_{\mathscr{C}} \\ R \subseteq \Omega}} \sum_{R' \in \mathscr{A}_{j,0}(R)} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}\\
&\qquad + \sum_{k \in \mathbb{Z}_+} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R \in \mathcal{R}^d_{\mathscr{C}} \\ R \subseteq \Omega}} \sum_{R' \in \mathscr{A}_{0,k}(R)} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}\\
&\qquad + \sum_{j,k \in \mathbb{Z}_+} \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{\substack{R \in \mathcal{R}^d_{\mathscr{C}} \\ R \subseteq \Omega}} \sum_{R' \in \mathscr{A}_{j,k}(R)} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}\\
&\quad \equiv I + II + III + IV.
\end{aligned}$$

We only estimate the terms I and IV, as terms II and III can be handled similarly. To simplify notation, in the sequel, we always assume $R, R' \in \mathcal{R}^d_{\mathscr{C}}$.

Estimates for I Denote $\mathscr{B}_{0,0} = \{R' \in \mathcal{R}^d_{\mathscr{C}} : 3R' \cap \Omega^{0,0} \neq \varnothing\}$. For any $R' \notin \mathscr{B}_{0,0}$, we have $3R' \cap \Omega^{0,0} = \varnothing$. Thus for every $R \subseteq \Omega$, $3R' \cap 3R = \varnothing$, and thus $R' \notin \mathscr{A}_{0,0}(R)$.

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This implies that $\bigcup_{R\subseteq\Omega}\mathscr{A}_{0,0}\subseteq\mathscr{B}_{0,0}$. Hence

$$I\leq\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{R'\in\mathscr{B}_{0,0}(R)}\sum_{\substack{R:R\subseteq\Omega\\R'\in\mathscr{A}_{0,0}(R)}}|R'|^2w(R')^{-1}r(R,R')P(R,R')T_{R'}.$$

For each integer $h\geq 1$, let $\mathcal{F}_h^{0,0}=\{R'\in\mathscr{B}_{0,0},|3R'\cap\Omega^{0,0}|\geq 1/2^h|3R'|\}$. Denote $\mathcal{D}_h^{0,0}=\mathcal{F}_h^{0,0}\backslash\mathcal{F}_{h-1}^{0,0}$ and $\Omega_h^{0,0}=\bigcup_{R'\in\mathcal{D}_h^{0,0}}R'$. Observe that $\mathscr{B}_{0,0}=\bigcup_{h\geq 1}\mathcal{D}_h^{0,0}$ and that $P(R,R')\leq 1$ for any pair of dyadic acceptable rectangles (R,R') with $R'\in\mathscr{B}_{0,0}$ and $R'\in\mathscr{A}_{0,0}(R)$. Thus

$$I\leq\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{h\geq1}\sum_{R'\subseteq\Omega_h^{0,0}}\sum_{\substack{R:R\subseteq\Omega\\R'\in\mathscr{A}_{0,0}(R)}}|R'|^2w(R')^{-1}r(R,R')T_{R'}.\tag{3.5}$$

We now estimate

$$\sum_{\substack{R:R\subseteq\Omega\\R'\in\mathscr{A}_{0,0}(R)}}r(R,R')$$

for each $h\in\mathbb{Z}_+$ and $R'\subseteq\Omega_h^{0,0}$. Note that $R'\in\mathscr{A}_{0,0}(R)$ implies $3R\cap 3R'\neq\varnothing$. Using an idea of Chang and Fefferman in [CF1], for such R, we consider the following three cases:

Case 1: $|I_1'|\geq|I_1|,|I_2'|\geq|I_2|,\dots,|I_m'|\geq|I_m|$;
Case 2: $|I_i'|\geq|I_i|$ for $i\in A$ and $|I_i'|<|I_i|$ for $i\in B$, where A,B are nonempty subsets of $\{1,\dots,m\}$ and $B=\{1,\dots,m\}\setminus A$;
Case 3: $|I_1'|<|I_1|,|I_2'|<|I_2|,\dots,|I_m'|<|I_m|$.

We first consider Case 1. In this case, we have

$$|R|\leq|3R'\cap 3R|\leq|3R'\cap\Omega^{0,0}|\leq 2^{1-h}|3R'|\leq 2^{1-h+2N}|R'|,$$

which implies that $|R'|=2^{h-2N-1+\theta}|R|$ for some integer $\theta\geq 0$. For each fixed θ, the number of such R's must be less than $C(\theta+h)^N2^{\theta+h}$. Consequently,

$$\sum_{R\in\text{Case 1}}r(R,R')\leq C\sum_{\theta\geq0}\Big(\frac{1}{2^{\theta+h}}\Big)^L(\theta+h)^N2^{\theta+h}\leq C2^{-hL'},$$

where $L'=L-(N+1)>0$.

We next deal with Case 2. For $A=\{i_1,\dots,i_l\}$ and $B=\{i_{l+1},\dots,i_m\}$, we denote $I_A=I_{i_1}\times\dots\times I_{i_l}$, $I_B=I_{i_{l+1}}\times\dots\times I_{i_m}$ and similarly for I_A' and I_B'. Thus $R=I_A\times I_B$ and $R'=I_A'\times I_B'$. It is easy to see that

$$\frac{|I_A|}{|I_{A'}|}\frac{|3R'|}{2^{2N}}\leq\frac{|I_A|}{|3I_A'|}|3R'|\leq|3R\cap3R'|\leq 2^{1-h}|3R'|,$$

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which implies that $|I'_A| = 2^{\theta+h-2N-1}|I_A|$ for some integer $\theta \geq 0$. For each fixed θ, the number of such I_A's must be less than $C(\theta+h)^N \cdot 2^{\theta+h}$. Similarly, $|I_B| = 2^{\lambda}|I'_B|$ for some integer $\lambda \geq 0$. For each fixed λ, $3I_B \cap 3I'_B \neq \varnothing$ implies that the number of such I_B's is less than 5^N. It follows that

$$\sum_{R\in\text{Case 2}} r(R,R') \leq C \sum_{\theta\geq 0}\sum_{\lambda\geq 0}\Big(\frac{1}{2^{\theta+h+\lambda}}\Big)^{L}(\theta+h)^N 2^{\theta+h} \leq C2^{-hL'}.$$

We finally handle Case 3. In this case, we have

$$|R'| \leq |3R\cap 3R'| \leq |3R'\cap \Omega^{0,0}| \leq 2^{1-h}|3R'| \leq 2^{1-h+2N}|R'|,$$

which implies that $h \leq 2N+1$. Since in this case $|R'| \leq |R|$, we have $|R'| = 2^{\theta}|R|$ for some integer $\theta \geq 0$. For each fixed θ, the number of such R's must be less than 5^N. Therefore,

$$\sum_{R\in\text{Case 3}} r(R,R') \leq C\sum_{\theta\geq 0}\Big(\frac{1}{2^{\theta}}\Big)^{L} 2^{\theta} \leq C.$$

Now we rewrite the right side of (3.5) as

$$\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{h\geq 1}\sum_{R'\subseteq\Omega_h^{0,0}}\Big(\sum_{R\in\text{Case 1}}+\sum_{R\in\text{Case 2}}+\sum_{R\in\text{Case 3}}\Big) r(R,R')\frac{|R'|^2}{w(R')}T_{R'} \equiv I_1+I_2+I_3.$$

Note that for $x\in\Omega_h^{0,0}$, there exists a dyadic acceptable rectangle $R\subseteq\Omega_h^{0,0}$ such that $x\in R$. Therefore, $\mathcal{M}_{\mathscr{C}}(\chi_{\Omega^{0,0}})(x) \geq |3R'\cap\Omega^{0,0}|/|3R'| \geq 2^{-h}$. Applying the $L^q_w(\mathbb{R}^N)$ boundedness of $\mathcal{M}_{\mathscr{C}}$ with $q\in(q_w,\frac{pL}{2-p})$ and Lemma 2.1 yields

$$w(\Omega_h^{0,0}) \leq w\big(\{x : \mathcal{M}_{\mathscr{C}}(\chi_{\Omega^{0,0}})(x)\geq 2^{-h}\}\big) \lesssim 2^{qh}w(\Omega^{0,0}) \lesssim 2^{qh}w(\Omega).$$

This, together with the estimates in Case 1 and Case 2, yields

$$\begin{aligned}
I_1+I_2 &\leq C\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{h\geq 1}\sum_{R'\subseteq\Omega_h^{0,0}} 2^{-hL'}\frac{|R'|^2}{w(R')}T_{R'}\\
&\leq C\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{h\geq 1}2^{-hL'}[w(\Omega_h^{0,0})]^{\frac{2}{p}-1}\frac{1}{[w(\Omega_h^{0,0})]^{\frac{2}{p}-1}}\sum_{R'\subseteq\Omega_h^{0,0}}\frac{|R'|^2}{w(R')}T_{R'}\\
&\leq C\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}}\sum_{h\geq 1}2^{-hL'}(2^{qh})^{\frac{2}{p}-1}[w(\Omega)]^{\frac{2}{p}-1}\sup_{\overline{\Omega}}\frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}}\sum_{R'\subseteq\overline{\Omega}}\frac{|R'|^2}{w(R')}T_{R'}\\
&\leq C\sup_{\overline{\Omega}}\frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}}\sum_{R'\subseteq\overline{\Omega}}\frac{|R'|^2}{w(R')}T_{R'},
\end{aligned}$$

where in the last inequality we have chosen L' large enough so that

$$\sum_{h\geq 1} 2^{-hL'}(2^{qh})^{\frac{2}{p}-1} \leq C.$$

For I_3, note that in this case, h must be less than $C = 2N+1$. Hence,

$$\begin{aligned}
I_3 &\leq C\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{1\leq h\leq C} \sum_{R'\subseteq\Omega_h^{0,0}} \frac{|R'|^2}{w(R')} T_{R'} \\
&\leq C\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{1\leq h\leq C} [w(\Omega_h^{0,0})]^{\frac{2}{p}-1} \frac{1}{[w(\Omega_h^{0,0})]^{\frac{2}{p}-1}} \sum_{R'\subseteq\Omega_h^{0,0}} \frac{|R'|^2}{w(R')} T_{R'} \\
&\leq C\frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{1\leq h\leq C} (2^{qh})^{\frac{2}{p}-1}[w(\Omega)]^{\frac{2}{p}-1} \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R'\subseteq\overline{\Omega}} \frac{|R'|^2}{w(R')} T_{R'} \\
&\leq C \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R'\subseteq\overline{\Omega}} \frac{|R'|^2}{w(R')} T_{R'}.
\end{aligned}$$

Altogether, this yields

$$I \leq C \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R'\subseteq\overline{\Omega}} \frac{|R'|^2}{w(R')} T_{R'}.$$

Estimates for IV For $j,k \geq 1$, set

$$a_{j,k} \equiv \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{R\subseteq\Omega} \sum_{R'\in\mathscr{A}_{j,k}(R)} |R'|^2 w(R')^{-1} r(R,R')P(R,R')T_{R'},$$

and

$$\mathscr{B}_{j,k} \equiv \{R' : 3R'_{j,k} \cap \Omega^{0,0} \neq \varnothing\}.$$

Recall that

$$\mathscr{A}_{j,k}(R) = \{R' \in \mathcal{R}^d_{\mathscr{C}} : 3R'_{j,k} \cap 3R \neq \varnothing,\ 3R'_{j-1,k} \cap 3R = \varnothing,\ \text{and } 3R'_{j,k-1} \cap 3R = \varnothing\}.$$

For any $R' \notin \mathscr{B}_{j,k}$, we have $3R'_{j,k} \cap \Omega^{0,0} = \varnothing$. Thus for every $R \subset \Omega$, $3R'_{j,k} \cap 3R = \varnothing$, which implies that $R' \notin \mathscr{A}_{j,k}(R)$. Therefore $\bigcup_{R\subseteq\Omega} \mathscr{A}_{j,k}(R) \subseteq \mathscr{B}_{j,k}$. Hence,

$$a_{j,k} \leq \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{R'\in\mathscr{B}_{j,k}} \sum_{\substack{R:R\subseteq\Omega \\ R'\in\mathscr{A}_{j,k}(R)}} |R'|^2 w(R')^{-1} r(R,R')P(R,R')T_{R'}.$$

Let

$$\mathcal{F}_h^{j,k} = \{R' \in \mathscr{B}_{j,k} : |3R'_{j,k} \cap \Omega^{0,0}| \geq 1/2^h |3R'_{j,k}|\},\ h \geq 0,$$
$$\mathcal{D}_h^{j,k} = \mathcal{F}_h^{j,k} \setminus \mathcal{F}_{h-1}^{j,k},\ h \geq 1, \quad \mathcal{D}_0^{j,k} = \varnothing, \text{ and}$$
$$\Omega_h^{j,k} = \bigcup_{R' \in \mathcal{D}_h^{j,k}} R',\ h \geq 1.$$

Note that $\mathscr{B}_{j,k} = \bigcup_{h \geq 1} \mathcal{D}_h^{j,k}$. Thus,

$$a_{j,k} \leq \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h \geq 1} \sum_{R' \in \mathcal{D}_h^{j,k}} \sum_{\substack{R : R \subseteq \Omega \\ R' \in \mathscr{A}_{j,k}(R)}} |R'|^2 w(R')^{-1} r(R,R') P(R,R') T_{R'}. \tag{3.6}$$

Note that $R' \in \mathscr{A}_{j,k}(R)$ implies that

$$|x_{I_i} - x_{I'_i}| > 2^{j_i \vee k_i} \ell(I'_i) \vee \ell(I_i), \quad \text{for } i = 1, \dots, m.$$

As for the estimates for I, we consider three cases:
Case 1: $|2^{j_1 \vee k_1} I'_1| \geq |I_1|, \dots, |2^{j_m \vee k_m} I'_m| \geq |I_m|$;
Case 2: $|2^{j_i \vee k_i} I'_i| \geq |I_i|$ for $i \in A$, and $|2^{j_i \vee k_i} I'_i| < |I_i|$ for $i \in B$, where A, B are nonempty subsets of $\{1, \dots, m\}$ and $A \cup B = \{1, \dots, m\}$;
Case 3: $|2^{j_1 \vee k_1} I'_1| < |I_1|, \dots, |2^{j_m \vee k_m} I'_m| < |I_m|$.

We rewrite (3.6) as

$$\begin{aligned} a_{j,k} &\leq \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h \geq 1} \sum_{R' \in \mathcal{D}_h^{j,k}} |R'|^2 w(R')^{-1} T_{R'} \\ &\quad \times \Big(\sum_{R \in \text{Case 1}} + \sum_{R \in \text{Case 2}} + \sum_{R \in \text{Case 3}} \Big) r(R,R') P(R,R') \\ &\equiv a_{j,k,1} + a_{j,k,2} + a_{j,k,3}. \end{aligned}$$

We first handle the term $a_{j,k,2}$. For each $h \geq 1$ and $R' \in \mathcal{D}_h^{j,k}$, we estimate

$$\sum_{R \in \text{Case 2}} r(R,R') P(R,R').$$

Let $A = \{i_1, \dots, i_l\}$ and $B = \{i_{l+1}, \dots, i_m\}$. Set

$$I_A = I_{i_1} \times \cdots \times I_{i_l}, \quad I_B = I_{i_{l+1}} \times \cdots \times I_{i_m},$$

and similarly for I'_A and I'_B. Thus $R = I_A \times I_B$ and $R' = I'_A \times I'_B$. For each $j, k \geq 0$, set

$$I'_{A,j,k} = 2^{j_1 \vee k_1} I'_{i_1} \times \cdots \times 2^{j_l \vee k_l} I'_{i_l},$$
$$I'_{B,j,k} = 2^{j_{l+1} \vee k_{l+1}} I'_{i_{l+1}} \times \cdots \times 2^{j_m \vee k_m} I'_{i_m}.$$

Then $R'_{j,k} = I'_{A,j,k} \times I'_{B,j,k}$. Let $j_A = j_{i_1} n_{i_1} + \cdots + j_{i_l} n_{i_l}$ and $j_B = j_{i_{l+1}} n_{i_{l+1}} + \cdots + j_{i_m} n_{i_m}$ and similarly for $k_A, k_B, (j \vee k)_A$ and $(j \vee k)_B$.

Note first that $|I_A \times I'_{B,j,k}| \le |3R'_{j,k} \cap 3R|$. Thus

$$\frac{|I_A|}{|3I'_{A,j,k}|}|3R'_{j,k}| \le |3R'_{j,k} \cap 3R| \le |3R'_{j,k} \cap \Omega^{0,0}| \le \frac{1}{2^{h-1}}|3R'_{j,k}|,$$

which yields

$$2^{h-1}|I_A| \le 3^N |I'_{A,j,k}| \le 2^{2N+(j\vee k)_A}|I'_A|.$$

We now consider two subcases.

Subcase 2.1: $|I'_A| \ge |I_A|$. In this subcase, since $2^{h-1-(j\vee k)_A}|I_A| \lesssim |I'_A|$, we have $|I'_A| \sim 2^{h-1-(j\vee k)_A+n}|I_A|$ for some integer $n \ge 0$. And for each fixed n, the number of such I_A's must be less than $\lesssim (n+h)^N 2^{n+h}$.

Subcase 2.2: $|I'_A| < |I_A|$. In this subcase, we have $|I'_A| < |I_A| \le |I'_{A,j,k}|$. So $2^{\overline{n}}|I'_A| = |I_A|$ for some integer $\overline{n}$ satisfying $1 \le \overline{n} \le (j \vee k)_A$. Moreover, for each $\overline{n}$, the number of such I_A's must be $\lesssim 1$. Moreover, we have $2^{h-1}2^{\overline{n}}|I'_A| = 2^{h-1}|I_A| \le 2^{(j\vee k)_A+2N}|I'_A|$, which implies that $h \le 2N + (j \vee k)_A - \overline{n}$. Note also that

$$\prod_{i=1}^{l}\left(\frac{|x_{I_i} - x_{I'_i}|}{\ell(I_i)}\right)^{n_i} = \prod_{i=1}^{l}\left(\frac{|x_{I_i} - x_{I'_i}|}{\ell(I'_i)}\frac{\ell(I'_i)}{\ell(I_i)}\right)^{n_i} \ge 2^{(j\vee k)_A}\frac{|I'_A|}{|I_A|} \ge 2^{(j\vee k)_A - \overline{n}}.$$

In Case 2, $|I'_{B,j,k}| \le |I_B|$ implies that $2^{(j\vee k)_B+\kappa}|I'_B| \sim |I_B|$ for some $\kappa \ge 0$. And for each fixed κ, the number of such I_B's must be $\lesssim 1$ since $3I'_{B,j,k} \cap 3I_B \ne \varnothing$. These considerations imply that for $M > L$,

$$\begin{aligned}
&\sum_{\text{Subcase 2.1}} r(R,R')P(R,R') \\
&\le \sum_{\text{Subcase 2.1}} \left(\frac{|I_A|}{|I'_A|}\right)^L \left(\frac{|I'_B|}{|I_B|}\right)^L \prod_{i=1}^{l}\left(1 + \frac{|x_{I_i} - x_{I'_i}|}{\ell(I'_i)}\right)^{-(1+M)} \\
&\lesssim \sum_{n,\kappa \ge 0} (n+h)^N 2^{n+h} 2^{-[h-1-(j\vee k)_A+n]L} 2^{-[(j\vee k)_B+\kappa]L} 2^{-(1+M)(j\vee k)_A} \\
&\lesssim 2^{-hL'} 2^{-[(j\vee k)_A][M-L]} 2^{-[(j\vee k)_B]L},
\end{aligned}$$

and that for $M > NM'$

$$\begin{aligned}
&\sum_{\text{Subcase 2.2}} r(R,R')P(R,R') \\
&\le \sum_{\text{Subcase 2.2}} \left(\frac{|I'_A|}{|I_A|}\right)^L \left(\frac{|I'_B|}{|I_B|}\right)^L \prod_{i=1}^{l}\left(1 + \frac{|x_{I_i} - x_{I'_i}|}{\ell(I_i)}\right)^{-n_i M'} \\
&\lesssim \sum_{\overline{n}=1}^{(j\vee k)_A} \sum_{\kappa \ge 0} 2^{-\overline{n}L} 2^{-[(j\vee k)_B+\kappa]L} 2^{-M'[(j\vee k)_A - \overline{n}]}
\end{aligned}$$

$$\lesssim \sum_{\overline{n}=1}^{(j\vee k)_A} 2^{-\overline{n}L} 2^{-M'[(j\vee k)_A-\overline{n}]} 2^{-(j\vee k)_B L}$$
$$\lesssim 2^{-L(j\vee k)_A} 2^{-L(j\vee k)_B}.$$

To estimate $a_{j,k,2}$, we write

$$\begin{aligned} a_{j,k,2} &= \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h\geq 1} \sum_{R'\in\mathcal{D}_h^{j,k}} |R'|^2 w(R')^{-1} T_{R'} \\ &\quad \times \Big(\sum_{R\in \text{Subcase } 2.1} + \sum_{R\in \text{Subcase } 2.2} \Big) r(R,R') P(R,R') \\ &\equiv a_{j,k,2.1} + a_{j,k,2.2}. \end{aligned}$$

Note that for $x \in \Omega_h^{j,k}$ there exists a dyadic acceptable rectangle $R \subseteq \Omega_h^{j,k}$ such that $x \in R$, and therefore $\mathcal{M}_{\mathscr{C}}(\chi_{\Omega^{0,0}})(x) \geq |R'_{j,k} \cap \Omega^{0,0}|/|R'_{j,k}| \geq 2^{-h}$. Thus, applying the L^q_w boundedness of $\mathcal{M}_{\mathscr{C}}$ with $q \in (q_w, \frac{pL}{2-p})$ and Lemma 2.1,

$$w(\Omega_h^{j,k}) \leq w\big(\{x : \mathcal{M}_{\mathscr{C}}(\chi_{\Omega^{0,0}})(x) \geq 2^{-h}\}\big) \lesssim 2^{qh} w(\Omega^{0,0}) \lesssim 2^{qh} w(\Omega).$$

Combining the above estimates yields

$$\begin{aligned} a_{j,k,2.1} &\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h\geq 1} 2^{-hL'} 2^{-[(j\vee k)_A][M-L]} 2^{-[(j\vee k)_B]L} [w(\Omega_h^{j,k})]^{\frac{2}{p}-1} \\ &\quad \times \frac{1}{[w(\Omega_h^{j,k})]^{\frac{2}{p}-1}} \sum_{R'\subseteq\Omega_h^{j,k}} |R'|^2 w(R')^{-1} T_{R'} \\ &\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h\geq 1} 2^{-hL'} 2^{-[(j\vee k)_A][M-L]} 2^{-[(j\vee k)_B]L} [2^{qh}]^{\frac{2}{p}-1} [w(\Omega)]^{\frac{2}{p}-1} \\ &\quad \times \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R'\subseteq\overline{\Omega}} |R'|^2 w(R')^{-1} T_{R'} \\ &\lesssim 2^{-[(j\vee k)_A][M-L]} 2^{-[(j\vee k)_B]L} \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R'\subseteq\overline{\Omega}} |R'|^2 w(R')^{-1} T_{R'} \end{aligned}$$

and

$$\begin{aligned} a_{j,k,2.2} &\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{h=1}^{2N+(j\vee k)_A} 2^{-[(j\vee k)_A]L} 2^{-[(j\vee k)_B]L} [w(\Omega_h^{j,k})]^{\frac{2}{p}-1} \\ &\quad \times \frac{1}{[w(\Omega_h^{j,k})]^{\frac{2}{p}-1}} \sum_{R'\subseteq\Omega_h^{j,k}} |R'|^2 w(R')^{-1} T_{R'} \end{aligned}$$

$$\lesssim \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} 2^{-[(j\vee k)_A][L-N-q(2/p-1)]} 2^{-[(j\vee k)_B]L} [w(\Omega)]^{\frac{2}{p}-1}$$

$$\times \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R'\subseteq\overline{\Omega}} |R'|^2 w(R')^{-1} T_{R'}$$

$$= 2^{-[(j\vee k)_A]L''} 2^{-[(j\vee k)_B]L} \sup_{\overline{\Omega}} \frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}} \sum_{R'\subseteq\overline{\Omega}} |R'|^2 w(R')^{-1} T_{R'}.$$

Combining these estimates yields that for $M > L > q(\frac{2}{p} - 1)$

$$\sum_{j,k\geq 1} a_{j,k,2} \leq \sum_{j,k\geq 1} a_{j,k,2.1} + \sum_{j,k\geq 1} a_{j,k,2.2} \lesssim \sup_{\overline{\Omega}} \frac{1}{|\overline{\Omega}|^{\frac{2}{p}-1}} \sum_{R'\subseteq\overline{\Omega}} |R'|^2 w(R')^{-1} T_{R'}.$$

Using the same skills as above, we can also get

$$\sum_{j,k\geq 1} a_{j,k,1} + a_{j,k,3} \lesssim \sup_{\overline{\Omega}} \frac{1}{|\overline{\Omega}|^{\frac{2}{p}-1}} \sum_{R'\subseteq\overline{\Omega}} |R'|^2 w(R')^{-1} T_{R'}.$$

This gives the desired estimate for IV, and hence Theorem 1.6 follows. ■

4 Duality of $H^p_{\mathscr{C},w}(\mathbb{R}^N)$ with $\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$

The purpose of this section is to prove Theorem 1.7. To this end, we first introduce weighted sequence spaces associated with different homogeneities.

Definition 4.1 Let $0 < p \leq 1$ and $w \in A^{\mathscr{C}}_\infty(\mathbb{R}^N)$. The weighted sequence space $s^p_w(\mathbb{R}^N)$ is defined to be the collection of all sequences $s = \{s_R\}$ such that

$$\|s\|_{s^p_w(\mathbb{R}^N)} \equiv \Big\| \Big\{ \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}^{j,k}_{\mathscr{C}}} |s_R|^2 |R|^{-1} \chi_R \Big\}^{\frac{1}{2}} \Big\|_{L^p_w(\mathbb{R}^N)} < \infty.$$

The weighted sequence space $c^p_w(\mathbb{R}^N)$ consists of all sequences $s = \{s_R\}$ such that

$$\|s\|_{c^p_w(\mathbb{R}^N)} \equiv \sup_{\Omega} \Big\{ \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}^{j,k}_{\mathscr{C}} \\ R\subseteq\Omega}} |s_R|^2 \frac{|R|}{w(R)} \Big\}^{\frac{1}{2}} < \infty,$$

where Ω runs over all open sets in $\mathbb{R}^N$ with $w(\Omega) < \infty$.

The duality theorem for the weighted sequence spaces is as follows.

Theorem 4.2 *Let $0 < p \leq 1$ and $w \in A^{\mathscr{C}}_\infty(\mathbb{R}^N)$. Then $(s^p_w(\mathbb{R}^N))^* = c^p_w(\mathbb{R}^N)$. More precisely, for every $t = \{t_R\} \in c^p_w(\mathbb{R}^N)$, the mapping*

$$s = \{s_R\} \longrightarrow \langle s, t\rangle \equiv \sum_R s_R \bar{t}_R$$

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defines a continuous linear functional on $s^p_w(\mathbb{R}^N)$ *with operator norm* $\|t\|_{(s^p_w(\mathbb{R}^N))^*} \approx \|t\|_{c^p_w(\mathbb{R}^N)}$, *and conversely, for every* $\ell \in (s^p_w(\mathbb{R}^N))^*$, *there exists a unique* $t \in c^p_w(\mathbb{R}^N)$ *such that* $\ell(\{s_R\}) = \langle s, \bar{t}\rangle$.

Proof We first prove that $c^p_w(\mathbb{R}^N) \subseteq (s^p_w(\mathbb{R}^N))^*$. Suppose $t = \{t_R\} \in c^p_w(\mathbb{R}^N)$. For $s \in s^p_w(\mathbb{R}^N)$, set

$$\mathcal{G}(s)(x) = \Big\{ \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}^{j,k}_{\mathscr{C}}} |s_R|^2 |R|^{-1} \chi_R(x) \Big\}^{\frac{1}{2}}.$$

For $i \in \mathbb{Z}$, set

$$\begin{aligned}
\Omega_i &= \{x \in \mathbb{R}^N : \mathcal{G}(s)(x) > 2^i\},\\
\widetilde{\Omega}_i &= \{x \in \mathbb{R}^N : \mathcal{M}_{\mathscr{C}}(\chi_{\Omega_i})(x) > 1/2\},\\
\mathcal{B}_i &= \{R \in \mathcal{R}^d_{\mathscr{C}} : |R \cap \Omega_i| > 1/2|R|,\ |R \cap \Omega_{i+1}| \le 1/2|R|\}.
\end{aligned}$$

If $x \in R \in \mathcal{B}_i$, then

$$\mathcal{M}_{\mathscr{C}}(\chi_{\Omega_i})(x) \ge \frac{1}{|R|}\int_R \chi_{\Omega_i}(y)dy = \frac{|R\cap\Omega_i|}{|R|} > \frac{1}{2},$$

which implies

$$\bigcup_{R\in\mathcal{B}_i} R \subseteq \widetilde{\Omega}_i. \tag{4.1}$$

Moreover, for $q \in (q_w, \infty)$, by the $L^q_w(\mathbb{R}^N)$ boundedness of $\mathcal{M}_{\mathscr{C}}$,

$$w(\widetilde{\Omega}_i) \lesssim w(\Omega_i), \tag{4.2}$$

and in view of Lemma 2.1, for each $R \in \mathcal{B}_i$,

$$\frac{w(R\cap(\Omega_i\setminus\Omega_{i+1}))}{w(R)} = \frac{w(R\setminus\Omega_{i+1})}{w(R)} \gtrsim \Big[\frac{|R\setminus\Omega_{i+1}|}{|R|}\Big]^q \ge \frac{1}{2^q}. \tag{4.3}$$

Suppose $t = \{t_R\} \in c^p_w(\mathbb{R}^N)$. Applying (4.1), (4.2), (4.3), and the Cauchy–Schwarz inequality yields

$$\begin{aligned}
&\Big|\sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{R}^{j,k}_{\mathscr{C}}} s_R\bar{t}_R\Big|\\
&\quad\lesssim \Big|\sum_{i\in\mathbb{Z}}\int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}}\sum_{R\in\mathcal{B}_i}|\bar{t}_R|\frac{|R|^{\frac{1}{2}}}{w(R)}|s_R||R|^{-\frac{1}{2}}\chi_R(x)w(x)dx\Big|\\
&\quad\le \sum_{i\in\mathbb{Z}}\Big\{\sum_{R\subseteq\Omega_i}|t_R|^2\frac{|R|}{w(R)}\Big\}^{\frac{1}{2}}\Big\{\int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}}\sum_{R\in\mathcal{B}_i}|s_R|^2|R|^{-1}\chi_R(x)w(x)dx\Big\}^{\frac{1}{2}}
\end{aligned}$$

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$$\lesssim \|t\|_{c^p_w} \sum_{i\in\mathbb{Z}} [w(\widetilde{\Omega}_i)]^{(\frac{2}{p}-1)\frac{1}{2}} \Big\{ \int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}} [\mathcal{G}(s)(x)]^2 w(x)dx \Big\}^{\frac{1}{2}}$$
$$\leq C\|t\|_{c^p_w} \sum_{i\in\mathbb{Z}} 2^i [w(\Omega_i)]^{\frac{1}{p}} \lesssim \|t\|_{c^p_w} \|\mathcal{G}(s)\|_{L^p_w} = \|t\|_{c^p_w}\|s\|_{s^p_w},$$

proving the inclusion $c^p_w(\mathbb{R}^N) \subseteq (s^p_w(\mathbb{R}^N))^*$.

The converse can be proved similarly to that given in [FJ] in the one-parameter setting. If $\ell \in (s^p_w(\mathbb{R}^N))^*$, then it is clear that $\ell(s) = \sum_R s_R\bar{t}_R$ for some $t = \{t_R\}$. Now fix an open set $\Omega \subseteq \mathbb{R}^N$ with $w(\Omega) < \infty$. Let μ be a measure of $\mathcal{R}^d_{\mathcal{C}}$ such that $\mu(R) = [w(\Omega)]^{1-2/p}|R|[w(R)]^{-1}$ if $R \subseteq \Omega$, and otherwise $\mu(R) = 0$. Set

$$\|\{s_R\}\|_{\ell^2(\Omega,\mu)} = \Big\{ \sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}^{j,k}_{\mathcal{C}}\\ R\subseteq\Omega}} |s_R|^2 [w(\Omega)]^{1-2/p}|R|[w(R)]^{-1} \Big\}^{\frac{1}{2}}.$$

Thus

$$\Big\{ \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}^{j,k}_{\mathcal{C}}\\ R\subseteq\Omega}} |t_R|^2 \frac{|R|}{w(R)} \Big\}^{\frac{1}{2}}$$
$$= \|t\|_{\ell^2(\Omega,\mu)} = \sup_{\|s\|_{\ell^2(\Omega,\mu)}\leq 1} \Big| \sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}^{j,k}_{\mathcal{C}}\\ R\subseteq\Omega}} s_R\bar{t}_R [w(\Omega)]^{1-2/p}|R|[w(R)]^{-1} \Big|$$
$$\leq \|\ell\| \sup_{\|s\|_{\ell^2(\Omega,\mu)}\leq 1} \Big\| s_R[w(\Omega)]^{1-2/p}|R|[w(R)]^{-1} \Big\|_{s^p_w},$$

where $s = \{s_R\}$, $s_R = 0$ if R is not contained in Ω. However, for such an s, by Hölder's inequality,

$$\Big\| s_R[w(\Omega)]^{1-2/p}|R|[w(R)]^{-1} \Big\|_{s^p_w(\mathbb{R}^N)}$$
$$= \Big\{ \int_\Omega \big[\sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}^{j,k}_{\mathcal{C}}\\ R\subseteq\Omega}} |s_R|^2 [w(\Omega)]^{2-4/p}|R|[w(R)]^{-2}\chi_R(x) \big]^{\frac{p}{2}} w(x)dx \Big\}^{\frac{1}{p}}$$
$$\leq [w(\Omega)]^{\frac{1}{p}-\frac{1}{2}} \Big\{ \int_\Omega \sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}^{j,k}_{\mathcal{C}}\\ R\subseteq\Omega}} |s_R|^2 [w(\Omega)]^{2-4/p}|R|[w(R)]^{-2}\chi_R(x) w(x)dx \Big\}^{\frac{1}{2}}$$
$$= \|s\|_{\ell^2(\Omega,\mu)} \leq 1.$$

This shows that $\|t\|_{c^p_w(\mathbb{R}^N)} \leq \|\ell\|$ and thus $t \in c^p_w(\mathbb{R}^N)$. Hence the proof of Theorem 4.2 is concluded. ∎

In order to use Theorem 4.2 to show Theorem 1.7, we define the *lifting operator* $\mathcal{L}(f)$ for $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^N)$ and the *projection operator* $\mathcal{T}(t)$ for a sequence $t = \{t_R\}$ respectively by

$$\mathcal{L}(f) = \{|R|^{\frac{1}{2}} \psi_{j,k} * f(x_R)\} = \{s_R\}$$

and

$$\mathcal{T}(t)(x) = \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}^{j,k}_{\mathscr{C}}} |R|^{\frac{1}{2}} \psi_{j,k}(x - x_R) t_R,$$

where $\psi_{j,k}$ and x_R are the same as in Definition 1.5.

To prove Theorem 1.7, we also need the following theorem.

Theorem 4.3 *Let $0 < p \le 1$ and $w \in A^{\mathscr{C}}_\infty(\mathbb{R}^N)$. Then the lifting operator $\mathcal{L}$ is bounded from $H^p_{\mathscr{C},w}(\mathbb{R}^N)$ to $s^p_w(\mathbb{R}^N)$ and from $\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$ to $c^p_w(\mathbb{R}^N)$, the projection operator $\mathcal{T}$ is bounded from $s^p_w(\mathbb{R}^N)$ to $H^p_{\mathscr{C},w}(\mathbb{R}^N)$ and from $c^p_w(\mathbb{R}^N)$ to $\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$. Moreover, $\mathcal{T} \circ \mathcal{L}$ is the identity on $H^p_{\mathscr{C},w}(\mathbb{R}^N)$ and $\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$.*

Proof The boundedness of $\mathcal{L}$ from $H^p_{\mathscr{C},w}(\mathbb{R}^N)$ to $s^p_w(\mathbb{R}^N)$ and from $\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$ to $c^p_w(\mathbb{R}^N)$ follows directly from Definition 4.1.

We next show that $\mathcal{T}$ is bounded from $s^p_w(\mathbb{R}^N)$ to $H^p_{\mathscr{C},w}(\mathbb{R}^N)$. The proof is similar to that of [Wu, Theorem 1.2]. Let $t = \{t_R\}$. Applying the Calderón reproducing formula in Lemma 2.3, the almost orthogonality estimates with $M > N[(q_w/p - 1) \vee 0]$ and $L = 10M$, Lemma 2.6 with $M_i = M$, and the $L^{p/\delta}_w(\ell^{2/\delta})$ boundedness of $\mathcal{M}_{\mathscr{C}}$ in Theorem 1.3, we have for some $N/(N+M) < \delta < (1 \wedge p/q_w)$,

$$\begin{aligned}
&\|\mathcal{T}(t)\|_{H^p_{\mathscr{C},w}} \\
&= \Big\| \Big\{ \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}^{j,k}_{\mathscr{C}}} \Big| \sum_{j',k'\in\mathbb{Z}} \sum_{R'\in\mathcal{R}^{j',k'}_{\mathscr{C}}} |R'| \psi_{j,k} * \psi_{j',k'}(x_R - x_{R'}) t_{R'} |R'|^{-\frac{1}{2}} \Big|^2 \chi_R \Big\}^{\frac{1}{2}} \Big\|_{L^p_w} \\
&= \Big\| \Big\{ \sum_{j,k\in\mathbb{Z}} \Big(\sum_{j',k'\in\mathbb{Z}} 2^{-5M\|\mathbf{j}-\mathbf{j}'\|} 2^{-5M\|\mathbf{k}-\mathbf{k}'\|} \Big) \\
&\qquad \times \Big(\sum_{j',k'\in\mathbb{Z}} 2^{-5M\|\mathbf{j}-\mathbf{j}'\|} 2^{-5M\|\mathbf{k}-\mathbf{k}'\|} \big(\mathcal{M}_{\mathscr{C}}[(\sum_{R'\in\mathcal{R}^{j',k'}_{\mathscr{C}}} t^2_{R'} |R'|^{-1} \chi_{R'})^{\frac{\delta}{2}}] \big)^{\frac{2}{\delta}} \Big) \Big\}^{\frac{1}{2}} \Big\|_{L^p_w} \\
&\lesssim \Big\| \Big\{ \sum_{j',k'\in\mathbb{Z}} \{ \mathcal{M}_{\mathscr{C}} [\sum_{R'\in\mathcal{R}^{j,k}_{\mathscr{C}}} t^2_{R'} |R'|^{-1} \chi_{R'}]^{\delta/2} \}^{\frac{2}{\delta}} \Big\}^{\frac{1}{2}} \Big\|_{L^p_w} \\
&\lesssim \Big\| \Big\{ \sum_{j',k'\in\mathbb{Z}} \sum_{R'\in\mathcal{R}^{j,k}_{\mathscr{C}}} t^2_{R'} |R'|^{-1} \chi_{R'} \Big\}^{\frac{1}{2}} \Big\|_{L^p_w} = \|s\|_{s^p_w}.
\end{aligned}$$

Finally, we prove that the operator $\mathcal{T}$ is bounded from $c^p_w(\mathbb{R}^N)$ to $\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$.

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Suppose $t=\{t_R\}\in c_w^p(\mathbb{R}^N)$. We have

$$\sum_{j,k\in\mathbb{Z}}\sum_{\substack{R\in\mathcal{R}_{\mathcal{C}}^{j,k}\\ R\subseteq\Omega}}|\psi_{j,k}*\mathcal{T}(t)(x_R)|^2\frac{|R|^2}{w(R)}=$$

$$\sum_{j,k\in\mathbb{Z}}\sum_{\substack{R\in\mathcal{R}_{\mathcal{C}}^{j,k}\\ R\subseteq\Omega}}\Big(\sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{R}_{\mathcal{C}}^{j,k}}|\psi_{j,k}*\psi_{j',k'}(x_R-x_{R'})|\cdot t_{R'}\cdot|R'|^{\frac{1}{2}}\Big)^2\frac{|R|^2}{w(R)}.$$

Using the same skills as in Theorem 1.6, we can obtain

$$\|\mathcal{T}(t)\|_{\mathrm{CMO}^p_{\mathcal{C},w}(\mathbb{R}^N)}\lesssim\sup_{\overline{\Omega}}\Big\{\frac{1}{[w(\overline{\Omega})]^{\frac{2}{p}-1}}\sum_{R'\subseteq\overline{\Omega}}|t_{R'}|^2\frac{|R'|^2}{w(R')}\Big\}^{\frac{1}{2}}\approx\|t\|_{c_w^p(\mathbb{R}^N)}.$$

The fact that $\mathcal{T}\circ\mathcal{L}$ is the identity on $H^p_{\mathcal{C},w}(\mathbb{R}^N)$ and $\mathrm{CMO}^p_{\mathcal{C},w}(\mathbb{R}^N)$ follows directly from the discrete Calderón reproducing formula in Lemma 2.3. This concludes the proof of Theorem 4.3. ∎

Now we are ready to give the proof of Theorem 1.7.

Proof of Theorem 1.7 We first prove the inclusion $\mathrm{CMO}^p_{\mathcal{C},w}(\mathbb{R}^N)\subseteq(H^p_{\mathcal{C},w}(\mathbb{R}^N))^*$. Let $g\in\mathrm{CMO}^p_{\mathcal{C},w}(\mathbb{R}^N)$. For $f\in\mathcal{S}_\infty(\mathbb{R}^N)$, define the mapping $\ell_g=\langle f,g\rangle$. By Lemma 2.3 and Theorems 4.2 and 4.3,

$$\begin{aligned}|\ell_g(f)|=|\langle f,g\rangle|&=|\sum_{j,k\in\mathbb{Z}}\sum_{R\subseteq\mathcal{R}_{\mathcal{C}}^{j,k}}|R|^{\frac{1}{2}}\psi_{j,k}*f(x_R)|R|^{\frac{1}{2}}\psi_{j,k}*g(x_R)|\\ &=|\langle\mathcal{L}(f),\mathcal{L}(g)\rangle|\le C\|\mathcal{L}(f)\|_{s_w^p(\mathbb{R}^N)}\|\mathcal{L}(g)\|_{c_w^p(\mathbb{R}^N)}\\ &\le C\|f\|_{H^p_{\mathcal{C},w}(\mathbb{R}^N)}\|g\|_{\mathrm{CMO}^p_{\mathcal{C},w}(\mathbb{R}^N)},\end{aligned}$$

where we have chosen $\psi^{(1)}(-x)=\psi^{(1)}(x)$ and $\psi^{(2)}(-x)=\psi^{(2)}(x)$. Since $\mathcal{S}_\infty(\mathbb{R}^N)$ is dense in $H^p_{\mathcal{C},w}(\mathbb{R}^N)$ (see [Wu, Corollary 3.1]), this implies that the mapping $\ell_g(f)=\langle f,g\rangle$ can be extended to a continuous linear functional on $H^p_{\mathcal{C},w}(\mathbb{R}^N)$ and $\|\ell_g\|\le C\|g\|_{\mathrm{CMO}^p_{\mathcal{C},w}(\mathbb{R}^N)}$.

Conversely, let $\ell\in(H^p_{\mathcal{C},w}(\mathbb{R}^N))^*$ and $\ell_1=\ell\circ\mathcal{T}$. By Theorem 4.3,

$$|\ell_1(s)|=|\ell(\mathcal{T}(s))|\le\|\ell\|\cdot\|\mathcal{T}(s)\|_{H^p_{\mathcal{C},w}(\mathbb{R}^N)}\le C\|\ell\|\cdot\|s\|_{s_w^p(\mathbb{R}^N)},\quad\text{for } s\in s_w^p(\mathbb{R}^N),$$

which implies that $\ell_1\in(s_w^p(\mathbb{R}^N))^*$. Then by Theorem 4.2, there exists $t=\{t_R\}\in c_w^p(\mathbb{R}^N)$ such that $\ell_1(s)=\sum_R s_R\bar{t}_R$ for all $s=\{s_R\}\in s_w^p(\mathbb{R}^N)$ and $\|t\|_{c_w^p(\mathbb{R}^N)}\lesssim\|\ell_1\|\lesssim\|\ell\|$. Again by Theorem 4.2, $\ell=\ell\circ\mathcal{T}\circ\mathcal{L}=\ell_1\circ\mathcal{L}$. Hence,

$$\ell(f)=\ell_1(\mathcal{L}(f))=\langle\mathcal{L}(f),t\rangle=\langle f,g\rangle,$$

where

$$g(x) = \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathscr{C}}^{j,k}} |R|^{\frac{1}{2}}\, t_R\, \psi_{j,k}(x_R - x).$$

This implies that $\ell = \ell_g$ and, by Theorem 4.2, $\|g\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)} \leq C\|t\|_{c^p_w(\mathbb{R}^N)} \leq C\|\ell_g\|$. The proof of Theorem 1.7 is concluded. ■

5 Boundedness of Compositions of Singular Integrals on $\mathrm{CMO}^p_{\mathscr{C},w}$

In this section, we give the proof of Theorem 1.8. As mentioned in Section 1, to show the boundedness of $T_1 \circ T_2$ on $\mathrm{CMO}^p_{\mathscr{C},w}$, we first need to define $T_1 \circ T_2$ on $\mathrm{CMO}^p_{\mathscr{C},w}$. To this end, we need the following weak density result.

Lemma 5.1 *Let $0 < p \leq 1$ and $w \in A^{\mathscr{C}}_\infty(\mathbb{R}^N)$. Then $L^2(\mathbb{R}^N)\cap\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$ is dense in $\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$ in the weak topology $\langle H^p_{\mathscr{C},w}(\mathbb{R}^N), \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)\rangle$. More precisely, for any $f \in \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$, there exists a sequence*

$$\{f_n\} \subseteq L^2(\mathbb{R}^N) \cap \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$$

such that $\|f_n\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)} \leq C\|f\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)}$ and for any $g \in H^p_{\mathscr{C},w}(\mathbb{R}^N)$,

$$\langle f_n, g\rangle \to \langle f, g\rangle, \quad \text{as } n \to \infty,$$

where the constant C is independent of n and f.

Proof Suppose $f \in \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$. Set

$$f_n(x) = \sum_{\substack{|j|\leq n\\ |k|\leq n}} \sum_{R\subseteq \mathbf{B}_n} 2^{j_1\vee k_1+\cdots+j_m\vee k_m} \psi_{j,k} * f(x_R)\psi_{j,k}(x - x_R),$$

where $\psi_{j,k}$ is the same as in Lemma 2.3 and $\mathbf{B}_n = \{x : |x_1| \leq n, \ldots, |x_m| \leq n\}$.

It is easy to see that $f_n \in L^2(\mathbb{R}^N)$. Applying the same proof as Theorem 1.6 implies that $\|f_n\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)} \leq C\|f\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)}$ and thus $f_n \in L^2(\mathbb{R}^N) \cap \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$. For any $g \in \mathcal{S}_\infty(\mathbb{R}^N)$, by the discrete Calderón reproducing formula in Lemma 2.3,

$$\begin{aligned}\langle f - f_n, g\rangle &= \Big\langle \sum_{\substack{|j|>n \text{ or } |k|>n\\ \text{or } R\subsetneq \mathbf{B}_n}} 2^{j_1\vee k_1+\cdots+j_m\vee k_m} \psi_{j,k} * f(x_R)\psi_{j,k}(\cdot - x_R), g\Big\rangle\\ &= \Big\langle f, \sum_{\substack{|j|>n \text{ or } |k|>n\\ \text{or } R\subsetneq \mathbf{B}_n}} 2^{j_1\vee k_1+\cdots+j_m\vee k_m} \psi_{j,k} * g(x_R)\psi_{j,k}(\cdot - x_R)\Big\rangle.\end{aligned}$$

By a result in [Wu], the function

$$\sum_{\substack{|j|>n \text{ or } |k|>n\\ \text{or } R\subsetneq \mathbf{B}_n}} 2^{j_1\vee k_1+\cdots+j_m\vee k_m} \psi_{j,k} * g(x_R)\psi_{j,k}(x - x_R)$$

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belongs to $H^p_{\mathscr{C},w}(\mathbb{R}^N)$ and its $H^p_{\mathscr{C},w}(\mathbb{R}^N)$ norm tends to 0 as n goes to ∞. Therefore, Theorem 1.7 yields that $\langle f - f_n, g\rangle$ tends to zero as n gets to ∞. This concludes the proof of Lemma 5.1 after a standard density argument (since $\mathcal{S}_\infty(\mathbb{R}^N)$ is dense in $H^p_{\mathscr{C},w}(\mathbb{R}^N)$). ■

Now let us show how the composition $T_1 \circ T_2$ acts on $\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$ functions. Given $f \in \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$, by Lemma 5.1, there is a sequence $\{f_n\} \subseteq L^2(\mathbb{R}^N) \cap \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$ such that $\|f_n\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)} \le C\|f\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)}$, and for any $g \in L^2(\mathbb{R}^N) \cap H^p_{\mathscr{C},w}(\mathbb{R}^N)$, $\langle f_n, g\rangle \to \langle f, g\rangle$ as $n \to \infty$. Thus, for $f \in \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$, we define

$$\langle T_1 \circ T_2(f), g\rangle = \lim_{n\to\infty} \langle T_1 \circ T_2(f_n), g\rangle, \text{ for any } g \in L^2(\mathbb{R}^N) \cap H^p_{\mathscr{C},w}(\mathbb{R}^N). \tag{5.1}$$

To see that the limit exists, write $\langle (T_1 \circ T_2)(f_l - f_n), g\rangle = \langle f_l - f_n, (T_1 \circ T_2)^*(g)\rangle$ since both $f_l - f_n$ and g belong to $L^2(\mathbb{R}^N)$, and $T_1 \circ T_2$ is bounded on $L^2(\mathbb{R}^N)$. By a result in [Wu], $(T_1 \circ T_2)^*$ is bounded on $H^p_{\mathscr{C},w}(\mathbb{R}^N)$, thus $(T_1 \circ T_2)^*(g) \in L^2(\mathbb{R}^N) \cap H^p_{\mathscr{C},w}(\mathbb{R}^N)$. Therefore, by Lemma 5.1, $\langle f_l - f_n, (T_1 \circ T_2)^*(g)\rangle$ tends to zero as $l, n \to \infty$. It is also easy to verify that the definition of $T_1 \circ T_2(f)$ is independent of the choice of the sequence f_n satisfying the conditions in Lemma 5.1.

To finish the proof of Theorem 1.8, we only need to show the boundedness of $T_1 \circ T_2$ on $L^2(\mathbb{R}^N) \cap \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$. For this purpose, we establish a discrete Calderón-type identity on $L^2(\mathbb{R}^N)$.

Let $\phi^{(1)}$ be a Schwartz function supported in the unit ball in $\mathbb{R}^N$ with

$$\int_{\mathbb{R}^N} \phi^{(1)}(x) x^{\alpha_1}\, dx = 0, \text{ for } 0 \le |\alpha_1| \le M_0,$$

where M_0 is a large positive integer which will be determined later, and

$$\sum_{j\in\mathbb{Z}} \widehat{\phi^{(1)}}(2^j \circ_1 \xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^N \backslash \{0\}$$

with $\phi^{(2)}$ satisfying similar conditions with $\circ_1$ replaced by $\circ_2$. For $j, k \in \mathbb{Z}$, let $\phi^{(1)}_j(x) = 2^{-jN_1}\phi^{(1)}(2^{-j} \circ_1 x)$, $\phi^{(2)}_k(x) = 2^{-kN_2}\phi^{(2)}(2^{-k} \circ_2 x)$, and $\phi_{j,k}(x) = \phi^{(1)}_j * \phi^{(2)}_k(x)$.

Lemma 5.2 *Suppose $0 < p \le 1$ and $w \in A^{\mathscr{C}}_\infty(\mathbb{R}^N)$. Let $\phi_{j,k}$ be defined as above with $M_0 \ge 10(N[q_w/(1 \wedge p) - 1] + 1)$. Then for any $f \in L^2(\mathbb{R}^N) \cap \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$, there exists $h \in L^2(\mathbb{R}^N) \cap \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$ such that for a sufficiently large $K \in \mathbb{N}$,*

$$f(x) = \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}^{j-K,k-K}_{\mathscr{C}}} |R|\phi_{j,k}(x - x_R)\phi_{j,k} * h(x_R),$$

where x_R denotes the minimal corner of R and the series converges in $L^2(\mathbb{R}^N)$. Moreover,

$$\|f\|_{L^2(\mathbb{R}^N)} \sim \|h\|_{L^2(\mathbb{R}^N)}. \tag{5.2}$$

and

$$\|f\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)} \sim \|h\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)}. \tag{5.3}$$

Proof Applying Coifman's decomposition of the identity operator, we have

$$\begin{aligned} f(x) &= \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathscr{C}}^{j-K,k-K}} |R|\phi_{j,k} * f(x_R)\phi_{j,k}(x-x_R) + S_K(f)(x) \\ &\equiv T_K(f)(x) + S_K(f)(x), \end{aligned}$$

where

$$\begin{aligned} &S_K(f)(x) \\ &\quad= \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathscr{C}}^{j-K,k-K}} \int_R \phi_{j,k}(x-x')(\phi_{j,k}*f)(x') - \phi_{j,k}(x-x_R)(\phi_{j,k}*f)(x_R)dx' \\ &\quad= \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathscr{C}}^{j-K,k-K}} \int_R \big[\phi_{j,k}(x-x') - \phi_{j,k}(x-x_R)\big](\phi_{j,k}*f)(x')dx' \\ &\qquad+ \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathscr{C}}^{j-K,k-K}} \int_R \phi_{j,k}(x-x')\big[(\phi_{j,k}*f)(x') - (\phi_{j,k}*f)(x_R)\big]dx' \\ &\quad\equiv S_K^1(f)(x) + S_K^2(f)(x). \end{aligned}$$

Now we claim that for $l = 1, 2$,

$$\|R_K^l(f)\|_{L^2(\mathbb{R}^N)} \leq C2^{-K}\|f\|_{L^2(\mathbb{R}^N)} \tag{5.4}$$

and

$$\|R_K^l(f)\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)} \leq C2^{-K}\|f\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)}, \tag{5.5}$$

where C is a constant independent of f and K.

Assume the claim for the moment, then, by choosing sufficiently large K, $T_K^{-1} = \sum_{n=0}^{\infty}(S_K)^n$ is bounded on both $L^2(\mathbb{R}^N)$ and $\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$. For any $f \in L^2(\mathbb{R}^N) \cap \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$, set $h = T_K^{-1}(f)$, then the estimates in (5.4) and (5.5) imply (5.2) and (5.3), respectively. Moreover,

$$f(x) = T_K(T_K^{-1}(f))(x) = \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathscr{C}}^{j-K,k-K}} |R|\phi_{j,k}(x-x_R)(\phi_{j,k}*h)(x_R),$$

where the series converges in $L^2(\mathbb{R}^N)$.

Thus, to finish the proof of Theorem 5.2, it suffices to verify the claim. We only prove (5.5), since (5.4) has been established in [Wu]. Since the proofs for S_K^1 and S_K^2 are similar, we only give the proof for S_K^1. Roughly speaking, the proof is similar

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to Theorem 1.6. To see this, let $f \in L^2(\mathbb{R}^N) \cap \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$. Applying Calderón's discrete reproducing formula in Lemma 2.3 yields

(5.6)

$$\begin{aligned}
&\psi_{j',k'} * S^1_K(f)(x)\\
&\quad= \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}^{j-K,k-K}_{\mathscr{C}}} \int_R \psi_{j',k'} * [\phi_{j,k}(\cdot - x') - \phi_{j,k}(\cdot - x_R)](x)(\phi_{j,k} * f)(x')dx'\\
&\quad= \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}^{j-K,k-K}_{\mathscr{C}}} \int_R \psi_{j',k'} * [\phi_{j,k}(\cdot - x') - \phi_{j,k}(\cdot - x_R)](x)\\
&\quad\times \Big(\sum_{j'',k''\in\mathbb{Z}} \sum_{R''\in\mathcal{R}^{j-K,k-K}_{\mathscr{C}}} |R''| \cdot \psi_{j'',k''} * f(x_{R''})\phi_{j,k} * \psi_{j'',k''}(x' - x_{R''}) \Big) dx',
\end{aligned}$$

where $x_{R''} = (x_{I''_1}, \dots, x_{I''_m})$ is the minimal corner of R''.

Set $\widetilde{\phi}_{j,k}(u) = \phi_{j,k}(u - x') - \phi_{j,k}(u - x_R)$. Applying Lemma 2.4 (particularly Remark 2.5) with $M = N[q_w/(1 \wedge p) - 1] + 1$ and $L = 10M$, we obtain that for some constant C (depending only on M, ψ and ϕ, but independent of K),

$$\begin{aligned}
|\psi_{j',k'} * \widetilde{\phi}_{j,k}(x)| &\le C2^{-K}2^{-10M\|\mathbf{j}-\mathbf{j}'\|}2^{-10M\|\mathbf{k}-\mathbf{k}'\|} \prod_{i=1}^m \frac{2^{(j_i\vee j'_i\vee k_i\vee k'_i)M}}{(2^{j_i\vee j'_i\vee k_i\vee k'_i} + |x_i - x'_i|)^{n_i+M}}\\
&\le C2^{-K}2^{-3M\|\mathbf{j}-\mathbf{j}'\|}2^{-3M\|\mathbf{k}-\mathbf{k}'\|} \prod_{i=1}^m \frac{2^{(j'_i\vee k'_i)M}}{(2^{j'_i\vee k'_i} + |x_i - x'_i|)^{n_i+M}},
\end{aligned}$$

where the last inequality follows from $2^{j_i\vee j'_i\vee k_i\vee k'_i} \le 2^{\|\mathbf{j}-\mathbf{j}'\|}2^{\|\mathbf{k}-\mathbf{k}'\|}2^{j'_i\vee k'_i}$. Similarly,

$$\begin{aligned}
&|\phi_{j,k} * \psi_{j'',k''}(x' - x''_R)| \le\\
&\qquad C2^{-K}2^{-3M\|\mathbf{j}-\mathbf{j}''\|}2^{-3M\|\mathbf{k}-\mathbf{k}''\|} \prod_{i=1}^m \frac{2^{(j''_i\vee k''_i)M}}{(2^{j''_i\vee k''_i} + |x'_i - x_{I''_i}|)^{n_i+M}}.
\end{aligned}$$

Inserting these estimates into the last term in (5.6) yields

$$\begin{aligned}
&|\psi_{j',k'} * S^1_K(f)(x)|\\
&\quad\lesssim \sum_{j'',k''\in\mathbb{Z}} \sum_{R''\in\mathcal{R}^{j-K,k-K}_{\mathscr{C}}} |R''||\psi_{j'',k''} * f(x_{R''})|\\
&\qquad\times \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}^{j-K,k-K}_{\mathscr{C}}} \int_R 2^{-K} \prod_{i=1}^m 2^{-\|\mathbf{j}-\mathbf{j}'\|3M}2^{-\|\mathbf{k}-\mathbf{k}'\|3M} \frac{2^{(j'_i\vee k'_i)M}}{(2^{j'_i\vee k'_i} + |x_i - x'_i|)^{n_i+M}}\\
&\qquad\times 2^{-\|\mathbf{j}-\mathbf{j}''\|3M}2^{-\|\mathbf{k}-\mathbf{k}''\|3M} \prod_{i=1}^m \frac{2^{(j''_i\vee k''_i)M}}{(2^{j''_i\vee k''_i} + |x'_i - x_{I''_i}|)^{n_i+M}} dx'
\end{aligned}$$

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$$\lesssim 2^{-K} \sum_{j'',k''\in\mathbb{Z}} \sum_{R''\in\mathcal{R}_{\mathscr{C}}^{j-K,k-K}} 2^{-\|\mathbf{j}'-\mathbf{j}''\|3M} 2^{-\|\mathbf{k}'-\mathbf{k}''\|3M} |R''|$$

$$\times \Big\{ \prod_{i=1}^{m} \frac{2^{(j_k'\vee j_k''\vee k_i'\vee k_i'')M}}{(2^{j_i'\vee j_i''\vee k_i'\vee k_i''} + |x_i - x_{I_i''}|)^{n_i+M}} \Big\} |\psi_{j'',k''} * f(x_{R''})|.$$

Taking $x = x_{R'}$, adding up all the terms, and multiplying by $|R'|^2/w(R')$ over $j', k' \in \mathbb{Z}$, $R' \in \mathcal{R}_{\mathscr{C}}^{j'-K,k'-K}$, $R' \subseteq \Omega'$ and applying Lemma 2.2, we obtain

$$\sup_{\Omega'} \Big\{ \frac{1}{[w(\Omega')]^{\frac{2}{p}-1}} \sum_{j',k'\in\mathbb{Z}} \sum_{\substack{R'\in\mathcal{R}_{\mathscr{C}}^{j'-K,k'-K}\\ R\subseteq\Omega'}} |R'|^2 w(R')^{-1} |\psi_{j',k'} * S_K^1(f)(x)|^2 \Big\} \lesssim$$

$$2^{-K} \sup_{\Omega'} \Big\{ \frac{1}{[w(\Omega')]^{\frac{2}{p}-1}} \sum_{j',k'\in\mathbb{Z}} \sum_{\substack{R'\in\mathcal{R}_{\mathscr{C}}^{j'-K,k'-K},\\ R'\subseteq\Omega'}} \sum_{j'',k''\in\mathbb{Z}}$$

$$\sum_{R''\in\mathcal{R}_{\mathscr{C}}^{j''-K,k''-K}} \frac{|R''|^2}{w(R'')} r(R', R'') P(R', R'') T_{R''} \Big\},$$

where

$$r(R', R'') = 2^{-L(\|\mathbf{j}'-\mathbf{j}''\|+\|\mathbf{k}'-\mathbf{k}''\|)}$$

and

$$P(R', R'') = \prod_{i=1}^{m} \frac{2^{(j_k'\vee j_k''\vee k_i'\vee k_i'')M}}{(2^{j_i'\vee j_i''\vee k_i'\vee k_i''} + |x_i' - x_{I_i''}|)^{n_i+M}}.$$

Repeating the same proof as in Theorem 1.6, we can get (5.5). Thus the claim is concluded, and Theorem 5.2 follows. ∎

We point out that in the discrete Calderón reproducing formula of Lemma 2.3 the series converges in L^2, $\mathcal{S}_\infty$, and $\mathcal{S}'/\mathcal{P}$, while in the above Calderón-type identity, the series only converges in L^2. However, the $\phi_{j,k}$ in Lemma 5.2 have compact supports, but $\psi_{j,k}$ in Lemma 2.3 do not. The fact that the $\phi_{j,k}$ have compact supports enables us to derive the key estimates of the kernels (see Lemma 5.4).

Repeating the same argument as in Lemma 5.2, we have the following corollary.

Corollary 5.3 *Let $0 < p \le 1$ and $w \in A_\infty^{\mathscr{C}}(\mathbb{R}^N)$. Suppose that $\phi_{j,k}$ satisfy the same conditions as in Lemma 5.2. Then for a large integer K as in Lemma 5.2 and $f \in L^2(\mathbb{R}^N) \cap \mathrm{CMO}_{\mathscr{C},w}^p(\mathbb{R}^N)$,*

$$\|f\|_{\mathrm{CMO}_{\mathscr{C},w}^p(\mathbb{R}^N)} \approx \sup_{\Omega} \Big\{ \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}_{\mathscr{C}}^{j-K,k-K}\\ R\subseteq\Omega}} |\phi_{j,k} * h(x_R)|^2 |R|^2 [w(R)]^{-1} \Big\}^{\frac{1}{2}},$$

where x_R denotes the minimal corner of R and the implicit constants are independent of f.

The following lemma provides key estimates for the kernels.

Lemma 5.4 *Let $\phi_j^{(1)}, \phi_{j'}^{(1)}, \phi_k^{(2)}, \phi_{k'}^{(2)}$ satisfy the same conditions as in Lemma 5.2 with $M_0 \geq 10M$. Then*

$$|\phi_j^{(1)} * \mathcal{K}_1 * \phi_{j'}^{(1)}(x)| \leq C2^{-10M\|\mathbf{j}-\mathbf{j}'\|} \prod_{i=1}^{m} \frac{2^{(j_i \vee j_i')M_1^i}}{(2^{j_i \vee j_i'} + |x_i|)^{n_i + M_1^i}} \tag{5.7}$$

and

$$|\phi_k^{(2)} * \mathcal{K}_2 * \phi_{k'}^{(2)}(x)| \leq C2^{-10M\|\mathbf{k}-\mathbf{k}'\|} \prod_{i=1}^{m} \frac{2^{(k_i \vee k_i')M_2^i}}{(2^{k_i \vee k_i'} + |x_i|)^{n_i + M_2^i}}, \tag{5.8}$$

where $M_k^i = Mn_i/N_k$ for $i = 1, \dots, m$ and $k = 1, 2$.

Proof We borrow an idea from [FS1]. We only show (5.7), as (5.8) can be proved similarly. By the classical almost orthogonality estimate,

$$\phi_j^{(1)} * \phi_{j'}^{(1)}(u) = C2^{-10M\|\mathbf{j}-\mathbf{j}'\|} \varphi_{j \vee j'}(u),$$

where $\varphi_{j\vee j'}(u) = 2^{-(j\vee j')N_1} \varphi(2^{-j\vee j'} \circ_1 u)$ and φ is a Schwartz function in $\mathbb{R}^N$ supported in $\{|u|_1 \leq 2\}$ with the same moment conditions as $\phi^{(1)}$. If we can show

$$|\mathcal{K}_1 * \varphi(x)| \lesssim \frac{1}{(1 + |x|_1)^{N_1 + M}}, \tag{5.9}$$

then a dilation argument would yield

$$\begin{aligned} |\phi_j^{(1)} * \mathcal{K}_1 * \phi_{j'}^{(1)}(x)| &\lesssim 2^{-10M\|\mathbf{j}-\mathbf{j}'\|} 2^{-(j\vee j')N_1} \frac{1}{(1 + 2^{-j\vee j'}|x|_1)^{N_1+M}} \\ &\leq 2^{-10M\|\mathbf{j}-\mathbf{j}'\|} 2^{-(j\vee j')N_1} \prod_{i=1}^{m} \frac{1}{(1 + 2^{-j_i \vee j_i'}|x_i|)^{n_i + M_1^i}} \\ &= 2^{-10M\|\mathbf{j}-\mathbf{j}'\|} \prod_{i=1}^{m} \frac{2^{(j_i \vee j_i')M_1^i}}{(2^{j_i \vee j_i'} + |x_i|)^{n_i + M_1^i}}, \end{aligned}$$

which gives (5.7). Thus, to finish the proof it suffices to verify (5.9).

We consider two cases. If $|x|_1 \geq 4$, then applying the cancellation condition of φ and smoothness condition of $\mathcal{K}_1$ (via the stratified Taylor inequality in [FS2, (1.44)]),

$$\begin{aligned} |\mathcal{K}_1 * \varphi(x)| &= \left| \int [\mathcal{K}_1(x-u) - P_M(x)]\varphi(u)du \right| \lesssim \int \frac{|u|_1^M}{|x|_1^{N_1+M}} |\varphi(u)| du \\ &\lesssim \frac{1}{(1 + |x|_1)^{N_1+M}}, \end{aligned}$$

where P_M denote the $(M-1)$-th order Taylor's polynomial of $\mathcal{K}_1$ at x.

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If $|x|_1 \le 4$, then write

$$|\mathcal{K}_1 * \varphi(x)| = \left| \int_{|u|_1 \le 6} \mathcal{K}_1(u)[\varphi(x-u) - \varphi(x)]du \right| + |\varphi(x)| \cdot \left| \int_{|u|_1 \le 6} \mathcal{K}_1(u)du \right|.$$

The estimate for the first term can be derived by the use of the size condition of $\mathcal{K}_1$ and the smoothness condition of φ. The second term can be handled by using the cancellation condition of $\mathcal{K}_1$. This concludes the proof of (5.9), and Lemma 5.4 follows. ■

Remark 5.5 Let $\widetilde{M} = \min_{\substack{i=1,\dots,m \\ k=1,2}} \{M_k^i\}$. Then by Lemma 5.4,

$$|\phi_j^{(1)} * \mathcal{K}_1 * \phi_{j'}^{(1)}(x)| \le C 2^{-10\widetilde{M}\|\mathbf{j}-\mathbf{j}'\|} \prod_{i=1}^m \frac{2^{(j_i \vee j_i')\widetilde{M}}}{(2^{j_i \vee j_i'} + |x_i|)^{n_i + \widetilde{M}}}$$

and

$$|\phi_k^{(2)} * \mathcal{K}_2 * \phi_{k'}^{(2)}(x)| \le C 2^{-10\widetilde{M}\|\mathbf{k}-\mathbf{k}'\|} \prod_{i=1}^m \frac{2^{(k_i \vee k_i')\widetilde{M}}}{(2^{k_i \vee k_i'} + |x_i|)^{n_i + \widetilde{M}}}.$$

Moreover, the above inequalities indeed hold for arbitrary $\widetilde{M} > 0$, since M_0 can be chosen arbitrarily large.

We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8 We first show that for $f \in L^2(\mathbb{R}^N) \cap \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$,

$$\|T_1 \circ T_2(f)\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)} \le C\|f\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)},$$

where the constant C is independent of f. In view of Corollary 5.3, this would follow if we show that for any open set Ω,

$$\text{(5.10)} \quad \left\{ \frac{1}{[w(\Omega)]^{\frac{2}{p}-1}} \sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}_{\mathscr{C}}^{j-K,k-K} \\ R\subseteq\Omega}} |\phi_{j,k} * (T_1 \circ T_2)(f)(x_R)|^2 \frac{|R|^2}{w(R)} \right\}^{\frac{1}{2}} \lesssim \|f\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)},$$

where $\phi_{j,k}$ and K are the same as in Theorem 5.2, and the constant C is independent of f.

By the discrete Calderón-type identity given in Theorem 5.2, we write

$$\sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}_{\mathscr{C}}^{j-K,k-K}\\ R\subseteq\Omega}} |\phi_{j,k} * (T_1\circ T_2)(f)(x_R)|^2|R|^2[w(R)]^{-1}$$
$$= \sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}_{\mathscr{C}}^{j-K,k-K}\\ R\subseteq\Omega}} \Big| \sum_{j',k'\in\mathbb{Z}} \sum_{R'\in\mathcal{R}_{\mathscr{C}}^{j'-K,k'-K}} t_{R'}|R'|^{\frac12}$$
$$\times\, \phi_{j,k} * (\mathcal{K}_1 * \mathcal{K}_2) * (\phi_{j',k'})(x_R - x_{R'})\Big|^2 |R|^2[w(R)]^{-1},$$

where $t_{R'} = \phi_{j',k'} * h(x_{R'})|R'|^{\frac12}$ and $\|h\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)} \approx \|f\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)}$. Noticing that by Lemma 5.4 (particularly Remark 5.5), $\phi_{j,k} * (\mathcal{K}_1 * \mathcal{K}_2) * \phi_{j',k'}$ satisfy the same almost orthogonality estimates as $\psi_{j,k} * \varphi_{j',k'}$ in Lemma 2.4. Repeating the same argument as in the proof of Theorem 1.6, we conclude that for $f \in L^2(\mathbb{R}^N) \cap \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$,

$$\Big\{ \frac{1}{[w(\Omega)]^{\frac2p-1}} \sum_{j,k\in\mathbb{Z}} \sum_{\substack{R\in\mathcal{R}_{\mathscr{C}}^{j-K,k-K}\\ R\subseteq\Omega}} |\phi_{j,k} * (T_1\circ T_2)(f)(x_R)|^2|R|^2[w(R)]^{-1} \Big\}^{\frac12}$$
$$\le C\|h\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)} \le C\|f\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)},$$

which is just (5.10).

For $f \in \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$, let $\{f_n\} \subseteq L^2(\mathbb{R}^N) \cap \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$ be the sequence as in (5.1). By the definition of $T_1\circ T_2(f)$ and the boundedness of $T_1\circ T_2$ on $L^2(\mathbb{R}^N) \cap \mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)$,

$$\|T_1\circ T_2(f)\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)} \le \liminf_{n\to\infty} \|T_1\circ T_2(f_n)\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)}$$
$$\le C \liminf_{n\to\infty} \|f_n\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)} \le C\|f\|_{\mathrm{CMO}^p_{\mathscr{C},w}(\mathbb{R}^N)}.$$

This concludes the proof of Theorem 1.8. ■

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