FOUR-DIMENSION EQUIVALENCES

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The object of this paper is to establish the equivalence of four function-related dimension concepts in arbitrary topological spaces. These concepts involve stability of functions (**3**, p. 74), the modification of covering dimension involving basic covers (**1**, p. 243) (which is equivalent to Yu. M. Smirnov's definition using normal covers), the definition involving essential mappings (**2**, p. 496), and a modification of the closed set separation characterization of dimension in (**3**, p. 35).

In the following, as convenience demands, I^{n+1} will denote any of the sets $\{x \in E^{n+1}: |x| \leqslant 1\}$, $\{x \in E^{n+1}: -1 \leqslant x_i \leqslant 1, \quad i = 1, \ldots, n+1\}$, or the unit $(n+1)$-simplex in E^{n+2}, and S^n will correspondingly be $\{x \in E^{n+1}: |x| = 1\}$, $\{x \in E^{n+1}: |x_i| = 1, \text{ for some } i\}$, or the union of the n-faces of the unit $(n+1)$-simplex. The reader is referred to (**1**) for the terminology and the elementary properties of zero and cozero sets.

Definition 1. For the topological space X, take $d_1(X)$ to be the least integer n such that for each mapping $f\colon X \to {}^{n+1}$ and each $y \in I^{n+1}$ y is an unstable value of f (i.e. for each $\epsilon > 0$ there is a mapping $g\colon X \to I^{n+1}$ such that $|f - g| < \epsilon$ and $y \notin g(X)$).

Definition 2. Let $d_2(X)$ be the least integer n such that every basic cover of X has a basic refinement of order $\leqslant n$.

Definition 3. Let $d_3(X)$ be the least integer n such that for each mapping $f\colon X \to I^{n+1}$ there is a mapping $F\colon X \to S^n$ such that $F(x) = f(x)$ for $x \in f^{-1}(S^n)$.

Definition 4. Let $d_4(X)$ denote the least integer n such that given $n+1$ disjoint pairs C_i, C'_i, $i = 1, \ldots, n+1$, of zero sets of X there exist $n+1$ zero sets A_i such that C_i and C'_i are separated in $X - A_i$, and

$$A_1 \cap \ldots \cap A_{n+1} = \emptyset.$$

The functions d_2 and d_4 are easily seen to be equivalent in normal spaces to their associated dimension functions referred to in the first paragraph. The equivalence of d_2, d_3, and d_4 in normal Hausdorff spaces is therefore a consequence of results of Hemmingsen (**2**). The equivalence of d_2 and d_3 has been established by Smirnov (**4**, p. 19) for completely regular spaces.

THEOREM. *In a topological space X, $d_1(X) = d_2(X) = d_3(X) = d_4(X)$.*

Proof. $d_1(X) \geqslant d_3(X)$. Let $d_1(X) = n$, and let f be a mapping from X to I^{n+1}, and let $C = f^{-1}(S^n)$. As $d_1(X) = n$ there is a mapping $g\colon X \to I^{n+1}$ such that

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$|f(x) - g(x)| < \frac{1}{4}$, $x \in X$, and $g(X)$ does not contain the origin. $C_1 = \{x \in X: |g(x)| \geqslant \frac{3}{4}\}$ and $C_2 = \{x \in X: |g(x)| \leqslant \frac{1}{4}\}$ are disjoint zero sets of X and hence are completely separated. Let $k: X \to [0, 1]$ be a mapping which is 1 on C_1 and 0 on C_2, and let $f'(x) = k(x)f(x) + (1 - k(x))g(x)$, for $x \in X$. $f'(X)$ does not contain the origin and $f' = f$ on C. If $F(x)$ is taken as the radial projection of $f'(x)$ onto S^n, then $F: X \to S^n$ is continuous and agrees with f on C.

$d_3(X) \geqslant d_2(X)$. The proof is identical with that in Theorem 3.1b (**2**, p. 499), as the refining cover obtained in that proof is basic.

$d_2(X) \geqslant d_1(X)$. Let $d_2(X) = n$ and f be a mapping from X into I^{n+1} (the unit $(n + 1)$-simplex). It is sufficient to show that, for $\epsilon > 0$, there is an ϵ-approximation g to f such that $g(X)$ does not contain the barycentre q of I^{n+1}.

Let K be the complex consisting of the faces of I^{n+1} and let m be an integer sufficiently large to have the mesh of $Sd^m K < \epsilon/2$. There is a homeomorphism h defined on I^{n+1} which is composed of a radial shrinking of I^{n+1} about q followed by a translation, so that q is not in $h(s)$ for any n-simplex $s \in Sd^m K$, so that the simplex $h(I^{n+1}) \subset I^{n+1}$, and so that $|h(x) - x| < \epsilon/2$, for $x \in I^{n+1}$. Let L denote the complex composed of the faces of the simplex $h(I^{n+1})$. Then q is in no n-simplex of $Sd^m L$ and mesh $Sd^m L < \epsilon/2$. Let $p_1, \ldots, p_j$ denote the vertices of $Sd^m L$.

$\{V_1, \ldots, V_j\}$, where $V_i = (hf)^{-1}(\text{star } p_i)$, is a basic cover of X and hence has a basic refinement of order $\leqslant n$. If W_1 is the union of the elements of this refinement contained in V_1, and W_i, $i = 2, \ldots, j$, is the union of those elements of the refinement in V_i but not in any of $V_1, \ldots, V_{i-1}$, then $\{W_1, \ldots, W_j\}$ is a basic cover of order $\leqslant n$ with each $W_i \subset V_i$. There exist mappings $g_i: X \to [0, 1]$ such that $g_i^{-1}(0) = X - W_i$, $i = 1, \ldots, j$, and

$$\sum_{i=1}^{j} g_i(x) = 1, \qquad \text{for } x \in X.$$

Let $g: X \to I^{n+1}$ be defined by

$$g(x) = \sum_{i=1}^{j} g_i(x) p_i, \qquad x \in X.$$

Then g is continuous. For $x \in X$, $g_i(x) \neq 0$ for at most $n + 1$ subscripts i, and $g(x)$ is contained in some n-simplex of $Sd^m L$ having the corresponding p_i's as vertices since $hf(x)$ is common to the stars of these vertices. Thus $g(x) \neq q$ for any $x \in X$ and $|g(x) - hf(x)| < \epsilon/2$. Therefore

$$|g(x) - f(x)| \leqslant |g(x) - hf(x)| + |hf(x) - f(x)| < \epsilon.$$

$d_3(X) \geqslant d_4(X)$. A minor modification of the proof by Hemmingsen of Theorem 6.1 (**2**, pp. 502f.) yields this result since disjoint zero sets are completely separated.

$d_4(X) \geqslant d_1(X)$. Assume that $d_4(X) = n$. Let f, a mapping from X to I^{n+1}, and $\epsilon > 0$ be given.

$$C_i = f^{-1}\{x \in I^{n+1}: x_i \geqslant \epsilon_0\} \quad \text{and} \quad C'_i = f^{-1}\{x \in I^{n+1}: x_i \leqslant -\epsilon_0\},$$

where $\epsilon_0 = \epsilon/\{2(n+1)\}$, are disjoint zero sets of X for $i = 1, \ldots, n+1$. Hence there exist $n+1$ zero sets A_i, with

$$\bigcap_{i=1}^{n+1} A_i = \emptyset,$$

such that for each i there is a separation $X - A_i = U_i \cup U'_i$, where $C_i \subset U_i$ and $C'_i \subset U'_i$.

As A_k and

$$C_k \cup C'_k \cup \bigcap_{\substack{i=1 \\ i\neq k}}^{n+1} A_i$$

are disjoint zero sets, for each $k = 1, \ldots, n+1$, there exists a mapping $h_k : X \to [0, \epsilon_0]$ with $A_k = h_k^{-1}(0)$ and

$$C_k \cup C'_k \cup \bigcap_{\substack{i=1 \\ i\neq k}}^{n+1} A_i = h_k^{-1}(\epsilon_0).$$

Let $g_k: X \to [-\epsilon_0, \epsilon_0]$ be defined by: $g_k|U_k = h_k|U_k$, $g_k|U'_k = -h_k|U'_k$, and $g_k|A_k = 0$. Since g_k is continuous on each of the closed sets $U_k \cup A_k$ and $U'_k \cup A_k$, it is continuous on X. Note that $A_k = g_k^{-1}(0)$ for each $k = 1, \ldots, n+1$.

For $i = 1, \ldots, n+1$, let f_i denote the ith component of f and define $g'_i: X \to [-1, 1]$ by

$$g'_i|(C_i \cup C'_i) = f_i|(C_i \cup C'_i)$$

and

$$g'_i|(X - (C_i \cup C'_i)) = g_i|(X - (C_i \cup C'_i)).$$

Since f_i and g_i have the same value on the boundary of $C_i \cup C'_i$, g'_i is continuous. Then $g: X \to I^{n+1}$, having the g'_i as components, is continuous and since

$$\bigcap_{i=1}^{n+1} A_i = \emptyset,$$

$g(X)$ does not contain the origin. Also

$$|f(x) - g(x)| \leqslant \sum_{i=1}^{n+1} |f_i(x) - g'_i(x)| < (n+1)(2\epsilon_0) = \epsilon, \qquad \text{for all } x \in X.$$

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