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CHARACTERIZATIONS OF FINITE LATTICES THAT ARE BOUNDED-HOMOMORPHIC IMAGES OR SUBLATTICES OF FREE LATTICES

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1. Introduction. In [**8**], McKenzie introduced the notion of a bounded homomorphism between lattices, and, using this concept, proved several deep results in lattice theory. Some of these results were intimately connected with the work of Jónsson and Kiefer in [**6**] where an attempt was made to characterize finite sublattices of free lattices. McKenzie's characterization and others that followed (see [**7**] and [**5**]) still have not answered the (now) celebrated Jónsson conjecture:

A finite lattice is a sublattice of a free lattice if and only if it satisfies $(SD_{\vee})$, $(SD_{\vee})$ *and* (W).

(The properties mentioned here are defined in the text.)

Several of the known characterizations have been of the form $\Phi_{\vee}$, $\Phi_{\wedge}$ and (W), where $\Phi_{\wedge}$ is the lattice-theoretical dual of the property $\Phi_{\vee}$. Moreover in these cases one had $\Phi_{\vee}$ implies $(\mathrm{SD}_{\vee})$ (and therefore $\Phi_{\wedge}$ implied $(\mathrm{SD}_{\wedge})$). Considering the different formulations of the various $\Phi_{\vee}$'s, it was somehow surprising when all were found to be equivalent.

The purpose of this paper is to supply another such $\Phi_{\vee}$ and to inch slightly closer to Jónsson's conjecture by showing that being a finite sublattice of a free lattice is equivalent to $\Phi_{\vee}$, $(\mathrm{SD}_{\wedge})$ and (W). A structural representation is also given for these lattices.

2. Preliminaries. Let **L** be the class of all finite lattices. We will be considering several different properties of (finite) lattices and the corresponding classes they define. As a notational convenience, for any property (P) we let **P** be the class of finite lattices having (P), i.e. for $L \in \mathbf{L}$, $L \in \mathbf{P}$ if and only if $L \models (\mathrm{P})$.

$$\mathbf{SD}_{\wedge} = \{L \in \mathbf{L} : \text{for all } a, b, c \in L(a \wedge b = a \wedge c \text{ implies } a \wedge b = a \wedge (b \vee c))\}$$

$$\mathbf{SD}_{\vee} = \{L \in \mathbf{L} : \text{for all } a, b, c \in L(a \vee b = a \vee c \text{ implies } a \vee b = a \vee (b \wedge c))\}$$

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These classes correspond to the special distributive laws introduced in [**6**].

$$\mathbf{W} = \{L \in \mathbf{L} : \text{for all } a, b, c, d \in L(a \wedge b \leqq c \vee d \text{ implies } \{a, b, c, d\} \cap [a \wedge b, c \vee d] \neq \emptyset)\}.$$

A quadruple in $L\ (\in \mathbf{L})$ which does not satisfy the defining condition of (W) will be called a *W-failure* in L.

$$\mathbf{F} = \{L \in \mathbf{L} : L \text{ is a sublattice of a free lattice}\}.$$

$\mathbf{LB} = \{L \in \mathbf{L} : L$ is a lower bounded-homomorphic image of a free lattice$\}$. (A homomorphism $\alpha : L \rightarrow M$ is called *lower bounded* if the inverse image of each member of M is either empty or has a least member. An upper-bounded homomorphism and the class **UB** is defined dually and $\mathbf{B} = \mathbf{LB} \cap \mathbf{UB}$.)

We also need to recall some definitions due to Jónsson which may be found at least in [**7**] (and [**8**] under an alias).

Let L be a finite lattice. A non-trivial cover of $a \in L$ is a non-empty subset $C \subseteq L$ such that $a \leqq \bigvee C$ and $a \nleqq c$ for all $c \in C$. For any two subsets C and D of L, we say C is a *lower refinement* of D (written $C \ll D$) if every member of C is less than or equal to some member of D. A sequence of subsets of L, $S_k(L)$ is defined inductively by:

(i) $x \in S_0(L)$ if and only if x has no non-trivial covers;

(ii) $x \in S_{k+1}(L)$ if and only if for any non-trivial cover $C \subseteq L$ of x there exists a lower refinement $D \subseteq S_k(L)$ of C which is also a (non-trivial) cover of x;

(iii) $S(L) = \bigcup S_k(L)$.

The dual definitions of dual covers ($x \geqq \bigwedge C$), upper refinements, the sequence $P_k(L)$, and $P(L)$ are left to the reader.

Finally we need a generalization, due to Sivak [**10**], of a construction given by the author in [**4**]. (See [**5**] for the ultimate such generalization).

For a finite lattice, L, $K \subseteq L$ is called a *lower pseudo-interval* if $K = \bigcup_i^{1,n} [u, v_i]$ for some $u, v_i, \ldots, v_n \in L$ with $u \leqq v_i$ (all i). An *upper pseudo-interval* of L is defined dually. Moreover, a subset $I \subseteq L$ is an *interval* if and only if it is both an upper and a lower pseudo-interval.

Now let K be a lower pseudo-interval of a (finite) lattice L. We define a new lattice $L[K] = (L \backslash K) \,\dot{\cup}\, (K \times \mathbf{2})$ obtained by doubling every element in K with the product order and "fitting" this inside $L \backslash K$ using the original (and first projection) order on L. Clearly this construction can also be done for intervals, upper pseudo-intervals and (cf. [**5**]) for arbitrary convex subsets of L.

A class, **K**, of (finite) lattices is said to be *closed under the splitting of intervals* (lower or upper pseudo-intervals) if for every $L \in \mathbf{K}$ and interval (respectively lower or upper pseudo-interval) $K \subseteq L$, $L[K] \in \mathbf{K}$.

We need the following results.

(2.1) THEOREM (McKenzie [8]). $F = \mathbf{B} \cap \mathbf{W}$.

(2.2) THEOREM ([10] and [4]). **LB** *is closed under the splitting of lower pseudo-intervals and* **B** *is closed under the splitting of intervals.*

3. Cycles and congruences. Since, in any finite lattice, every element is the join of the join-irreducibles less than or equal to it, if $b \nleqq a$ holds in L then there is a join-irreducible u in $J(L)$ with $u \leqq b$ and $u \nleqq a$. Moreover if u is a minimal such join-irreducible than $u \wedge a = u_*$, the largest element less than u. Dually, the same remarks can be made about meet irreducibles and we will call a join-irreducible or meet-irreducible related to a pair (a, b), $b \nleqq a$, in the above respective manner an *associated join-(resp.meet-) irreducible.*

This "association" is of greater interest when a is covered by b ($a \prec b$). Here, any associated join-irreducible u and meet-irreducible m satisfy the following equations:

$$u \vee a = b, \quad u \wedge a = u_*, \quad m \wedge b = a \quad \text{and } m \vee b = m^*,$$

the least element of L greater than m.

Being more particular still, there is a relation $\rho \subseteq M(L) \times J(L)$ defined by:

$$m\rho u \text{ if and only if } u \vee m = m^* \text{ and } u \wedge m = u_*$$

An interesting observation is the following:

(3.1) LEMMA. *A finite lattice L satisfies* ($\mathrm{SD}_\vee$) *if and only if ρ is* (*the graph of*) *a function.*

Proof. Suppose L satisfies ($\mathrm{SD}_\vee$) and assume also that $m\rho u_1$ and $m\rho u_2$. We have then $m \vee u_1 = m^* = m \vee u_2$ and therefore $m^* = m \vee (u_1 \wedge u_2)$. But this forces for both i, $u_{i*} < u_1 \wedge u_2 \leqq u_i$. Therefore $u_1 = u_1 \wedge u_2 = u_2$.

Conversely, assume ρ is a function and take $a, b_1, b_2 \in L$ with $a \vee b_1 = a \vee b_2$. If $a \vee (b_1 \wedge b_2) < a \vee b_1$ then there is an associated meet-irreducible $m \in M(L)$ with $a \vee (b_1 \wedge b_2) \leqq m$ and $a \wedge b_i \nleqq m$ and also then $b_i \nleqq m$ for $i = 1, 2$.

Now if for each $i = 1, 2$, u_i is a join-irreducible associated with b_i and m, we have $m\rho u_1$ and $m\rho u_2$. Since ρ is a function,

$$u = u_1 = u_2 \leqq b_1 \wedge b_2 \leqq a \vee (b_1 \wedge b_2) \leqq m,$$

a contradiction. Therefore $a \vee b_i = a \vee (b_1 \wedge b_2)$ and L satisfies ($\mathrm{SD}_\vee$).

By transitivity, any congruence relation, θ, on a finite lattice is determined uniquely by the prime quotients it collapses and therefore also by the join-irreducibles u for which $u_*\theta u$. We wish characterizations of the sets of prime quotients and sets of join-irreducibles so obtained. The first characterization is an easy exercise.

(3.2) LEMMA. *For a finite lattice L, and $Q(L)$ its set of prime quotients then $Q \subseteq Q(L)$ is precisely the set of prime quotients collapsed by some congruence* *relation on L if and only if for $b/a \in Q$ and $d/c \in Q(L)$ if either $a \leqq c < d \leqq b \vee c$ or $a \wedge d \leqq c < d \leqq b$ hold then $d/c \in Q$.*

The next characterization requires a definition essentially due to Pudlák and Tuma in [**9**].

(3.3) *Definition.* For a finite lattice L and join-irreducibles $u, v \in J(L)$, uCv if and only if for some $p \in L$ the following hold:

($C1$) $u \leqq v \vee p$ and $u \nleqq v_* \vee p$;
($C2$) $u \nleqq v \vee x$ for all $x < p$.

One perhaps should note that uCv implies $u \nleqq v$ and that this together with the existence of some $q \in L$ satisfying ($C1$) will produce ($C2$) by taking a minimal such q.

(3.4) LEMMA. *Let S be a set of join-irreducibles in a finite lattice L. Then the following are equivalent:*

(1) $S = \{u \in J(L) : u\theta u_*\}$ *for some* $\theta \in \text{Con}(L)$
(2) uCv *and* $v \in S$ *imply* $u \in S$.

Proof. (2) follows easily from (1) since if $v\theta v_*$ and uCv we have

$$u = u \wedge (p \vee v)\theta u \wedge (p \vee v_*) \quad \text{with} \quad u \wedge (p \vee v_*) \leqq u_* < u.$$

Conversely let S satisfy (2) and let Q be the set of all prime quotients of L which have associated join-irreducibles in S. We will show one part of (3.2) and leave the other for the reader. Assume then that $b/a \in Q$ and d/c is another prime quotient with $a \leqq c < d \leqq b \vee c (= b \vee d)$. Let $v \in S$ satisfy $v \vee a = b$ and $v \wedge a = v_*$, and take $u \in J(L)$ with $u \vee c = d$ and $u \wedge c = u_*$. Now $c \vee v = c \vee a \vee v = c \vee b \geqq u$ and $c \vee u_* = c \ngeqq u$. Therefore for any minimal member, p, of $\{x \in L : x \leqq c, v \vee x \geqq u \text{ and } u_* \vee x \ngeqq u\}$ we have $u \cdot Cv$. By (2) this gives $d/c \in Q$, and the proof is complete.

(3.5) COROLLARY. *For all $u, v \in J(L)$, con $(u_*, u) \subseteq$ con (v_*, v) if and only if there exists a sequence $u = u_0, \ldots, u_n = v$ with u_iCu_{i+1} for all $i < n$.*

Proof. The set of all $u \in J(L)$ for which such a sequence exists with $u_n = v$ clearly satisfies (2).

(3.6) *Definition.* A *C-cycle* in a finite lattice L is a non-singleton set $\{u_i : 0 \leqq i < n\} \subseteq J(L)$ such that for all $i < n$, u_iCu_{i+1} (where $i + 1$ is computed modulo n).

4. The condition ($\mathbf{P}_\vee$). This property was considered by Pudlák and Tuma in [**9**] and is precisely the condition needed on a finite lattice to make their graph-theoretical proof of the Gratzer-Schmidt theorem produce a *finite* algebra with the given lattice as its congruence lattice. We require some of their results and will present here lattice-theoretic proofs.

Now for any finite lattice L, the function $u \mapsto \mathrm{con}_L(u_*, u)$ provides a natural surjection from $J(L)$ onto $J(\mathrm{Con}(L))$.

(4.1) *Definition.* L satisfies $(\mathrm{P}_\vee)$ if and only if for any $u, v \in J(L)$, $\mathrm{con}_L(u_*, u) = \mathrm{con}_L(v_*, v)$ implies $u = v$.

This is perhaps the best definition for generalizations to non-finite lattices; however proofs in the finite case would require only the checking of the respective cardinalities of $J(L)$ and $J(\mathrm{Con}(L))$. The following is an obvious corollary of (3.1).

(4.2) LEMMA. $\mathbf{P}_\vee \subseteq \mathbf{SD}_\vee$

(4.3) THEOREM [**9**]. *The class* $\mathbf{P}_\vee$ *is non-empty and closed under* **H**, **S** *and* $\mathbf{P}_{\mathrm{fin}}$.

Proof. Clearly **2** belongs to $\mathbf{P}_\vee$ and as lattices have factorable congruences (i.e. $\mathrm{Con}(L \times M) \simeq \mathrm{Con}\, L \times \mathrm{Con}\, M$), $\mathbf{P}_\vee$ is closed under finite products.

Now take $B \in \mathbf{P}_\vee$ with $A \leqq B$. Furthermore let a and b be join-irreducible in A with $a_0 = \bigvee \{x \in A : x < a\}$ and b_0 defined similarly, and suppose $\mathrm{con}_A(a_0, a) = \mathrm{con}_A(b_0, b)$. Passing to B we obtain $\mathrm{con}_B(a_0, a) = \mathrm{con}_B(b_0, b)$.

Now for any $x < y$ in any finite lattice L,

$$\mathrm{con}_L(x, y) = \bigvee \{\mathrm{con}_L(u_*, u) : u \in J(L), u \leqq y \text{ and } u \nleqq x\}.$$

Moreover one may take an irredundant set of such join-prime congruences ($\mathrm{Con}(L)$ is distributive!). Therefore there are sets $J_a, J_b \subseteq J(B)$ such that for $x \in \{a, b\}$:

(1) $u \in J_x$ implies $u \leqq x$ and $u \nleqq x_0$;
(2) $\mathrm{con}_B(x_0, x) = \bigvee \{\mathrm{con}_B(u_*, u) : u \in J_x\}$ irredundantly.

Now $\mathrm{con}_B(a_0, a) = \mathrm{con}_B(b_0, b)$ and $B \in \mathbf{P}_\vee$ force $J_a = J_b$ and hence there is a $w \in J(B)$ (actually in $J_a = J_b$) with $w \leqq a \wedge b$, $w \nleqq a_0$ and $w \nleqq b_0$. We now have in A, $a, b \in J(A)$, $a \wedge b \nleqq a_0$ and $a \wedge b \nleqq b_0$. Therefore $a = b$ and $\mathbf{S}(\mathbf{P}_\vee) \subseteq \mathbf{P}_\vee$.

Now since any epimorphism $\phi : A \twoheadrightarrow B$ between finite algebras can be factored into a composition of epimorphisms $\phi_i : A_i \twoheadrightarrow A_{i+1}$ with $\mathrm{Ker}\, \phi_i$ an atom in $\mathrm{Con}(A_i)$ we need only consider $A \in \mathbf{P}_\vee$ and epimorphism $\phi : A \twoheadrightarrow B$ with Δ_A covered by $\mathrm{Ker}\, \phi$ in $\mathrm{Con}(A)$. Since A is a finite lattice ϕ is bounded and an easy argument shows that the lower bound $\vee$-monomorphism $\alpha : B \rightarrowtail A$ given by $b \mapsto \wedge \phi^{-1}[b]$ takes $J(B)$ into $J(A)$. Moreover, since $\mathrm{Ker}\, \phi$ collapses at least one pair (u_*, u) for $u \in J(A)$ (exactly one since $A \in \mathbf{P}_\vee$) we obtain

$$|J(B)| < |J(A)|.$$

But since $\mathrm{Ker}\, \phi$ is an atom in a finite distributive lattice $\mathrm{Con}(A)$ and since

Con $(B) \cong [\text{Ker } \phi, \Delta_A]_{\text{Con}(A)}$ we obtain

$$|J(\text{Con } (A))| = |J(\text{Con } (B))| + 1.$$

This forces $|J(B)| = |J(\text{Con } (B))|$ since $A \in \mathbf{P}_\vee$.

(4.4) LEMMA. $\mathbf{P}_\vee$ *is closed under the splitting of lower pseudo-intervals.*

Proof. Let K be a lower pseudo-interval of $L \in \mathbf{P}_\vee$ and suppose $K = \bigcup_i^{1,n} [u, v_i]$. Then it is easily seen that $x \in J(L[K])$ if and only if one of the following hold:

(1) $x \in L\backslash K$ and $x \in J(L)$;
(2) $x = (s, 0)$ and $s \in J(L)$;
(3) $x = (u, 1)$.

Since the kernel of the natural epimorphism $\kappa : L[K] \twoheadrightarrow L$ is an atom in Con $(L[K])$ we can conclude as in (4.3) that

$$|J(\text{Con } (L[K])| = 1 + |J(\text{Con } (L))|.$$

Since $L \in \mathbf{P}_\vee$, the result follows.

5. The characterization theorems. The main results will be derived from the following theorem. For the creditations for some parts of this theorem, the reader should consult the historical notes in § 7.

(5.1) THEOREM. *For a finite lattice, L, the following are equivalent:*

(1) $L \in \mathbf{P}_\vee$.
(2) *There is a sequence of lattices $L_0 = \mathbf{1}, L_1, \ldots, L_{n+1} = L$ together with a sequence of lower pseudo-intervals $K_0, \ldots, K_n$ with $K_i \subseteq L_i$, $i \leqq n$ such that for all $i \leqq n$, $L_{i+1} \cong L_i[K_i]$.*
(3) $L \in \mathbf{LB}$.
(4) $S(L) = L$.
(5) *L has no C-cycle.*

Proof. (1) implies (2): Let θ be an atom in Con (L). By $(P_\vee)$, we have a unique $u \in J(L)$ with $\theta = \text{con } (u_*, u)$. We need only show that L can be obtained from L/θ by finding a suitable lower pseudo-interval in L/θ. This is easily done by examining the congruence classes of θ.

Let $\{m_1, \ldots, m_n\}$ be the set of meet-irreducibles associated with u.

Claim 1. *For $x \in [u_*, m_i]$, $(v \vee x) \wedge m_i = x$.*

If $x < (u \vee x) \wedge m_i$, then we can find a $v \in J(L)$ such that $v \leqq (u \vee x) \wedge m_i$ and $v \nleqq x$. Now $u_* \theta u$ implies $x\theta(u \vee x) \wedge m_i$ which implies in turn $v_* \theta v$. By $(P_\vee)$ this forces $u = v \leqq m_i$, a contradiction.

Similarly one obtains

Claim 2. *For $x \in [u, m_i^*]$, $u \vee (x \wedge m_i) = x$.*

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Claim 3. $x\theta y$ *if and only if* $x = y$ *or for some* $i \in \{1, \ldots, n\}$ *there exists* $z \in [u_*, m_i]$ *with* $\{x, y\} = \{z, u \vee z\}$.

We need only check the claim for pairs in θ of the form $x \prec y$. But in this case, u must be the associated join-irreducible. Therefore the associated meet-irreducible must be some m_i and $\{x, y\} = \{x, u \vee x\}$.

(1) implies (2) now follows by considering K to be the image of $\bigcup_i^{1,n} [u_*, m_i]$.

(2) implies (3): This is (2.2).

(3) implies (4): See Jónsson and Nation [**7**] where the proof is given for the finitely generated case.

(4) implies (5): Suppose L has a C-cycle $u_1Cu_2C \ldots Cu_nCu_1$ and without loss of generality let u_1 have the least S-rank. (That is: $u_1 \in S_k(L)$ for some k and for all $i \in \{2, \ldots, n\}$, $u_i \notin S_l(L)$ for all $l < k$).

Now u_1Cu_2 implies there exists an $x \in L$ with $u_1 \leqq x \vee u_2$, $u_1 \leqq x \vee u_{2*}$ and for all $y < x$ $u_1 \nleqq y \vee u_2$. This implies immediately that $k \neq 0$ and if $k = l + 1$ then there exists $M \subseteq D_l(L)$ such that $u_1 \leqq \bigvee M$ and $M \ll \{u_2, x\}$. Now the above stated inequalities force $u_2 \in M \subseteq D_l(L)$, a contradiction on the minimality of $k = l + 1$.

(5) implies (1): If con $(u_*, u) =$ con (v_*, v) holds in L then there exists C-chains $u = w_0Cw_1C \ldots w_{n-1}Cv$ and $v = s_0Cs_1C \ldots s_mCu$. But then one would have a proper C-cycle and hence $u = v$.

(5.2) Corollary. *The only simple lattice in* $\mathbf{P}_\vee$ *is* $\mathbf{2}$.

(5.3) Theorem. $\mathbf{B} = \mathbf{P}_\vee \cap \mathbf{SD}_\wedge$.

Proof. By (5.1) and (4.2), $\mathbf{B} = \mathbf{P}_\vee \cap \mathbf{P}_\wedge \subseteq \mathbf{P}_\vee \cap \mathbf{SD}_\wedge$. Conversely in the proof of (5.1); (1) $\rightarrow$ (2), if $L \in \mathbf{SD}_\wedge$ then there is a unique meet-irreducible m associated with the given join-irreducible u. This implies that $L \cong A[I]$ for some interval $I \subseteq A$ and the proof is complete via (2.2) and induction on the cardinality of L.

(5.4) Corollary. *For a finite lattice* L, $L \in \mathbf{B}$ *if and only if there exists a sequence of lattices* $L_0 = \mathbf{1}, L_1, \ldots L_{n+1} = L$ *together with a sequence of intervals* $I_i \leqq L_i$, $0 \leqq i \leqq n$ *such that for all* $i \leqq n$, $L_{i+1} \simeq L_i[I_i]$.

(5.5) Corollary. $\mathbf{F} = \mathbf{P}_\vee \cap \mathbf{SD}_\wedge \cap \mathbf{W}$.

6. B-reflections. Since $\mathbf{B}$ is closed under finite subdirect products, we can measure how far a (finite) lattice is from being in $\mathbf{B}$ in the following way. For $A \in \mathbf{L}$, let $\theta_\mathbf{B} = \wedge \{\theta \in \text{Con}(A) : A/\theta \in \mathbf{B}\}$ and define $R_\mathbf{B}(A) = A/\theta_\mathbf{B}$ with $\rho A : A \twoheadrightarrow R_\mathbf{B}(A)$ the canonical epimorphism. Clearly $R_\mathbf{B}(A)$ is the largest homomorphic image of A in $\mathbf{B}$ and we have for any $\phi : A \rightarrow B$ with $B \in \mathbf{B}$ there exists a unique $\bar{\phi} : R_\mathbf{B}(A) \rightarrow B$ with $\phi = \bar{\phi} \circ \rho A$. In this section we want to give a description of the $\mathbf{B}$-reflection, $R_\mathbf{B}(A)$, and of Ker ρA for lattices of particular interest, namely those in $\mathbf{SD}_\wedge \cap \mathbf{SD}_\vee \cap \mathbf{W}$.

To ease notation for the rest of this section, let $\rho : L \to B$ be the **B**-reflection of $L \in$ L.

(6.1) LEMMA. *If $L \in \mathbf{W}$, then $B \in \mathbf{W}$.*

Proof. If $B \notin \mathbf{W}$, let (a, b, c, d) be a quadruple in B witnessing a W-failure in B and take $I = [a \wedge b, c \vee d]$. Now since (a, b, c, d) is a W-failure the canonical $\kappa : B[I] \twoheadrightarrow B$ is such that no proper sublattice of $B[I]$ has B as its image under κ. But since $L \in \mathbf{W}$, there exists a lifting $\bar{\rho} : L \to B[I]$ such that $\kappa \circ \bar{\rho} = \rho$ (c.f. Davey and Sands [**1**]). Therefore $\bar{\rho}$ is an epimorphism and $B[I] \in \mathbf{B}$ supplies a contradiction to the fact that B is the **B**-reflection of L.

(6.2) LEMMA. *If $L \in \mathbf{W}$, then $S(L)$ is a sublattice of L and the map $\alpha : L \to S(L)$ given by $x \mapsto \bigvee \{s \in S(L) : s \leqq x\}$ is a lattice homomorphism.*

Proof. The fact that $S(L)$ is a sublattice in the presence of (W) is an old (then-) unpublished result of Jónsson circa 1960. Also the proof that α is a homomorphism is essentially in [**7**, 3.3]. We will here provide a proof of the second part.

Since $\alpha(x) \leqq x$ for all $x \in L$ and $S(L)$ is a sublattice it follows that α preserves meets. In order to show α preserves joins it is enough to show that for all $s \in S(L)$, $s \leqq x \vee y$ implies $s \leqq \alpha(x) \vee \alpha(y)$.

Clearly for $s \in S_0(L)$ this property holds as $S_0(L)$ is the set of join-prime elements of L. Now for $s \in S_{k+1}(L)$, $s \leqq x \vee y$ implies there exists a lower refinement $C \ll \{x, y\}$ with $C \subseteq S_k(L)$ and $s \leqq \vee C$. Now $C \ll \{x, y\}$ implies $C = C_x \cup C_y$ where $c \in C_z$ if and only if $c \leqq z$ $(z \in \{x, y\})$. Therefore

$$s \leqq \bigvee C = \bigvee C_x \vee \bigvee C_y \leqq \alpha(x) \vee \alpha(y).$$

Dually we obtain:

(6.3) LEMMA. *For $L \in W$, then $P(L)$ is a sublattice of L and the map $\beta : L \to P(L)$ given by $x \mapsto \wedge \{p \in P(L) : x \leqq p\}$ is a lattice homomorphism.*

Another easy consequence of these definitions is the following, also known to Jónsson.

(6.4) LEMMA. *Let A and B be finite lattices and $\phi : A \twoheadrightarrow B$ be an epimorphism with lower and upper bounds $\alpha : B \rightarrowtail A$ and $\beta : B \rightarrowtail A$ respectively. Then for every $k < \omega$,*

$$\alpha[S_k(B)] \subseteq S_k(A) \quad \textit{and} \quad \beta[P_k(B)] \subseteq P_k(A).$$

(6.5) THEOREM. *If $\rho : L \twoheadrightarrow B$ is the B-reflection of L and*

$$L \in \mathbf{SD}_\wedge \cap \mathbf{SD}_\vee \cap \mathbf{W},$$

then

(1) *$B \simeq S(L) \simeq P(L)$ under the restriction of ρ;*

(2) *The congruence classes of* Ker *ρ are of the form $[\alpha(x), \beta(x)]$, $x \in L$ where α and β are as in* (5.3) *and* (5.4).

Proof. Let $\sigma : B> \rightarrow L$ and $\delta : B> \rightarrow L$ be the respective lower and upper bounds for ρ. Since $B \in \mathbf{B}$, $B = S(B) = P(B)$, we have by (6.4) that $\sigma[B] \subseteq S(L)$ and $\delta[B] \subseteq P(L)$. Therefore each congruence class of ρ contains at least one member of $S(L)$ (its smallest) and at least one member of $P(L)$ (its largest). Also $\rho|S(L)$ and $\rho|P(L)$ are surjective.

Now $\mathbf{SD}_\wedge \cap \mathbf{SD}_\vee$ is closed under $\tilde{S}$ and furthermore $S(S(L)) = S(L)$; therefore by (5.3), $S(L) \in \mathbf{B}$ (and in fact $S(L) \in \mathbf{F}$). Since $\rho : L \rightarrow > B$ is the B-reflection, there exists $\bar{\alpha} : B \rightarrow S(L)$ such that $\bar{\alpha} \circ \rho = \alpha$. Since $\alpha|S(L)$ is the identity, we have $\rho|S(L)$ is injective.

Dually $\rho|P(L)$ is injective and therefore $\rho|S(L)$ and $\rho|P(L)$ are isomorphisms onto B.

It is clear then that the equivalence classes of ρ are of the form $[\alpha(x), \beta(x)]$, $x \in L$.

(6.6) COROLLARY. *For $L \in \mathbf{SD}_\wedge \cap \mathbf{SD}_\vee \cap \mathbf{W}$, $L \in \mathbf{F}$ if and only if $S(L) = P(L)$.*

Since clearly $\alpha(1) = 1$ and $\beta(0) = 0$, knowing what B is provides us with some skeletal structure for L. For example, if $L \in \mathbf{SD}_\wedge \cap \mathbf{SD}_\vee \cap \mathbf{W}$ and its $\mathbf{B}$-reflection is $\mathbf{2} \times \mathbf{2}$ then we know that L looks roughly like figure (i) and an easy analysis will show that necessarily $L = S(L) = P(L) = \mathbf{2} \times \mathbf{2} \in \mathbf{F}$. The author is not able at this time to use the $\mathbf{B}$-reflection in general to answer the conjecture $\mathbf{F} = \mathbf{SD}_\wedge \cap \mathbf{SD}_\vee \cap \mathbf{W}$.

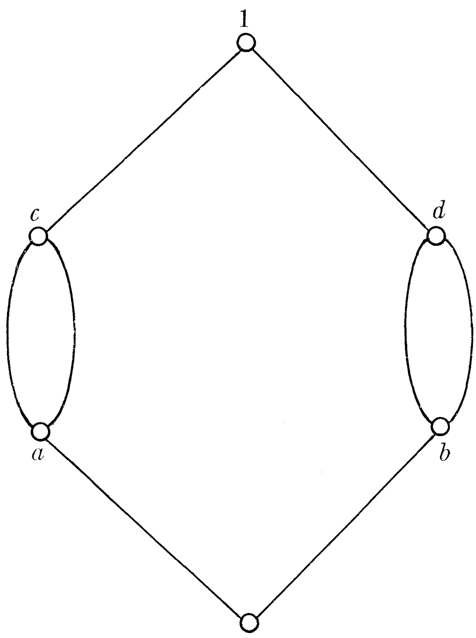

FIGURE (i).

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7. Historical comments and acknowledgements. For the general history of Jónsson's conjecture and other references we refer the reader to [**7**].

The main results of this paper (5.3), (5.4) and § 6) were found in the fall of 1975 and announced in [**4**]. They stemmed directly from discussions with Prof. Pavel Goralcík at the Czechoslovakian Summer School (1975) and the author wishes to thank the organizers of that conference for the opportunities that were made available to him there.

A pre-publication manuscript of these results was circulated in early 1976 and since that time the connections between $(P_\vee)$ and the other known results were established. This has necessitated a rather drastic revision of the original manuscript. A proof of (5.3) will appear in [**7**] which, though it does not give (5.4), supplies the non-structural equivalences of (5.1) under the extra assumption that $L \in \mathbf{SD}_\wedge \cap \mathbf{SD}_\vee$. Sivák's notion in [**10**] of a small congruence is precisely the splitting of a lower pseudo-interval. Moreover he also had the equivalence of (5.1 : 1) and (5.1 : 2) and that these implied (5.1 : 3). The equivalence of (5.1 : 1) and (5.1 : 5) is due to Pudlák and Tuma. We refer the reader to [**7**] for the history of the other implications and equivalences.

Some of the lattice theoretical proofs in § 4 and § 5 of the results of Pudlák and Tuma were obtained in consultation with Professors Ralph Freese, J. B. Nation and Ivan Rival. To them and especially to Professor Bjarni Jónsson the author expresses much thanks. Finally many thanks to the C. M. C. Summer School on Lattice Theory (1977) where these results were presented in their present form and to Professors Herb Gaskill and Craig Platt for their comments.

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