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LINEAR FUNCTIONAL-DIFFERENTIAL EQUATIONS IN A BANACH ALGEBRA*

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The theory of analytic differential systems in Banach algebras has been investigated by E. Hille and others, see for instance Chapter 6 in [4].

In this paper we show how a projection method used by W. A. Harris, Jr., Y. Sibuya, and L. Weinberg [3] can be applied to study a class of functional differential equations in this setting. The method, based on functional analysis, had been used extensively by L. Cesari [1] in similar forms for boundary value problems, and by J. K. Hale, S. Bancroft, and D. Sweet [2]. We also obtain as corollaries several results for ordinary differential equations in Banach algebras which were proved in a different way by Hille.

Let $\mathcal{B}$ be a noncommutative Banach algebra with unit element e. It is always possible to introduce a norm $|\cdot|$ such that $|e|=1$, and we assume that this has been done. Assume further that there exists a resolution of the identity with idempotents e_i, $i=1,\ldots,n$, such that

$$e=\sum_{i=1}^{n} e_i \quad \text{and} \quad e_ie_j=e_je_i=\delta_{ij}e_i.$$

THEOREM. *Let $P(z)$, $Q(z)$ and $R(z)$ be $\mathcal{B}$-valued functions holomorphic at $z=0$, let $D=\sum_{i=1}^{n} d_ie_i$ with nonnegative integers d_i, and let α, $0<|\alpha|<1$, be a complex constant. For every N sufficiently large, and every $\mathcal{B}$-valued polynomial $\phi(z)$ with $z^D\phi(z)$ of degree N, there exists a $\mathcal{B}$-valued polynomial $f(z;\phi)$ (depending on P, Q, R, α, and N) of degree $N-1$ such that the linear neutral-differential equation*

$$z^D\frac{dy}{dz}=P(z)y(z)+Q(z)y(\alpha z)+R(z)y'(\alpha z)+f(z;\phi) \tag{1}$$

has a $\mathcal{B}$-valued solution $y(z)$ holomorphic at $z=0$. Further, f and y are linear and homogeneous in ϕ, and

$$z^D(y-\phi)=O(z^{N+1}) \quad as \quad z\to 0.$$

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Proof. Because of the structure of D, z^D has the form

$$z^D=\sum_{j=1}^{n} e_j z^{d_j}.$$

For some $\delta>0$, let X be the set of all $\mathscr{B}$-valued functions $f\equiv f(z)\equiv \sum_{k=0}^{\infty} f_k z^k$ such that the series $\sum_{k=0}^{\infty} |f_k|\,\delta^k$ converges. For $f\in X$, define $f_k^j=e_jf_k$, $f^j=\sum_{k=0}^{\infty} f_k^j z^k$, and $\|f\|=\sum_{k=0}^{\infty}|f_k|\,\delta^k$. Note that $(X, \|\cdot\|)$ is also a Banach space.

For N sufficiently large, define the mapping $\mathscr{L}_N: X\to X$ by

$$\mathscr{L}_N y=g,\ y=\sum_{k=0}^{\infty} y_k z^k,\ g^j=\sum_{k=N}^{\infty}\frac{y_k^j}{k+1-d_j}\,z^{k+1-d_j}.$$

Then

$$\|\mathscr{L}_N y\|\le \sum_{j=1}^{n}\sum_{k=N}^{\infty}\frac{\delta^{1-d_j}}{N+1-d_j}\,|y_k^j|\,\delta^k.$$

Since there is an $M>0$ such that $|e_j|\le M$ for $j=1,2,\ldots,n$, $|y_k^j|\le M\,|y_k|$ and

(2) $$\|\mathscr{L}_N y\|\le M\sum_{j=1}^{n}\frac{\delta^{1-d_j}}{N+1-d_j}\,\|y\|.$$

Define $\hat{y}(z)=y(\alpha z)$ and $y^*(z)=\sum_{k=0}^{\infty}(k+1)\alpha^k y_{k+1}z^k$, and note that $\hat{y}\in X$ and $y^*\in X$ wherever $y\in X$. It is clear that

(3) $$\|\hat{y}\|\le\|y\|.$$

Furthermore, set $\chi(z)=\sum_{k=0}^{\infty}|y_k|\,z^k$, $|z|\le\delta$, to obtain

$$\chi'(|\alpha|\,z)=\sum_{k=1}^{\infty} k\,|\alpha|^{k-1}\,|y_k|\,z^{k-1},\ |z|\le\delta.$$

By the Cauchy integral formula,

$$|\chi'(|\alpha|\,z)|\le\max_{|\xi|=\delta}\frac{|\chi(\xi)|}{\delta(1-|\alpha|)^2}=\frac{\|y\|}{\delta(1-|\alpha|)^2}.$$

Hence

(4) $$\|y^*\|=|\chi'(|\alpha|\,\delta)|\le\frac{\|y\|}{\delta(1-|\alpha|)^2}.$$

For any function $A\in X$, and for each $f\in X$, note that $Af\in X$ and $\|Af\|\le \|A\|\,\|f\|$.

Let $\phi(z)=\sum_{i=0}^{N}\phi_i z^i$ be a $\mathscr{B}$-valued polynomial with $z^D\phi$ of degree N. Then

$$\phi^j(z)=\sum_{i=0}^{N-d_j}\phi_i^j z^i.$$

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Consider the functional equation in X

$$y = \phi + T_N[y], \tag{5}$$

where $T_N[y] = \mathscr{L}_N(Py + Q\hat{y} + Ry^*)$. The estimates (2)–(4) imply that for N sufficiently large, $\|T_N\| < 1$, and thus there exists a unique solution $y \in X$,

$$y(\cdot, \phi) = (e - T_N)^{-1}\phi. \tag{6}$$

It follows that the holomorphic solution of the functional equation (5) satisfies equation (1), where

$$\begin{aligned} f(z;\phi) = \sum_{k=0}^{N-1} f_k z^k = z^D \frac{d\phi}{dz} - \sum_{k=0}^{N-1} Py(\cdot, \phi)_k z^k \\ - \sum_{k=0}^{N-1} Qy(\cdot\,;\phi)_k z^k - \sum_{k=0}^{N-1} Ry^*(\cdot\,;\phi)_k z^k. \end{aligned} \tag{7}$$

Since the coefficients of $y(\cdot\,;\phi)$ are linear in the coefficients of ϕ, the f_k are also linear in the coefficients of ϕ; this completes the proof.

COROLLARY 1. *With notation as in the above theorem let* $P(z) = \sum_{k=0}^{\infty} a_k z^k$, $Q(z) = \sum_{k=0}^{\infty} b_k z^k$, *and* $R(z) = \sum_{k=0}^{\infty} c_k z^k$ *be convergent for* $|z| < a$, *and let* $y(z) = \sum_{k=0}^{\infty} y_k z^k$ *be a formal solution of*

$$z\frac{dy}{dz} = P(z)y(z) + Q(z)y(\alpha z) + R(z)y'(\alpha z). \tag{8}$$

Then $y(z)$ *is convergent for* $|z| < a$.

Proof. Since here $D = e$, we can choose $n = 1$, $d_1 = 1$, and thus $\phi = \sum_{i=0}^{N-1} \phi_i z^i$ and $y = \phi + O(z^N)$. Holomorphic solutions of

$$z\frac{dy}{dz} = P(z)y(z) + Q(z)y(\alpha z) + R(z)y'(\alpha z)$$

can be inferred from solutions of the determining equation

$$f(z;\phi) \equiv 0. \tag{9}$$

In this case (9) corresponds to the first N equations for the existence of a formal solution. Since for a formal solution $y = \sum_{k=0}^{\infty} y_k z^k$ the coefficients y_k are determined uniquely by the preceding coefficients if k is sufficiently large (since the spectral radius $\rho(a_0) \le \|a_0\|$), every formal solution is convergent.

REMARK 1. If $a_0 = 0$, then $z = 0$ is an ordinary point for the differential equation $z(dy/dz) = P(z)y$ and the equations (9) are recursive; hence we may choose $\phi_0 = e$ and obtain a fundamental solution, i.e., a solution $y(z)$ holomorphic at $z = 0$ such that every solution $w(z)$ holomorphic at $z = 0$ can be

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written in the form

$$w(z)=y(z)w_1, \quad \text{for some } w_1 \in \mathcal{B}.$$

COROLLARY 2. *Let* $P(z)=\sum_{k=0}^{\infty} a_k z^k$ *be convergent for* $|z|<a$, *and let* a_0 *satisfy one of the following conditions*: *i*) a_0 *belongs to the center of* $\mathcal{B}$; *ii*) *no two spectral values of* a_0 *differ by an integer. We can then determine a solution of*

$$z\frac{dy}{dz}=P(z)y \tag{10}$$

of the form

$$y(z)=\sum_{k=0}^{\infty} y_k z^{k+a_0}$$

such that $y_0=e$. *The series solution for* $y(z)$ *converges in norm in* $0<|z|<a$, *and is a fundamental solution.*

Proof. Let $y(z)=w(z)z^{a_0}$. Then equation (10) becomes

$$zw'(z)+w(z)a_0=P(z)w(z).$$

Case i). If a_0 is in the center of $\mathcal{B}$, $z=0$ is an ordinary point of the equation and Remark 1 applies. The fact that the solution is fundamental in $0<|z|<a$ follows as in Hille [2].

Case ii). If a_0 is not in the center of $\mathcal{B}$, equation (10) becomes

$$zw(z)=\left(\mathcal{C}_{a_0}+\sum_{k=1}^{\infty} a_k z^k\right)w(z),$$

where $\mathcal{C}_{a_0}$ is the commutator of a_0. In this case, (9) becomes

$$(ne-\mathcal{C}_{a_0})\phi_n=\sum_{k=1}^{n} a_k\phi_{n-k}, \qquad n=0,1,\ldots,N-1.$$

Since $ne-\mathcal{C}_{a_0}$ is regular for all n, this system can be solved recursively, and by Corollary 1 the solution is convergent. The equation which determines y_0 is $\mathcal{C}_{a_0}y_0=0$, hence we may choose $y_0=e$. It again follows as in Hille [4] that the solution is fundamental.

REMARK 2. Corollary 2 was proved by Hille [4], who proved convergence by majorant series arguments.

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