ALMOST COMMUTATIVE BANDS

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1. Introduction. To find a " description of the structure of bands which is complete modulo semilattices " (from page 25 of [**1**]) seems to be a very difficult problem. As far as the author is aware, the only class of bands (except for rectangular bands) for which this problem has been solved (see [**4**] and [**3**]) is the class of all bands satisfying a generalization of commutativity, namely the condition that $efgh = egfh$ for all elements e, f, g and h.

The purpose of this paper is to give a solution to this problem for a further class of bands, which we call the class of almost commutative bands: a band is called *almost commutative* if any pair of elements are either $\mathcal{J}$-related or commute. It is easily seen from [**1**, Theorem 4.6] that a band B is almost commutative if and only if, for all $e, f \in B$, either $ef = fe$ or both $efe = e$ and $fef = f$.

Three examples of almost commutative bands played a major role in [**2**, §4] in the solution of two problems posed in [**6**]. The author was helped in the writing of this paper by an expository thesis of J. Pippey [**5**], which included the results given here.

2. Preliminaries. We use wherever possible, and usually without comment, the notations of Clifford and Preston [**1**].

Let B be any band. Then from [**1**, Theorem 4.6] B is a semilattice of rectangular bands; that is to say, for some semilattice Y and rectangular subbands $\{E_\alpha : \alpha \in Y\}$ of B, $B = \bigcup_{\alpha \in Y} E_\alpha$, and for all $\alpha, \beta \in Y$, $E_\alpha \cap E_\beta = \square$ if $\alpha \neq \beta$, and $E_\alpha E_\beta \subseteq E_{\alpha\beta}$; further, each E_α is a $\mathcal{J}$-class of B. If $e \in E_\alpha$, then we shall sometimes denote E_α by $E(e)$. It is clear that for any $e, f \in B$

$$eE(ef)e = \{g \in E(ef) : g \leqq e\},$$

where as usual $g \leqq e$ means $eg = g = ge$.

3. Almost commutative bands.

LEMMA 1. *For any elements e, f in the band B,*

$$|[eE(ef)e] \cap [fE(ef)f]| \leqq 1.$$

Proof. Suppose that the set $[eE(ef)e] \cap [fE(ef)f] \neq \square$ and take any element g of this set. Then $g \leqq e$ and $g \leqq f$, from which we easily see that $g \leqq ef$. But $ef, g \in E(ef)$, a rectangular band; so $g = ef$. Therefore the set above contains at most one element, namely ef (further, we may easily see that it contains ef if and only if $ef = fe$).

LEMMA 2. *The band B is almost commutative if and only if, for any $\alpha, \beta \in Y$ and any $e \in E_\alpha$, $f \in E_\beta$, $\beta < \alpha$ implies $f < e$.*

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Proof. Suppose that B is almost commutative and take any $\alpha, \beta \in Y$ such that $\beta < \alpha$. Then, for any $e \in E_\alpha$, $f \in E_\beta$, we have $ef = fe \in E_{\alpha\beta} = E_\beta$. But, since $ef = fe$, we see that $ef \leqq f$, whence $ef = f$, since E_β is a rectangular band. Thus $f < e$.

Conversely suppose, for all $\alpha, \beta \in Y$ and for all $e \in E_\alpha$, $f \in E_\beta$, that $\beta < \alpha$ implies $f < e$. Take any elements $g, h \in B$ such that $J_g \neq J_h$, i.e. $E(g) \neq E(h)$. Then $g \in E_\gamma$ and $h \in E_\delta$ for some $\gamma, \delta \in Y$ with $\gamma \neq \delta$. If $\gamma < \delta$, then $g < h$, whence $gh = g = hg$ and, similarly, if $\delta < \gamma$, then $gh = h = hg$. Let us consider then the remaining case, namely when γ and δ are not comparable; then $\gamma\delta < \gamma$ and $\gamma\delta < \delta$. Since $gh, hg \in E_{\gamma\delta}$, we have that both gh and hg are less than both g and h; so, from Lemma 1, $gh = hg$. Thus B is almost commutative.

REMARK 1. Lemma 2 contrasts with the case when B satisfies $efgh = egfh$ for all e, f, g, $h \in B$, for this is true of B if and only if $|eE_\beta e| = 1$ for all $\beta \in Y$, $e \in B$ [5, Theorem 6].

From Lemmas 1 and 2 we see that, if B is almost commutative, then, for any $\alpha, \beta \in Y$, $\alpha \neq \alpha\beta \neq \beta$ implies $|E_{\alpha\beta}| = 1$. We have thus proved already the final statement of the following theorem.

THEOREM 1. *Let now Y be any semilattice and let $\{E_\alpha : \alpha \in Y\}$ be any set of pairwise disjoint rectangular bands such that $|E_\alpha| = 1$ if $\alpha = \beta\gamma$ for some $\beta, \gamma \in Y$ and $\beta \neq \alpha \neq \gamma$; if this is the case, then let e_α denote the only element of E_α. Let the multiplication in each E_α be denoted by juxtaposition. Put $B = \bigcup_{\alpha \in Y} E_\alpha$ and define a multiplication $\circ$ for B as follows: for any $\alpha, \beta \in Y$ and for any $e \in E_\alpha$, $f \in E_\beta$, define*

$$e \circ f = \begin{cases} e & \text{if } \alpha < \beta, \\ ef \text{ as in } E_\alpha & \text{if } \alpha = \beta, \\ f & \text{if } \alpha > \beta, \\ e_{\alpha\beta} & \text{if } \alpha \neq \alpha\beta \neq \beta. \end{cases}$$

Then B is an almost commutative band. Conversely any almost commutative band is obtained in this way.

Proof. To show that $\circ$ is associative we shall only assume that the E_α $(\alpha \in Y)$ are semigroups and not necessarily rectangular bands.

Take any $e, f, g \in B$. Then $e \in E_\alpha$, $f \in E_\beta$ and $g \in E_\gamma$ for some $\alpha, \beta, \gamma \in Y$. It is clear from the definition of $\circ$ that $e \circ f \in E_{\alpha\beta}$, $(e \circ f) \circ g \in E_{\alpha\beta\gamma}$ and $e \circ (f \circ g) \in E_{\alpha\beta\gamma}$. Hence, if $|E_{\alpha\beta\gamma}| = 1$, then $(e \circ f) \circ g = e \circ (f \circ g)$.

Suppose then that $|E_{\alpha\beta\gamma}| > 1$. Then $\alpha\beta$ and γ are comparable.

Case 1: $\gamma < \alpha\beta$. Then $(e \circ f) \circ g = g$ and, since $\gamma < \beta$ and $\gamma < \alpha$, $e \circ (f \circ g) = e \circ g = g$, giving $(e \circ f) \circ g = e \circ (f \circ g)$.

Case 2: $\gamma = \alpha\beta$. Then $|E_{\alpha\beta}| = |E_{\alpha\beta\gamma}| > 1$, whence α and β are comparable.

Case 2(a): $\alpha < \beta$. Then $\gamma = \alpha\beta = \alpha < \beta$, whence

$$(e \circ f) \circ g = e \circ g = e \circ (f \circ g).$$

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Case 2(b): $\alpha = \beta$. Then $\alpha = \beta = \gamma$ and clearly $(e \circ f) \circ g = e \circ (f \circ g)$.

Case 2(c): $\alpha > \beta$. Then $\alpha > \beta = \alpha\beta = \gamma$, whence

$$(e \circ f) \circ g = f \circ g = fg = e \circ (fg) = e \circ (f \circ g).$$

Case 3: $\gamma > \alpha\beta$. Once again $|E_{\alpha\beta}| = |E_{\alpha\beta\gamma}| > 1$, whence α and β are comparable.

Case 3(a): $\alpha < \beta$. Then $\alpha = \alpha\beta < \gamma$ and $(e \circ f) \circ g = e \circ g = e$. Also $\alpha \leqq \beta\gamma$. If $\alpha = \beta\gamma$, then $\beta \neq \beta\gamma \neq \gamma$, giving $|E_{\alpha\beta\gamma}| = |E_{\beta\gamma}| = 1$, a contradiction. Hence $\alpha < \beta\gamma$, giving $e \circ (f \circ g) = e = (e \circ f) \circ g$.

Case 3(b): $\alpha = \beta$. Then $\beta = \alpha\beta < \gamma$, giving

$$(e \circ f) \circ g = e \circ f = e \circ (f \circ g).$$

Case 3(c): $\alpha > \beta$. Then $\beta = \alpha\beta < \gamma$ and

$$(e \circ f) \circ g = f \circ g = f = e \circ f = e \circ (f \circ g).$$

We now have that $\circ$ is associative, and clearly then B is an almost commutative band.

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