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LINEAR FUNCTIONALS ON SOME WEIGHTED BERGMAN SPACES

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The weighted Bergman space $A^{p,\alpha}$, $0 < p < 1$, $\alpha > -1$ of analytic functions on the unit disc Δ in $\mathbb{C}$ is an F-space. We determine the dual of $A^{p,\alpha}$ explicitly.

INTRODUCTION

Let Δ be the unit disc in $\mathbb{C}$. For a function analytic in Δ, we write

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{1/p}, \quad 0 < p < \infty$$

$$M_\infty(r, f) = \max_{0 \leqslant \theta < 2\pi} \left|f\left(re^{i\theta}\right)\right|.$$

It is well known that $M_p(r, f)$ $(0 < p \leqslant \infty)$ is an increasing function of r $(0 \leqslant r < 1)$.

The Hardy space H^p $(0 < p \leqslant \infty)$ is the class of analytic functions f in Δ and

$$\|f\|_p = \sup_{0 \leqslant r < 1} M_p(r, f) < \infty.$$

The weighted Bergman space $A^{p,\alpha}$, $p > 0$, $\alpha > -1$, is the class of analytic functions in Δ for which

$$\|f\|_{p,\alpha} = \left(\frac{\alpha+1}{\pi}\iint_\Delta (1-|z|)^\alpha |f(z)|^p \, dx dy\right)^{1/p} < \infty.$$

$A^{p,\alpha}$ $1 \leqslant p < \infty$, $\alpha > -1$ is known to be a Banach space and a Fréchet space with the metric

$$\|f\|_{p,\alpha}^p = \frac{\alpha+1}{\pi}\iint_\Delta (1-|z|)^\alpha |f(z)|^p \, dx dy$$

for $0 < p < 1$. Although $A^{p,\alpha}$, $0 < p < 1$, $\alpha > -1$ is not locally convex, it nevertheless has enough continuous linear functionals to separate points [8]. It is clear that $A^{p,0} = A^p$ where A^p is the usual Bergman space; also $H^p \subset A^{p(\alpha+2),\alpha}$ [6].

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Duren, Romberg and Shields [2, Theorem 1] have studied linear functionals on H^p over the unit disc Δ for $0 < p < 1$. Shapiro computed the dual space $A^{p,\alpha}$ $(0 < p < 1,\ \alpha > -1)$ by determining the Mackey topology of $A^{p,\alpha}$ [8]. Motivated by their work we shall prove a main theorem in Section 3 which gives the explicit dual of $A^{p,\alpha}$ spaces with $0 < p < 1$ and $\alpha > -1$.

In Section 4 we prove a theorem which says that $(A^{p,\alpha})^*$ is topologically equivalent to a certain Banach space $\Lambda_\gamma^{m-2}(\Lambda_*^{m-2})$.

Throughout this paper C denotes a positive constant, not necessarily the same at each occurrence.

2. Preliminaries

Let $F(z)$ be analytic in Δ. Then $F(z)$ is said to belong to the Lipschitz space Λ_β if

$$\sup_{|t-s|<h} \left|F(e^{it}) - F(e^{is})\right| = O(h^\beta) \quad (0 \leqslant \beta < 1).$$

A continuous function $F(z)$ is said to belong to the class Λ_* if

$$|F(t+h) - 2F(t) + F(t-h)| = O(h)$$

uniformly in t. If $\beta > 0$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in Δ, the fractional derivative of order β is

$$f^{[\beta]}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\beta)}{n!} a_n z^n,$$

the fractional integral of order β is

$$f_{[\beta]}(z) = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+1+\beta)} a_n z^n,$$

and $f^{[\beta]}$, $f_{[\beta]}$ are analytic in Δ [2].

Let $A(\Delta)$ denote the class of analytic functions in Δ and continuous on $\overline{\Delta}$. For analytic functions $f(z)$ we write $f \in \Lambda_\beta(\Lambda_*)$ to indicate that $f \in A$ and the boundary $f(e^{i\theta})$ is in $\Lambda_\beta(\Lambda_*)$.

We need the following theorems:

Theorem A. [2]. *Let f be analytic in Δ. Then $f \in \Lambda_\beta$ $(0 < \beta \leqslant 1)$ if and only if*

$$f'(z) = O\left(\frac{1}{(1-r)^{1-\beta}}\right).$$

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THEOREM B. [2]. *Let* f *be analytic in* Δ. *Then* $f \in \Lambda_*$ *if and only if*

$$f''(z) = O\left(\frac{1}{1-r}\right).$$

THEOREM C. *Let* $f \in A^{p,\alpha}$, $0 < p < q < \infty$. *Then*

$$\int_0^1 (1-\rho)^{(\alpha+2)q/p-2} M_q^q(\rho, f)\,d\rho < \infty.$$

The proof is a consequence of the following inequality

$$|f(z)| \leqslant \frac{C\,\|f\|_{p,\alpha}}{\left(1-|z|^2\right)^{(\alpha+2)/p}}, \tag{2.1}$$

which holds for $f \in A^{p,\alpha}$ [8].

By using Theorem C and (2.1) we have the following theorem:

THEOREM D. *Let* $f \in A^{p,\alpha}$, $0 < p < q < \infty$. *Then*

$$\int_0^1 (1-\rho)^{(\alpha+2)q/p-3} J(\rho, f)\,d\rho < \infty,$$

where $J(\rho, f) = \int_0^\rho M_q^q(r, f)\,dr$.

THEOREM E. [4]. *Let* f *be analytic in* Δ *and* $0 < q \leqslant 1$. *Then*

$$\lim_{s \to 1} J\left(s, f_{[\beta]}\right) \leqslant C \int_0^1 (1-\rho)^{q\beta-1} J(\rho, f)\,d\rho.$$

THEOREM F. *Let* $f \in A^{p,\alpha}$, $0 < p < q \leqslant 1$ *and* $0 < \beta < (\alpha+2)/p$. *Then* $f_{[\beta]} \in A^q$ *where* $q = 2p/((\alpha+2)-\beta p)$.

The proof follows from Theorems E and D.

THEOREM G. [2]. *If* f *is analytic in* Δ *and* $f'(z) = O(1/(1-r))$, *then*

$$f^{[1/2]}(z) = O\left(\frac{1}{(1-r)^{1/2}}\right).$$

THEOREM H. [9]. *Suppose* $\alpha > -1$ *and* $\gamma > 1+\alpha$; *then for* $0 < r$, $\rho < 1$,

$$\int_0^1 \frac{(1-r)^\alpha}{(1-\rho r)^\gamma}\,dr = O\left(\frac{1}{1-\rho}\right)^{\alpha-\gamma+1}.$$

THEOREM I. [10, p.128]. *Let* $f(z) = \sum_{k=0}^{\infty} a_n z^n \in A^p$, $0 < p \leqslant 1$ *and* $\alpha > -1$. *Then* $|a_n| \leqslant C n^{[(\alpha+2)/p]-1}$.

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3. Representation of bounded linear functionals

Let T be a linear bounded functional on $A^{p,\alpha}$ $(0 < p < 1, \alpha > -1)$. Then $T \in (A^{p,\alpha})^*$ if and only if

$$\|T\| = \sup_{\|f\|_{p,\alpha} < 1} |T(f)| < \infty.$$

It follows that

$$|T(f)| \leqslant \|T\| \, \|f\|_{p,\alpha}$$

for all $f \in A^{p,\alpha}$. Here $(A^{p,\alpha})^*$ is a Banach space.

Theorem 1 gives a representation for bounded linear functionals T on $A^{p,\alpha}$ $(0 < p < 1, \alpha > -1)$.

Theorem 1. *Let $T \in (A^{p,\alpha})^*$, $0 < p < 1$. Then there is a unique function $g \in A$ such that*

$$T(f) = \frac{1}{2\pi} \iint f(z) g(\overline{z}) dx dy. \tag{3.1}$$

If $(\alpha + 2)/(m + 1) < p < (\alpha + 2)/m$, $m = 2, 3, \ldots$, then

$$g^{(m-2)} \in \Lambda_{(\alpha+2)/p - m}$$

Conversely, for any g with $g^{(m-2)} \in \Lambda_{(\alpha+2)/p-m}$ the double integral (3.1) exists for all $f \in A^{p,\alpha}$ and defines a functional $T \in (A^{p,\alpha})^*$.

If $p = (\alpha + 2)/(n + 1)$, then $g^{(m-2)} \in \Lambda_*$.

Conversely, for any g with $g^{(m-2)} \in \Lambda_*$, the double integral (3.1) exists and represents a bounded linear functional on $A^{p,\alpha}$.

Theorem 1 of [**2**] for $0 < p < 1/2$ can be regarded as the limiting case of $\alpha = -1$ of our results and the question arises whether it holds for the case $1/2 < p < 1$. Also, this result generalises the announced result of Burchaev and Ryabykh [**1**] for A^p.

Proof: Suppose that $T \in (A^{p,\alpha})^*$ and $Tz^k = b_k/(2(k+1))$; then $\left|Tz^k\right| \leqslant \|T\| \left\|z^k\right\|_{p,\alpha}$. But

$$\left\|z^k\right\|_{p,\alpha} = \left(\frac{\alpha+1}{\pi} \int_0^1 \int_0^{2\pi} (1-r)^\alpha r^{pk+1} dr d\theta\right)^{1/p} \leqslant Ck^{-(1+\alpha)/p}$$

[**5**], so $|b_k| \leqslant C \|T\| / k^{(1+\alpha)/p-1}$ and hence $g(z) = \sum\limits_{k=0}^{\infty} b_k z^k$ is analytic in Δ. For each $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k \in A^{p,\alpha}$ and for fixed $\rho \in [0, 1)$ let $f_\rho(z) = f\left(\rho^2 z\right)$. Because the power series of f_ρ converges uniformly on Δ, and because T is continuous, we have

$$T(f_\rho(z)) = \sum_{k=0}^{\infty} a_k \frac{b_k}{2(k+1)} \rho^{2k}.$$

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As $\rho \to 1$, $f_\rho \to f$ in the $A^{p,\alpha}$ metric [8, p.197]

$$T(f) = \lim_{\rho\to 1} \frac{1}{2\pi} \iint_{|z|<\rho} \sum_{k=0}^{\infty} a_k z^k \sum_{k=0}^{\infty} b_k \bar{z}^k dx dy$$
$$= \frac{1}{2\pi} \iint_{|z|<1} f(z) g(\bar{z}) dx dy.$$

For fixed $\rho \in \Delta$, let $f_\xi(z) = 2/(1-\xi z)^2 = \sum_{n=0}^{\infty}(2n+2)z^n\xi^n$. Then

$$(3.2) \qquad |g(\xi)| = |T(f)| \leqslant \|T\| \left\| \frac{2}{(1-\xi z)^2} \right\|_{p,\alpha}$$

and hence $g \in H^\infty(\Delta)$; also $g \in A$, since $\lim_{\xi\to 1} g(\xi) = \lim_{\xi\to 1} T(f(z))$, and hence $\lim_{\xi\to 1} g(\xi) = T(f_1)$.

If $(\alpha+2)/(m+1) < p < (\alpha+2)/m$, $m = 2, 3, \ldots$, let

$$F(z) = \frac{d^m}{d\xi^m}\left(\frac{\xi}{1-\xi z}\right), \quad |\xi| < 1.$$

By a calculation, since $F \in A^{p,\alpha}$ we get

$$T(F) = \frac{1}{2} g^{(m-1)}(\xi).$$

It now follows from Theorem H that

$$(3.3) \qquad \left|g^{m-1}(\xi)\right| \leqslant 2\,\|T\|\,\|F\|_{p,\alpha} = O(1-|\xi|)^{(\alpha+2)/p-m-1},$$

so that $g^{(m-2)} \in \Lambda_\gamma$ where $\gamma = (\gamma+2)/p - m$, by Theorem A and $g \in A(\Delta)$.

If $p = (\alpha+2)/(m+1)$, let $F(z) = d^{m+1}/d\xi^{m+1}(\xi/(1-\xi z))$. By a similar argument one can show that

$$\left|g^{(m)}(\xi)\right| = O\left((1-|\xi|)^{-1}\right)$$

and $g^{(m-2)} \in \Lambda_*$ by Theorem B and $g \in A(\Delta)$.

To prove the converse we shall first show that if $g(z) = \sum_{k=0}^{\infty} b_k z^k \in \Lambda_\gamma$ where $\gamma = (\alpha+2)/p - m$, then for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A^{p,\alpha}$, $T(f)$ as defined in (3.1) exists.

If $(\alpha+2)/(m+1) < p < (\alpha+2)/m$, let $\psi(\rho^2) = \sum_{k=0}^{\infty} a_k(b_k/(2k+2))\rho^{2k}$. It is to be shown that $\psi(\rho^2)$ has a limit as $\rho \to 1$. We shall prove the existence of the limit by showing that

$$\int_0^1 \left|(\psi(\rho^2)\rho^2)'\right| d\rho < \infty. \tag{3.4}$$

Set $h(z) = z^{m-2}g(z)$; then

$$\begin{aligned}\frac{1}{\pi}\int_0^{2\pi}\int_0^{\rho} \bar{z} f_{[m-2]}(re^{i\theta})h^{m-1}(re^{-i\theta})\,dr\,d\theta &= \sum_{k=1}^{\infty} a_k b_k \rho^{2k+1} \\ &= \left(\sum_{k=0}^{\infty} \frac{a_k b_k \rho^{2k+2}}{2k+2}\right)' + a_0 b_0 \rho,\end{aligned}$$

so

$$(\psi(\rho^2)\rho^2)' = \int_0^{2\pi}\int_0^{\rho} e^{-i\theta} f_{[m-2]}(re^{i\theta})h^{(m-1)}(re^{-i\theta})\,dr\,d\theta - a_0 b_0 \rho. \tag{3.5}$$

Using the assumption that $g^{(m-2)} \in \Lambda_\gamma$ gives

$$\left|h^{(m-1)}(re^{i\theta})\right| \leqslant \frac{C}{(1-r)^{1-(\alpha+2)/p+m}}$$

by Theorem A, consequently (3.5) gives

$$\left|(\psi(\rho^2)\rho^2)'\right| \leqslant C\int_0^{\rho} (1-r)^{(\alpha+2)/p-m-1} \int_0^{2\pi} \left|f_{[m-2]}(re^{i\theta})\right| d\theta\,dr + |a_0 b_0|.$$

Hence by using Theorem F and D we have (3.4).

Finally, let $p = (\alpha+2)/(m+1)$ and $g^{(m-2)} \in \Lambda_*$; then (3.5) can be written in the form

$$(\psi(\rho^2)\rho^2)' = 2\int_0^{\rho}\int_0^{2\pi} G(re^{i\theta})H(re^{-i\theta})\,d\theta\,dr - a_0 b_0 \rho, \tag{3.6}$$

where $H(re^{i\theta}) = zh^{(m-1)}(re^{i\theta})$ and $G(re^{i\theta}) = f_{[m-2]}(re^{i\theta})$. By Theorem F, $G \in A^{2/3}$. Set $G(z) = \sum_{k=0}^{\infty} A_k z^k$; then by Theorem F again $G_{[1/2]} \in A^{4/5}$, so by Theorem D

$$\int_0^1 (1-\rho)^{-(1/2)} J(\rho, G_{[1/2]})\,d\rho < \infty. \tag{3.7}$$

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Now, since $g^{(m-2)} \in \Lambda_*$, we have $|H^{[1/2]}(re^{i\theta})| = O(1/(1-r))^{1/2}$, by Theorem B and Theorem G. Equation (3.6) can be written in the form

$$(\psi(\rho^2)\rho^2)' = 2\int_0^{\rho}\int_0^{2\pi} G_{[1/2]}(re^{i\theta})H^{[1/2]}(re^{i\theta})\,d\theta dr - a_0 b_0 \rho;$$

consequently

$$\left|(\psi(\rho^2)\rho^2)'\right| \leqslant C\int_0^{\rho}\int_0^{2\pi} \frac{1}{(1-r)^{1/2}}\left|G_{[1/2]}(re^{i\theta})\right| d\theta dr + |a_0 b_0|;$$

hence

$$(\psi(\rho^2)\rho^2)' \leqslant C(1-\rho)^{-(1/2)}\int_0^{\rho}\int_0^{2\pi} \left|G_{[1/2]}(re^{i\theta})\right| d\theta dr + |a_0 b_0|$$

and (3.7) gives (3.4).

To complete the proof, we need to show that if $g(z) = \sum_{k=0}^{\infty} b_k z^k$ is any analytic function for which $T(f)$ as defined in (3.1) exists for every $f(z) = \sum_{k=0}^{\infty} a_k z^k \in A^{p,\alpha}$, then $T \in (A^{p,\alpha})^*$. For fixed $\rho \in [0, 1)$, let $T(f) = \sum_{k=0}^{\infty}(a_k b_k)/(2k+2)\rho^{2k}$; $T_\rho(f)$ is a linear functional on $A^{p,\alpha}$. Also T_ρ is bounded for each ρ in $[0, 1)$ by Theorem 1. By hypothesis $\lim_{\rho\to 1} T_\rho(f)$ exists for each fixed $f \in A^{p,\alpha}$. Call this limit $T(f)$. By the uniform boundedness principle which holds for $A^{p,\alpha}$ [7, p.45], $\sup_{0\leqslant\rho<1} \|T_\rho\| = C < \infty$. Thus $|T_\rho(f)| \leqslant C\|f\|_{p,\alpha}$ and by the continuity of T_ρ in $[0, 1]$, $|T(f)| \leqslant C\|f\|_{p,\alpha}$. Therefore $f \in (A^{p,\alpha})^*$. ∎

4. Equivalence of two Banach spaces

Let Λ_α^n $(n = 0, 1, \ldots, 0 < \alpha \leqslant 1)$ be the space of analytic functions $f(z)$ in Δ with $f, f^1, \ldots, f^n \in A(\Delta)$ and $f^{(n)} \in \Lambda_\alpha$ with the form

$$\|f\| = \|f\| + \sup_{\substack{t,\theta\\ t>0}} \frac{\left|f^{(n)}(e^{i(\theta+t)}) - f^{(n)}(e^{i\theta})\right|}{t^\alpha};$$

Λ_α^n is a Banach space [2].

Let Λ_*^n be the Banach space of functions analytic in Δ with $f, f^*, \ldots, f^{(n)} \in A$ and $f^{(n)} \in \Lambda_*$, normed by

$$\|f\| = \|f\|_\infty + \sup_{\substack{t,\theta\\ t>0}} \frac{\left|f^{(n)}(e^{i(\theta+t)}) - 2f^{(n)}(e^{i\theta}) + f^{(n)}(e^{(\theta-t)})\right|}{t}.$$

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Two Banach spaces X and Y are said to be equivalent if there is one-to-one linear mapping L of X onto Y such that both L and L^{-1} are bounded. By the open mapping theorem it is sufficient that L is bounded.

We have the following theorem:

THEOREM 2. *If $(\alpha+2)/(m+1) < p < (\alpha+2)/m$, then the Banach space $(A^{p,\alpha})^*$ and Λ_γ^{m-2} with $\gamma = (\alpha+2)/p - m - 1$ are equivalent. If $p = (\alpha+2)/(m+1)$, then $(A^{p,\alpha})^*$ is equivalent to Λ_*^{m-2}.*

Theorem 2 of [2] is a limiting case of Theorem 2 for $0 < p < 1/2$, and the question arises whether it holds for the case $1/2 < p < 1$.

PROOF: Let $T \in (A^{p,\alpha})^*$. By Theorem 1 the mapping $T \to g$ where g is defined as in Theorem 1, is a one-to-one linear mapping L of $(A^{p,\alpha})^*$ onto $\Lambda_\gamma^{m-2}(\Lambda_*^{m-2})$. Then by (3.3)

$$\left|g^{(m-1)}(\xi)\right| \leqslant C\,\|T\|\,(1-|\xi|)^{\beta-1}$$

where $\beta = ((\alpha+2)/p) - m$. Hence the proof of Theorem 5.1 [3, p.74] shows that

$$\left|g^{(m-2)}\left(e^{i(\theta+t)}\right) - g^{(m-2)}\left(e^{i\theta}\right)\right| \leqslant C\left(1+\frac{2}{\beta}\right)\|T\|\,|t|^{\beta}.$$

We have $\|g\|_\infty = O(1)\,\|T\|$ by (3.2), so

$$\|g\| \leqslant C\,\|T\|.$$

Thus $g \in \Lambda_\gamma^{m-2}$ and $L(T) = g$, so

$$\|L\| = \sup_{\|T\|=1} \frac{|L(T)|}{\|T\|} = \sup \frac{\|g\|}{\|T\|} \leqslant C.$$

Thus L is a bounded linear functional from $(A^{p,\alpha})^*$ into Λ_γ^{m-2} and Theorem 2 is proved. ∎

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