On the properness of the moduli space of stable surfaces over $\mathbb{Z}[1/30]$

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ABSTRACT

We show the properness of the moduli stack of stable surfaces over $\mathbb{Z}[1/30]$, assuming the locally-stable reduction conjecture for stable surfaces. This relies on a local Kawamata–Viehweg vanishing theorem for 3-dimensional log canonical singularities at closed point of characteristic $p \neq 2, 3$ and 5, which are not log canonical centres.

Contents

1. Introduction

In [DM69, Theorem 5.2], Deligne and Mumford proved that the moduli stack of stable curves $\overline{\mathcal{M}}_g$ of given genus $g \geq 2$ is a *proper* Deligne-Mumford (DM) stack over $\mathbb{Z}$. By introducing stable curves (i.e. curves with at worst nodal singularities and ample canonical class) into the moduli problem, they were able to construct a natural compactification of the moduli of smooth curves of genus g, which led to interesting applications, such as the proof of irreducibility of $\mathcal{M}_g$ [DM69], and the proof of general semi-stable reduction for curves in [dJ97].

The natural higher-dimensional generalisation of smooth curves of genus at least 2 are smooth canonically polarised varieties. Hence, it is natural to look for a compactification of the moduli space of these. A possible approach has been proposed by Kollár and Shepherd-Barron in [KSB88] using the Minimal Model Program (MMP for short). According to Kollár and Shepherd-Barron, the correct generalisation of stable curves to arbitrary dimensions are *stable varieties*, projective varieties with semi-log canonical singularities and ample canonical class. We refer to the book [Kol23b] for a comprehensive treatment of the construction of the moduli space of stable varieties in characteristic 0.

The case of positive and mixed characteristic presents further difficulties. To mention a few: the MMP is still largely conjectural in dimension > 3, the invariance of plurigenera (even asymptotic) is known to fail [Bri23, Kol23a], the singularities of the MMP are cohomologically more complicated [CT19, Ber19] and other problems arise due to presence of inseparable morphisms [Kol23b, Section 8.8]. However the MMP for 3-folds in positive characteristic $p \geq 5$ and mixed characteristic $(0, p > 5)$ has now been established [HX15, CTX15, Bir16, BW17, DW22, HW22, TY23, BMP+23] and, following the strategy in characteristic 0, many of the steps needed for the construction of the moduli space $\overline{\mathcal{M}}_{2,v}$ of stable surfaces have been proven in [HK19, Pat17, BMP+23, Pos21a, Pos21b]. Nowadays, we know that $\overline{\mathcal{M}}_{2,v}$ exists as a separated Artin stack with finite diagonal over $\mathbb{Z}[1/30]$ by [BMP+23, Corollary 10.2], but whether it is proper remains still an open question. Our main result is the following, where locally-stable reduction means a weakening of semi-stable reduction; see Definition 4.11 for the precise definition.

Theorem 1.1. *Assume the existence of locally-stable reduction for surfaces. Then the moduli stack $\overline{\mathcal{M}}_{2,v}$ of stable surfaces of volume v is proper over $\mathbb{Z}[1/30]$.*

The main technical result needed to prove Theorem 1.1 concerns the depth of 3-dimensional log canonical singularities, which we briefly explain. In [Pos21b, Theorem 6.0.5], Posva reduced the valutative criterion of properness for the stack $\overline{\mathcal{M}}_{2,v}$ to two conjectures on families of stable surfaces over a DVR: roughly speaking, the existence of semi-stable reduction and the (S_2)-condition of the central fibre of a locally stable family of surfaces. To prove the (S_2)-condition of the central fibre, it is thus natural to study the (S_3)-condition at a closed point x of a log canonical 3-fold singularity.

Let us first explain what the tools used to prove the (S_2)-condition in characteristic 0 are, as we will mimic this approach. In [Kol13, Theorem 7.20], a local version of the Kawamata–Viehweg

vanishing theorem concerning the depth of divisorial sheaves on divisorially log terminal (dlt) and log canonical pairs in characteristic 0 is presented (similar results were obtained previously by Alexeev and Hacon [Ale08, AH12]).

THEOREM 1.2 (Local Kawamata–Viehweg vanishing for log canonical pairs). *Let* (X, Δ) *be a log canonical pair over a field of characteristic 0. Let* D *be a* $\mathbb{Z}$-*divisor such that* $D \sim_{\mathbb{Q}} \Delta'$ *where* $0 \le \Delta' \le \Delta$. *If* x *is a point that is not the generic point of an lc centre, then*

$$\mathrm{depth}_x \mathcal{O}_X(-D) \ge \min \{3, \mathrm{codim}_X x\}.$$

This local vanishing is one of the crucial ingredients for the properness of the moduli functor as shown in [Kol23b, Definition-Theorem 2.3], where 1.2 is used to prove the (S_2)-condition on the central fibre of a locally stable family. For this reason it is natural to consider whether Theorem 1.2 remains true in positive and in mixed characteristics. The examples of klt not CM 3-fold singularities (see [CT19, Ber21, ABL22]) show that Theorem 1.2 is false in equicharacteristic $p \le 5$. On the contrary, in [ABL22] the first two authors showed together with Lacini that 3-fold klt singularities are Cohen–Macaulay in characteristic $p > 5$, and this was later extended by the second author and Kollár in [BK23, Theorem 17] to a local Kawamata–Viehweg vanishing on 3-dimensional excellent dlt singularities whose residue field is perfect of characteristic $p > 5$ (analogue to [Kol13, Theorem 7.31]). Moreover, in [PS14, Theorem 3.8], the third author and Schwede prove a local Kawamata–Viehweg vanishing for sharply F-pure singularities. From all these results, it would be natural to expect an analogue of Theorem 1.2 for 3-dimensional log canonical singularities to hold, at least in large characteristics. Unfortunately, we show that this is not the case.

THEOREM 1.3 (See Section 5). *For every prime* $p > 0$, *there exist a 3-dimensional log canonical singularity* $x \in X$ *such that*

(i) *the residue field of the closed point* x *is perfect of characteristic* p;

(ii) x *is not a minimal log canonical centre;*

(iii) $\mathrm{depth}(\mathcal{O}_{X,x}) = 2$.

Nevertheless, we are able to obtain a weaker local Kawamata–Viehweg vanishing statement, which is sufficient to deduce the properness of the moduli space $\overline{\mathcal{M}}_{2,v}$. See Section 2.1 for the notion of pair used in the article.

THEOREM 1.4. *Let* $C \subset (X, \Delta)$ *be a 1-dimensional minimal log canonical centre of a 3-dimensional log canonical pair* $(X, X_0 + \Delta)$, *and let* $x \in C$ *be a closed point with perfect residue field of characteristic* $p \ne 2, 3$ *and* 5. *If* X_0 *is Cartier and* $x \in X_0$, *then* $\mathcal{O}_{X,x}$ *is* (S_3) *and* X_0 *is* (S_2) *at* x.

To prove Theorem 1.4, we find a clear geometric reason for the failure of the (S_3)-condition at a closed point x of a log canonical 3-dimensional singularity X, which is not a log canonical centre.

THEOREM 1.5. *Let* (X, Δ) *be a 3-dimensional log canonical pair on the spectrum of a local ring, such that the residue field of the closed point* x *is perfect of characteristic* $p \ne 2, 3, 5$. *Let* $C \subset X$ *be a 1-dimensional minimal log canonical centre for* (X, Δ). *Then there exists a proper birational modification* $f : Z \to X$ *such that*

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(i) *Z is Cohen–Macaulay,*

(ii) *the exceptional divisor E is (S_2), and for each point $c \in C$, E is normal at the generic points of the fibre E_c;*

(iii) $H_x^2(X, \mathcal{O}_X) \simeq H_x^0(C, R^1 f_* \mathcal{O}_E).$

In particular, if $\mathcal{O}_{X,x}$ is not (S_3), then $E \to C$ has a wild fibre over x.

We now give an overview of the article. In Section 2, we collect the various technical results on surfaces and 3-folds that we need for our proofs. In Section 3, using the MMP for 3-folds and the Kawamata–Viehweg vanishing for log canonical surfaces admitting a morphism to a curve, we show Theorem 1.5. In Section 4, we review the theory of wild fibers of a fibred surface $f \colon E \to C$, developed by Raynaud in [Ray70], which we apply in combination with Theorem 1.5 to conclude 1.4 in 4.2. In Subsection 4.3, we combine the previous results to show Theorem 1.1 and we also present an application to the asymptotic invariance of plurigenera for minimal models of log canonical surfaces of log general type. In Section 5, we show the counterexample 1.3 by applying a relative cone construction to an elliptic surface fibration with a wild fibre.

Remark 1.6. While completing this work, the first author has found an alternative construction [Arv23, Theorem 2] that can replace the role played by Theorem 1.5 in this work. This construction simplifies some of the technical arguments in this article. Indeed, the main technical difficulty in the present approach is that in Theorem 1.5, the modification Z together with the crepant bounday is not, in general, dlt, but only étale dlt. In [Arv23, Theorem 2] the first author proves an analogous statement using a possibly non-$\mathbb{Q}$-factorial dlt modification. We believe that the various vanishing statements discussed here may be of independent interest.

2. Preliminaries

2.1 Basic notation

Notation 2.1. Throughout this article we work over a fixed base ring R, and X, Y and Z always denote quasi-projective schemes of pure dimension n over R, unless otherwise stated.

The base ring R is always be assumed to be Noetherian, excellent, of finite Krull dimension, and admitting a dualising complex $\omega_R^\bullet$. Furthermore, R will always be assumed to be of pure dimension d. Here, and in general in the present article, dimension means the absolute dimension, not the relative dimension over R.

We normalise $\omega_R^\bullet$ as in [BMP$^+$23, Section 2.1]: $H^{-i}(\omega_R^\bullet) = 0$ if $i > d$ and with $H^{-d}(\omega_R^\bullet) \neq 0$. The first non-zero cohomology sheaf $\omega_R := H^{-d}(\omega_R^\bullet)$ is the dualising sheaf of R. For the upper-shriek functor, we follow the convention of [Sta, Tag 0A9Y]. By [Sta, Tag 0AA3], the complex $\omega_X^\bullet := \pi^! \omega_R^\bullet$ is a dualising complex for X, where $\pi \colon X \to \operatorname{Spec}(R)$ is the structure morphism. We then define the dualising sheaf ω_X of X to be the first non-zero cohomology sheaf of the complex $\omega_X^\bullet$.

We say that X is a curve (resp. a surface, a 3-fold) if it is a connected reduced scheme of dimension 1 (resp. 2, 3). We say a proper morphism $f \colon X \to Y$ is a *contraction* if $f_* \mathcal{O}_X = \mathcal{O}_Y$.

Given a closed subscheme Z of X, we denote by $\Gamma_{Z,X}$ the functor of global sections with support on Z. The induced right-derived functor is denoted by $R\Gamma_{Z,X}$ and its i-th cohomology by $R^i \Gamma_{Z,X}$ (or $H_Z^i(X, -)$). These groups are called the i-th local cohomology groups with support on Z.

A Weil $\mathbb{Q}$-divisor D on a connected reduced scheme X is a formal sum of codimension 1 integral subschemes with rational coefficients. As we will work with non-normal schemes, we recall the definition of the more restrictive class of Mumford divisors following [Kol23b].

DEFINITION 2.2. A Weil $\mathbb{Q}$-divisor B on X is called a Mumford $\mathbb{Q}$-divisor if X is regular at all generic points of $\operatorname{Supp} B$.

Equivalently, B is Mumford if $\operatorname{Supp} B$ does not contain any irreducible component of codimension 1 of the divisorial part of the conductor $D \subset X$. We say a Mumford $\mathbb{Q}$-divisor D is $\mathbb{Q}$-Cartier if there exists $n > 0$ such that nD is a Cartier divisor. We refer to [BMP$^+$23, Section 2.5] for the various notions of positivity (as ample, nef, big) for $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. When $R = k$ is a field and X is integral, then for a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L, the *numerical dimension* of L is $\nu(L) := \max \{n \geq 0 \mid L^n \neq 0\}$.

DEFINITION 2.3. We say (X, Δ) is a couple if

(i) X is a reduced, pure-dimensional, (S_2) and (G_1) (where the latter means that ω_X is locally free at codimension 1 points of X) scheme,

(ii) Δ is an effective Mumford $\mathbb{Q}$-divisor.

An open set $U \subset X$ is *big* if $\operatorname{codim}_X(X \setminus U) \geq 2$. If X is (S_2) and (G_1), then any reflexive sheaf is determined on big open sets, and a Mumford divisor D defines a reflexive sheaf $\mathcal{O}_X(D)$ [Har94]. By [Kol13, Paragraph 5.6], in this case the Mumford class group coincides with the group of isomorphism classes of reflexive sheaves of rank 1 that are locally free in codimension 1. As a consequence, if X is pure-dimensional, (S_2) and (G_1), then there exists a Mumford divisor K_X such that $\mathcal{O}_X(K_X) \simeq \omega_X$ (note that ω_X is reflexive by [Sta, Tag 0AWN]).

DEFINITION 2.4. We say a couple (X, Δ) is a pair if $K_X + \Delta$ is a Mumford $\mathbb{Q}$-Cartier divisor.

If (X, Δ) is a pair and X is normal, for every proper birational morphism of normal schemes $\pi \colon Y \to X$, we can write

$$K_Y + \pi_*^{-1}\Delta = \pi^*(K_X + \Delta) + \sum_i a(E_i, X, \Delta)E_i,$$

where E_i run through the π-exceptional divisors and $a(E_i, X, \Delta) \in \mathbb{Q}$ are called the *discrepancies* of E_i with respect to (X, Δ). We define $\Delta_Y := \pi_*^{-1}\Delta - \sum_i a(E_i, X, \Delta)E_i$ as the *crepant pull-back* of Δ on Y. We say that (X, Δ) is a *klt* (resp. log canonical or *lc*) pair if X is normal and if for every proper birational maps of normal schemes $\pi \colon Y \to X$, $\lfloor \Delta_Y \rfloor \leq 0$ (resp. the coefficients of $\lfloor \Delta_Y \rfloor$ are ≤ 1).

We say that a pair (X, D) of pure dimension n is *snc* (=simple normal crossings) if for every closed point $x \in X$, X is regular at x and if there exists local coordinates $t_1, \ldots, t_n$ such that $\operatorname{Supp}(D) \subset (t_1 \cdots t_n = 0)$. Note that being snc is a local property in the Zariski topology (but not in the étale topology). We denote by $\operatorname{nsnc}(X, \Delta)$ the non-snc locus of (X, Δ). We say that (X, Δ) is *dlt* if it is log canonical, and for every exceptional divisor E such that $\operatorname{cent}_X(E) \subset \operatorname{nsnc}(X, \Delta)$, we have $a(E_i, X, \Delta) > -1$.

DEFINITION 2.5. Let (X, Δ) be a pair. The étale-snc locus $\operatorname{etsnc}(X, \Delta)$ is the locus where (X, Δ) is snc in the étale topology. This is a Zariski open set of X.

We say that a pair (X, Δ) is *étale-dlt* if for every exceptional divisor E over X such that $\operatorname{cent}_X E \subseteq X \setminus \operatorname{etsnc}(X, \Delta)$ we have $a(E; X, \Delta) > -1$.

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2.2 Semi-log canonical singularities

If X is reduced, then we can consider its normalisation morphism $\pi\colon \overline{X} \to X$ (see [Sta, Tag 035N]). The *conductor* ideal of π is the largest ideal $\mathcal{I}$ of $\mathcal{O}_X$, which is also an ideal of $\mathcal{O}_{\overline{X}}$. It can be also defined explicitly in multiple ways:

$$\mathcal{I} = \big\{ s \in \mathcal{O}_X \mid s \cdot \pi_* \mathcal{O}_{\overline{X}} \subseteq \mathcal{O}_X \text{ as subsheaves of the field of total fractions} \big\} =$$

$$\mathrm{im}\Big(\big\{ \phi \in \mathrm{Hom}_{\pi_* \mathcal{O}_{\overline{X}}}(\pi_* \mathcal{O}_{\overline{X}}, \pi_* \mathcal{O}_{\overline{X}}) \mid \mathrm{im}\,\phi \subseteq \mathcal{O}_X \big\} \to \mathcal{O}_X \Big) = \mathrm{im}(\mathcal{H}om_{\mathcal{O}_X}(\pi_* \mathcal{O}_{\overline{X}}, \mathcal{O}_X) \to \mathcal{O}_X).$$

The *conductor subscheme* D of X (resp. $\overline{D}$ of $\overline{X}$) is the subscheme defined by $\mathcal{I}$ in X (resp. in $\overline{X}$).

We recall the definition of the singularities of the MMP for non-normal varieties, following [Kol13]. We start by explaining what a node is.

DEFINITION 2.6 [Kol13, 1.41]. We say that a scheme S has a node at a codimension 1 point $s \in S$ if $\mathcal{O}_{S,s} \simeq A/(f)$, where $(A, \mathfrak{m})$ is a regular local ring of dimension 2, $f \in \mathfrak{m}^2$ and f is not a square in $\mathfrak{m}^2 \setminus \mathfrak{m}^3$. Sometimes we equivalently say that S is nodal at $s \in S$.

Remark 2.7. Let $(A, \mathfrak{m})$ be a regular local ring of dimension 2, and let $f \in \mathfrak{m}^2$ such that $\mathrm{Spec}(A/(f))$ is a node. It is easy to see that the effective divisor given by $(f = 0)$ has multiplicity 2 at the closed point of $\mathrm{Spec}(A)$ and that f is not irreducible if and only if the pair $(W, D) := (\mathrm{Spec}(A), (f = 0))$ is snc. Examples where the pair is not snc are given in [Kol13, Examples, page 1]. If $\mathrm{char}(k(s)) \neq 2$, then it is clear that there exists an étale neighborhood V of W for which (V, D_V) is snc.

DEFINITION 2.8. The scheme X (quasi-projective over R as assumed in Notation 2.1) is said to be demi-normal if it is pure-dimensional, it satisfies Serre's condition (S_2) and its codimension 1 points are either regular or nodal.

If X is demi-normal, then D and $\overline{D}$ are reduced closed subschemes of pure codimension 1 (see [Kol13, Line 14 of page 189]. We use the following definition of semi-log canonical pairs in the present article:

DEFINITION 2.9. We say that (X, Δ) is a semi-log canonical pair (or slc) if

(i) X is demi-normal and (X, Δ) is a pair;
(ii) the normalised pair $(\overline{X}, \overline{D} + \overline{\Delta})$ is log canonical, where $\overline{D}$ is the conductor subscheme.

Note that in Definition 2.9, $(\overline{X}, \overline{D} + \overline{\Delta})$ is automatically a pair, as it is crepant to (X, Δ). As in this article we are interested in understanding the locus of strictly log canonical singularities, we recall the terminology on log canonical places and centres.

DEFINITION 2.10. Let (X, Δ) be a pair. We denote by $\mathrm{nklt}(X, \Delta)$ the non-klt locus of (X, Δ), which is the closed subset of X consisting of points x of X for which (X, Δ) is not klt near x.

Let (X, Δ) be a log canonical pair. We say that an irreducible exceptional divisor E for proper birational modification $f\colon Y \to X$ is a log canonical place if $a(E, X, \Delta) = -1$. A closed subset $Z \subset X$ is a log canonical centre if there exists a log canonical place E such that $\mathrm{cent}_X(E) = Z$.

We recall the construction of double covers of demi-normal varieties explained in [Kol13, 5.23]. This allows to reduce many questions to slc pairs whose irreducible components are regular in codimension 1.

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PROPOSITION 2.11 [Kol13, 5.23]. *Let (X, Δ) be an slc pair such that $\frac{1}{2} \in \mathcal{O}_X$. Then there exists a finite morphism $\pi \colon \widetilde{X} \to X$ of degree 2 such that*

(i) *$\widetilde{X}$ is (S_2);*

(ii) *π is étale in codimension 1;*

(iii) *the irreducible components of $\widetilde{X}$ are (R_1) (i.e. regular in codimension 1);*

(iv) *the normalisation of $\widetilde{X}$ is a disjoint union of two copies of the normalisation of X;*

(v) *if $K_{\widetilde{X}} + \Delta_{\widetilde{X}} = \pi^*(K_X + \Delta)$, the pair $(\widetilde{X}, \Delta_{\widetilde{X}})$ is slc.*

We need also a non-normal version of dlt-ness. We refer to [Kol13, Definition 1.10] for the definition of semi-snc pair.

DEFINITION 2.12. An slc pair (X, Δ) is semi-dlt if $a(E, X, \Delta) > -1$ for every exceptional divisor E such that the generic point of $\mathrm{cent}_X E$ is contained in the locus where (X, Δ) is not semi-snc, where "semi-snc" is defined in [Kol13, Def 1.10].

As for dlt, the notion of semi-dlt is not local in the étale topology.

2.3 Log canonical surface singularities

In this section we collect some results on 2-dimensional excellent surface singularities, relying on the classification scheme of [Kol13, Section 3.3].

Notation 2.13. Besides the assumptions on our base ring R stated in Notation 2.1, in the present section we suppose that R is integrally closed, local and of dimension 2 with maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$. Additionally, we set $X = \mathrm{Spec}\, R$, and we set Δ to be a $\mathbb{Q}$-divisor on X for which $(X, \Delta) = (\mathrm{Spec}(R), \Delta)$ is a pair. We denote by $x \in X$ the closed point of X.

DEFINITION 2.14. Let $\pi \colon Y \to X$ be a projective birational morphism of normal surfaces. We say that π is a log minimal resolution of (X, Δ) if

(i) *Y is regular and $\pi_*^{-1}\lfloor\Delta\rfloor$ is regular (as a closed subscheme);*

(ii) *$K_Y + \pi_*^{-1}\Delta$ is π-nef;*

(iii) *$\mathrm{mult}_y \pi_*^{-1}\Delta \le 1$ for every $y \in Y$;*

(iv) *the support of $\mathrm{Ex}(\pi) + \pi_*^{-1}\Delta$ has a node at every intersection point of $\mathrm{Ex}(\pi)$ and $\pi_*^{-1}\lfloor\Delta\rfloor$.*

Remark 2.15. The existence of a log minimal resolution for surfaces is proven in [Kol13, Theorem 2.25.a]. The construction goes as follows: if $f \colon W \to X$ is a projective log resolution of (X, Δ) such that $f_*^{-1}\Delta$ is regular, then Y is obtained as the output of a $(K_W + f_*^{-1}\Delta)$-MMP over X.

We need a slightly modified version of the above in the case of dlt surfaces.

LEMMA 2.16. *Assume that (X, Δ) is a dlt surface pair. Then there exists a projective birational morphism $\pi \colon Y \to X$ such that*

(i) *Y is regular and $K_Y + \pi_*^{-1}\Delta$ is π-nef;*

(ii) *if $K_Y + \Gamma \sim_{\mathbb{Q}} \pi^*(K_X + \Delta)$, then $\lfloor\Gamma\rfloor = \pi_*^{-1}\lfloor\Delta\rfloor$;*

(iii) *the support of $\mathrm{Ex}(\pi) + \pi_*^{-1}\Delta$ has a node at every intersection point of $\mathrm{Ex}(\pi)$ and $\pi_*^{-1}\lfloor\Delta\rfloor$.*

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Proof. Let $f\colon W \to X$ be a thrifty log resolution of (X, Δ) [Kol13, Lemma 2.79]. As X is a surface, f being thrifty means that it is an isomorphism at the nodes of $\lfloor \Delta \rfloor$. This we can achieve by running our resolution algorithm by excluding the nodes of $\lfloor \Delta \rfloor$.

Then run a $(K_W + f_*^{-1}\Delta)$-MMP over X ending with $\pi\colon Y \to X$ such that $K_Y + \pi_*^{-1}\Delta$ is π-nef. As this is also a K_W-MMP, we deduce that Y is regular by [Kol13, Theorem 2.29]. As f does not extract log canonical places, (ii) is immediate. To finally verify (iii), we argue as in the proof of [Kol13, Theorem 2.25.a]. □

Our next goal is a precise understanding of the exceptional divisor of a log minimal resolution of a log canonical singularity. We start by recalling [Kol13, Theorem 2.31] on the reduced boundary of 2-dimensional log canonical singularities.

THEOREM 2.17. *Assume that $(X, \Delta = E + D)$ is log canonical, where $E = \sum_i E_i$ has only coefficients 1. Then either:*

(i) *E is regular at x, or*

(ii) *E has a node at x, no components of the support of D contain x and every exceptional divisor of a minimal log resolution has discrepancy -1.*

We will need the following observation on conics.

LEMMA 2.18. *Let k be a separably closed field, and let C be a k-projective integral Gorenstein curve. Suppose that C is singular and char$k \neq 2$. Then $\deg_k \omega_{C/k} \geq 0$.*

Proof. Without loss of generality we can suppose $k = H^0(C, \mathcal{O}_C)$. Suppose by contradiction that $\deg_k \omega_{C/k} < 0$. By [Kol13, Lemma 10.6], C embeds as a conic in $\mathbb{P}_k^2$. Taking the base change to $\overline{k}$, we still get an embedding $C_{\overline{k}} \to \mathbb{P}_{\overline{k}}^2$. As k is separably closed, $C_{\overline{k}}$ is still an irreducible conic, and by the classification of conics over an algebraically closed field, either $C_{\overline{k}}$ is regular or it is a double line. Note that the case of a double line cannot appear as char$(k) \neq 2$ by [BT22, Lemma 2.17]. Finally, if $C_{\overline{k}}$ is regular, we deduce C is regular by descent for faithfully flat morphisms [Sta, Tag 033E], getting a contradiction. □

Example 2.19. The following examples show that the assumptions in Lemma 2.18 are sharp. Let k be a field, and consider the conic

$$C := \left\{ x^2 - uy^2 = 0 \right\} \subset \mathbb{P}_k^2 = \mathrm{Proj}\, k[x, y, z],$$

where $u \in k$. By the Jacobian criterion [Sta, Tag 07PF], it is easy to see that the only non-regular point of C is $p = [0:0:1]$. Note that $\deg_k \omega_{C/k} = -2$. This example shows that the assumptions of Lemma 2.17 are indeed necessary:

- If char$k \neq 2$ and $u \notin k^2$, then C is integral, singular and with $\deg_k \omega_{C/k} < 0$, but k is not separably closed.
- If char$k \neq 2$ and k is separably closed, then C is singular and with $\deg_k \omega_{C/k} < 0$, but C is not integral.
- If char$k = 2$, k is separably closed and $u \notin k^2$, then C is integral, singular and with $\deg_k \omega_{C/k} < 0$. Geometrically, $C_{\overline{k}}$ is a double line.

PROPOSITION 2.20. *Assume that (R, m) is strictly Henselian with char$k = p \neq 2$, and that (X, Δ) is log canonical such that Δ is a $\mathbb{Q}$-divisor. Let $\pi\colon Y \to (X, \Delta)$ be a log minimal resolution as in Definition 2.14. Then one of the following holds:*

(i) $\Delta = 0$ and there exists an irreducible nodal curve $E \subset \mathrm{Ex}(\pi)$. Then $\mathrm{Ex}(\pi) = E$, $K_Y + E \equiv_X 0$ and (Y, E) is étale-snc;

(ii) $(Y, \pi_*^{-1}\lfloor\Delta\rfloor + \mathrm{Ex}(\pi))$ is snc.

Proof. The proof is case by case.

$\boxed{\textit{There exists a singular exceptional curve } E \subset \mathrm{Ex}(\pi)\textit{:}}$ By Lemma 2.18, $\deg_k \omega_{C/k} \geq 0$ in this case. By [Kol13, 3.30.1], then $\Delta = 0$, and E is the unique exceptional divisor. Let $-1 \leq a \in \mathbb{Q}$ be the discrepancy of E. By adjunction we have

$$0 \leq \deg K_E \leq (K_X + E) \cdot E = (K_X - aE) \cdot E + (1+a)E^2 = (1+a)E^2 \leq 0. \qquad (1)$$

In particular, we have equality everywhere. Taking into account that $E^2 < 0$, this means that $a = -1$. We obtain that (Y, E) is log canonical. Taking into account that Y is regular, we deduce that (Y, E) is étale-snc. To see $K_Y + E \equiv_X 0$, we simply note that $(K_Y + E) \cdot E = 0$ by Equation (1).

$\boxed{\textit{All irreducible component } E_i \textit{ of } \mathrm{Ex}(\pi) \textit{ are regular:}}$ Note that $\pi_*^{-1}\lfloor\Delta\rfloor$ is regular and that $\mathrm{Ex}(\pi) + \pi_*^{-1}\lfloor\Delta\rfloor$ is snc at intersection points by construction of the minimal resolution, in this case. We are left to prove that $\mathrm{Ex}(\pi)$ is snc. We can suppose that all irreducible components are conics by [Kol13, 3.30.2], and we fix $r_i := \dim_k H^0(E_i, \mathcal{O}_{E_i})$.

$\boxed{\textit{All the } E_i \textit{ are regular, and } (E_i \cdot E_j) > \max\{r_i, r_j\} \textit{ for some } i \neq j\textit{:}}$ In this case $\Delta = 0$ and there are two exceptional curves E_1 and E_2, by [Kol13, 3.30.3]. Consider the following computation:

$$0 \leq E_1 \cdot E_2 + \deg K_{E_1} \leq (K_X + E_1 + E_2) \cdot E_1$$
$$= (K_X - a_1 E_1 - a_2 E_1) \cdot E_1 + (1+a_1)E_1^2 + (1+a_2)E_1 \cdot E_2 = (1+a_1)E_1^2 + (1+a_2)E_1 \cdot E_2$$

This implies that

$$-(1+a_1)E_1^2 \leq (1+a_2)E_1 \cdot E_2. \qquad (2)$$

By applying the same argument to E_2 instead of to E_1, we obtain

$$-(1+a_2)E_2^2 \leq (1+a_1)E_1 \cdot E_2. \qquad (3)$$

Multiplying (2) and (3) together, we obtain the following, where we are also using that both sides of the two inequalities are non-negative:

$$(1+a_2)(1+a_1)(E_1^2)(E_2^2) \leq (1+a_1)(1+a_2)(E_1 \cdot E_2)^2. \qquad (4)$$

In other words, either one of the a_i is equal to -1, or the determinant of the intersection matrix is non-positive. The latter contradicts the negative definiteness of the intersection matrix; hence, we obtain that one of the a_i is -1. By symmetry we can assume that $a_1 = -1$. However, then (3) says that $(1+a_2)E_2^2 \geq 0$. As $E_2^2 < 0$, this implies also that $a_2 = -1$.

In particular, $(Y, E_1 + E_2)$ are log canonical, and hence, by adjunction, so are $(E_1, E_1 \cap E_2)$ and $(E_2, E_1 \cap E_2)$. This means that the coefficients of $E_1 \cap E_2$ are 1 on both E_1 and E_2. As k is separably closed, all finite non-trivial extensions of k have degree divisible by p. As $E_1 \cdot E_2 = 2$ and $p \neq 2$, in fact $E_1 \cap E_2$ contains only points with residue field equal to k. As above we have seen that the coefficients of these points cannot be more than 1, so we obtain that $E_1 \cap E_2$ has two distinct points with coefficient 1 and, hence, that the intersection of E_1 and E_2 is transversal.

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In particular, $(Y, E_1 + E_2)$ is snc (and the singularity is a cusp with the exceptional divisor of the minimal resolution being a cycle of two conics).

All the E_i are regular, and $(E_i \cdot E_j) = \max\{r_i, r_j\}$ for all i and j: fix two components E_i and E_j. We may assume by symmetry that $r_i \geq r_j$. In particular, the intersection scheme $E_i \cap E_j$ is a length one Artinian scheme over $H^0(E_i, \mathcal{O}_{E_i})$. This implies that $E_i \cap E_j$ is reduced. Hence, the intersection of E_i and E_j is transversal, which concludes our proof. $\square$

The following is well-known:

LEMMA 2.21. *Let $f \colon Y \to Z$ be a projective birational morphism of normal surfaces over R, and let D be a nef $\mathbb{Q}$-Cartier divisor on Y. If $f_* D$ is $\mathbb{Q}$-Cartier, then it is nef.*

Proof. Let C be a curve on Z, mapping to a closed point of $\mathrm{Spec}(R)$. By projection formula for the Mumford pull-back, we conclude that $f_* D \cdot C = D \cdot f^* C \geq 0$. $\square$

COROLLARY 2.22. *Assume that the characteristic of k is $p \neq 2$ and that $(X = \mathrm{Spec}(R), \Delta)$ is log canonical. Then there exists a projective birational morphism $f \colon Z \to X$ such that*

(i) $(Z, f_*^{-1} \lfloor \Delta \rfloor + \mathrm{Ex}(f))$ *is étale-dlt,*
(ii) $K_Z + f_*^{-1} \Delta + \mathrm{Ex}(f) = f^*(K_X + \Delta)$, *and*
(iii) $-\mathrm{Ex}(f)$ *is nef over X.*

Proof. First, we may assume that R is strictly Henselian. Second, let $\pi \colon Y \to X$ be as in Definition 2.14. By Proposition 2.20, there are two cases. In case i, $\Delta = 0$ $(Y, \mathrm{Ex}(\pi))$ is étale-snc and $\mathrm{Ex}(\pi)$ is a single exceptional divisor; necessarily, then, anti-f-nef. In this case, $f := \pi$ satisfies the assertion of the theorem.

In case ii, the pair $(Y, \pi_*^{-1} \lfloor \Delta \rfloor + \mathrm{Ex}(\pi))$ is snc. By [Tan18, Theorem 1.1.(QF)], we can run a $(K_Y + \pi_*^{-1}\Delta + \mathrm{Ex}(\pi)) \equiv_X (\sum_i (1 + a(E_i, X, \Delta) E_i)$-MMP over X, denoted by $\rho \colon Y \to Z$, ending with a minimal model $f \colon Z \to X$. By a standard application of the negativity lemma [BMP$^+$23]*Lemma 2.16, the birational contraction ρ contracts exactly the π-exceptional divisors with discrepancy $a(E, X, \Delta) > -1$, and thus Z is a $\mathbb{Q}$-factorial surface with $(K_Z + f_*^{-1}\Delta + \mathrm{Ex}(f)) = f^*(K_X + \Delta)$. As a $(K_Y + \pi_*^{-1}\Delta + \mathrm{Ex}(\pi))$-MMP over X is a $(K_Y + \pi_*^{-1}\lfloor \Delta \rfloor + \mathrm{Ex}(\pi))$-MMP,[1] the pair $(Z, f_*^{-1}\lfloor \Delta \rfloor + \mathrm{Ex}(f))$ remains dlt. By the definition of the log minimal resolution, $-\sum_i a(E_i, X, \Delta) E_i$ is nef over X, and therefore, so is $-\mathrm{Ex}(f) = \rho_*(-\sum_i (1 + a(E_i, X, \Delta)) E_i)$ by Lemma 2.21. $\square$

2.4 Dlt modifications and log canonical centers

In this section, we recall dlt modifications and apply their existence to the study of log canonical centres of log canonical 3-folds. Since we will need the MMP developed in [BMP$^+$23], we suppose the following:

Notation 2.23. Besides the assumptions on our base ring R stated in Notation 2.1, we suppose that the characteristic of the residue fields of R are different from $2, 3$ and 5.

DEFINITION 2.24. Let (X, Δ) be a log canonical pair. A proper birational morphism $\pi \colon (Y, \Delta_Y) \to (X, \Delta)$ is a dlt modification (or a dlt blow-up) if (Y, Δ_Y) is dlt, where $K_Y + \Delta_Y \sim_{\mathbb{Q}}$

---

[1]This is because for any effective divisor D on a surface X, the strict transform $\pi_*^{-1}D$ is nef over X.

$\pi^*(K_X + \Delta)$ and $\Delta_Y = f_*^{-1}\Delta + E$, where E denotes the divisorial part of the exceptional locus of π.

The existence of a dlt modification for (X, Δ), extracting only divisors of discrepancies -1, is a standard consequence of the MMP (see [BMP+23, Corollary 9.21]).

PROPOSITION 2.25. *Let* (X, Δ) *be a log canonical 3-fold pair. Then there exists a dlt modification* $Y \to (X, \Delta)$.

We recall some properties of log canonical centres on log canonical excellent 3-fold pairs.

PROPOSITION 2.26. *Any intersection of log canonical centres of a 3-dimensional log canonical pair* (X, Δ) *is a union of log canonical centres.*

Proof. This is [FW20, Corollary 1.7] (see also [Pos21b, Corollary 5.2.16] for a proof in the case X is defined over $\mathbb{F}_p$). $\square$

We will need the following characterisation of plt pairs.

COROLLARY 2.27. *Let* $(X, \Delta = D + B)$ *be a 3-dimensional log canonical pair, where* D *is a prime divisor. If* D *is a minimal log canonical centre, then* (X, Δ) *is plt in a neighbourhood of* D.

Proof. Suppose by contradiction that (X, Δ) is not plt around D. By definition, there exists a log canonical centre S such that $Z := S \cap D$ is not-empty and that $\operatorname{codim}_X Z \geq \operatorname{codim}_X S > 1$. As Z is a union of log canonical centres by Proposition 2.26, this contradicts the minimality of D. $\square$

We will need the following technical result on dlt singularities:

LEMMA 2.28. *Let* $\pi \colon (Y, \Delta_Y) \to (X, \Delta)$ *be a proper crepant birational contraction of* $\mathbb{Q}$-*factorial pairs. If* (Y, Δ_Y) *is dlt and if* $a(E, X, \Delta) > -1$ *for every* π-*exceptional divisor* E, *then* (X, Δ) *is dlt as well.*

Proof. Let E be a log canonical place over (X, Δ). As (Y, Δ_Y) is dlt, the generic point of $Z = \operatorname{cent}_Y(E)$ is a stratum of $(Y, \Delta_Y^{=1})$, and E is already a log canonical place of $(Y, \Delta_Y^{=1})$. This in particular implies that $Z \not\subseteq \operatorname{Supp}\Delta_Y^{<1}$. However, as both X and Y are $\mathbb{Q}$-factorial, the exceptional locus of π is purely divisorial. So, putting the last two sentences and the assumption on the discrepancies of the π-exceptional divisors together, we obtain that $Z \not\subseteq \operatorname{Ex}(\pi)$. However, that means that π is an isomorphism around the generic point of $\pi(Z)$, and therefore $\pi(Z)$ is also a stratum of $(X, \Delta^{=1})$. $\square$

Note that the $\mathbb{Q}$-factoriality hypothesis in Lemma 2.27 is needed as shown in [Fuj07, Example 3.8.4].

2.5 A restriction sequence for pairs

In this section, we refine the short exact sequences used in [HW19, BK23]. We start by recalling some general properties of codimension 1 strata of dlt pairs.

LEMMA 2.29. *Let* $(X, \sum_{i \in I} E_i + \Delta)$ *be a dlt pair, where* E_i *are prime divisors and* $\lfloor \Delta \rfloor = 0$. *Then*

(i) E_i *is* (R_1) *(i.e.* E_i *is regular in codimension 1) for every* $i \in I$;

(ii) the normalisation $n\colon \bigcup E_i^n \to \bigcup E_i$ is the disjoint union of the (S_2)-ifications of the E_i's, and it factorises through the (S_2)-ification $\nu\colon E^\nu \to \bigcup E_i$;

(iii) if X is $\mathbb{Q}$-factorial, then $E_i^\nu \to E_i$ is a universal homeomorphism for every $i \in I$.

Proof. For i, it is sufficient to localise at codimension 1 points of E_i and apply [Kol13, Theorem 2.31]. Then ii follows immediately from i and iii is proven in [HW23, Lemma 2.1]. $\square$

We begin by studying the singularities of the étale-dlt surfaces.

LEMMA 2.30. *Assume that R is local with closed point $x \in X = \operatorname{Spec} R$ and that $(X, \Delta = E + D)$ is an étale-dlt surface pair such that $\lfloor \Delta \rfloor = E$. Then either*

(i) $(X, E + D)$ *is dlt at x; or*

(ii) X *is regular, E is irreducible with a node at x and $\Delta = 0$.*

In particular, X is $\mathbb{Q}$-factorial.

Proof. If $E = 0$, then $\lfloor \Delta \rfloor = 0$, and hence (X, Δ) is klt. This is covered by point 1. Hence, we may assume that $E \neq 0$. As we work in the local case, this means that $x \in \operatorname{Supp} E$. By [Kol13, Proposition 2.15] (X, Δ) has log canonical singularities. If there is an irreducible component of E that is singular, then E is an irreducible nodal curve and $\Delta = 0$ by Theorem 2.16. As (X, Δ) is étale-dlt, in this case X is regular, so we are in case ii. Thus we may also assume that every irreducible component of E is regular. We may also assume that $x \notin \operatorname{etsnc}(X, \Delta)$. Note that $\operatorname{Spec} R \setminus \{x\} \subseteq \operatorname{snc}(X, \Delta)$, and that all discrepancies over the point x are greater than -1 by the étale snc assumption. Hence, (X, Δ) is actually dlt at x.

For the assertion about $\mathbb{Q}$-factoriality, we conclude in case i by combining [Kol13, Proposition 2.28] and [Kol13, Proposition 10.9]; case ii is then immediate. $\square$

For étale-dlt surface pairs we need the following statement on the existence of a special resolution not extracting log canonical places.

LEMMA 2.31. *Let $(X, \Delta = E + \Gamma)$ be an étale-dlt surface pair such that $\lfloor \Delta \rfloor = E$. Then there exists a projective birational morphism $\pi\colon Y \to X$ such that*

(i) Y *is a regular surface, and*

(ii) *by setting $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$, we have $\Delta_Y \geq 0$ and $\lfloor \Delta_Y \rfloor = \pi_*^{-1}(E)$.*

Proof. Let $x \in (X, \Delta)$ be a closed point. We divide the proof into two cases. If (X, Δ) is dlt near x, we take the resolution of singularities at x constructed in Lemma 2.16. If (X, Δ) is not dlt near x, we do not perform any blow-up as X is already regular around x by Lemma 2.30. $\square$

We need the following generalisation of the short exact sequence of [HW19, Section 3] to étale-dlt surface pairs.

LEMMA 2.32. *Let (X, Δ) be a log canonical surface pair. Suppose $\Delta = E + \Delta'$, where E is a $\mathbb{Z}$-divisor and (X, E) is étale-dlt. Let D be a $\mathbb{Z}$-divisor on X such that $\operatorname{Supp} D$ does not contain any irreducible component of E or any point of $\operatorname{Sing} E$. Then there exists a canonically defined Mumford $\mathbb{Z}$-divisor D_E on E such that*

(i) $D_E \sim_{\mathbb{Q}} D|_E + \Gamma_E$ *for some Mumford $\mathbb{Q}$-divisor $0 \leq \Gamma_E \leq \operatorname{Diff}_E(0) \leq \operatorname{Diff}_E(\Delta')$;*

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(ii) there exists a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \omega_X(D) \to \omega_X(E+D) \to \omega_E(D_E) \to 0.$$

Proof. Recall that X is $\mathbb{Q}$-factorial by Lemma 2.30. By the assumption that no irreducible component of E and no point of the singular locus of E is contained in the support of D, the divisor $D|_E$ is a well-defined Mumford $\mathbb{Q}$-divisor on E.

Let $\pi \colon Y \to X$ be the resolution of the pair (X, E) given by Lemma 2.31, and write $K_Y + E_Y + \Gamma_Y = \pi^*(K_X + E)$, where $\lfloor E_Y + \Gamma_Y \rfloor = E_Y = \pi_*^{-1}(E)$. As π extracts no divisor of discrepancy -1, π is an isomorphism around the singular points of E. Hence, $\pi|_{E_Y} \colon E_Y \to E$ is an isomorphism. For similar reasons, $\lceil \pi^*D \rceil|_{E_Y}$ does make sense, i.e. the support of $\lceil \pi^*D \rceil$ intersects E_Y only along its regular locus. Let D_E be the divisor on E corresponding to the divisor $\lceil \pi^*D \rceil|_{E_Y}$ on E_Y via the isomorphism $\pi|_{E_Y}$.

As Y is regular, we have the following exact sequence on Y:

$$0 \to \omega_Y(\lceil \pi^*D \rceil) \to \omega_Y(E_Y + \lceil \pi^*D \rceil) \to \omega_{E_Y}(\lceil \pi^*D \rceil|_{E_Y}) \to 0. \tag{5}$$

Note the following properties:

(i) since π does not extract any divisor of discrepancy -1, we have $K_Y + \lceil \pi^*D \rceil \geq \lfloor K_Y + \Gamma_Y + \pi^*D \rfloor \geq \lfloor \pi^*(K_X + D) \rfloor$, so $\pi_*(\omega_Y(\lceil \pi^*D \rceil)) = \omega_X(D)$. Similarly, $\pi_*(\omega_Y(E_Y + \lceil \pi^*D \rceil)) = \omega_X(E+D)$;

(ii) By the above choice of D_E, we have $\pi_* \omega_{E_Y}(\lceil \pi^*D \rceil|_{E_Y}) \cong (\pi|_{E_Y})_* \omega_{E_Y}(\lceil \pi^*D \rceil|_{E_Y}) \cong \omega_E(D_E)$.

(iii) by GR vanishing for surfaces [Kol13, Theorem 10.4], $R^1\pi_*\omega_Y(\lceil \pi^*D \rceil) = 0$.

Thus, pushing forward 5 via π, we obtain the short exact sequence

$$0 \to \omega_X(D) \to \omega_X(E+D) \to \omega_E(D_E) \to 0.$$

We are left to check only that $D_E \sim_{\mathbb{Q}} D|_E + \Gamma_E$ for some $0 \leq \Gamma_E \leq \mathrm{Diff}_E(0)$. Note that via the isomorphism $\pi|_{E_Y}$, Γ_E identifies with $(\lceil \pi^*D \rceil - \pi^*D)|_{E_Y} \geq 0$. Let x be a point of X, and let i_x be the determinant of the dual graph of the minimal resolution of X at x. By possibly restricting to a neighbourhood of x, we have that $i_x D$ is Cartier by [Kol13, Prop 10.9.(3)]. Additionally, by [Kol13, Corollary 3.45], the following equality holds:

$$\mathrm{Diff}_E(0) = \begin{cases} \left(1 - \frac{1}{i_x}\right)x, & \text{if } (X, E) \text{ is plt near } x \\ x, & \text{if } (X, E) \text{ is not plt near } x, \end{cases}$$

Since $\lfloor \Gamma_E \rfloor = 0$ and $i_x \Gamma_E$ is integral, we finally conclude that $\Gamma_E \leq \mathrm{Diff}_E(0) \leq \mathrm{Diff}_E(\Delta')$. $\quad\square$

In higher dimension we deduce the following generalisation of [BK23, Lemma 5]:

Proposition 2.33. *Let (X, Δ) be a log canonical pair. Suppose $\Delta = E + \Delta'$, where E is a $\mathbb{Z}$-divisor and (X, E) is an étale-dlt pair. Let $\nu \colon E^\nu \to E$ be the (S_2)-ification of E. If D is a $\mathbb{Z}$-divisor on X, then there is a short exact sequence of $\mathcal{O}_X$-modules:*

$$0 \to \omega_X(D) \to \omega_X(E+D) \to^r \nu_*(\omega_{E^\nu}(D_{E^\nu})),$$

where $D_{E^\nu} \sim_{\mathbb{Q}} D|_{E^\nu} + \Gamma_E$ is a Mumford divisor on E^ν for some $\mathbb{Q}$-divisor $0 \leq \Gamma_{E^\nu} \leq \mathrm{Diff}_{E^\nu}(\Delta')$. Moreover, r is a surjection at all codimension 1 points in E, and, if $\omega_X(D)$ is S_3, then r is surjective.

Proof. By Lemma 2.30, at the codimension 2 singular points of E, the $\mathbb{Z}$-divisor D is Cartier. Hence, up to replacing D by another divisor in its linear equivalence class, we may assume that D does not contain any component of E and that it also does not contain any singular point of E that has codimension 2 in X. By localising at codimension 2 points of X and applying Lemma 2.32, there exists a canonically defined Mumford $\mathbb{Z}$-divisor D_E on E. As the irreducible components of E are (R_1), by taking the preimage of D in E^ν we obtain a globally defined Mumford $\mathbb{Z}$-divisor D_{E^ν} on the (S_2) surface E^ν.

Consider the natural exact sequence

$$0 \to \omega_X(D) \to \omega_X(E+D) \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is a sheaf supported on E. Note that $\mathcal{Q}$ is a torsion-free $\mathcal{O}_E$-module of rank 1 by [Kol13, Corollary 2.61], and therefore, the (S_2)-hull $\mathcal{Q} \to \mathcal{Q}^{(**)}$ is an injection. So we obtain the exact sequence

$$0 \to \omega_X(D) \to \omega_X(E+D) \to \mathcal{Q}^{**}.$$

We now claim that $\mathcal{Q}^{(**)}$ is isomorphic to $\nu_*(\omega_{E^\nu}(D_{E^\nu}))$. By construction of the residue map, there is a natural homomorphism $\psi\colon \mathcal{Q} \to \nu_*(\omega_{E^\nu}(D_{E^\nu}))$. As $\nu_*(\omega_{E^\nu}(D_{E^\nu}))$ is S_2, then there is a natural map $\mathcal{Q}^{(**)} \to \nu_*(\omega_{E^\nu}(D_{E^\nu}))$. As both $\mathcal{O}_X$-modules are (S_2), it is sufficient to show equality at codimension 1 points of E, which has been proved in Lemma 2.32. The linear equivalence $D_{E^\nu} \sim_{\mathbb{Q}} D|_{E^\nu} + \Gamma_E$ and $0 \le \Gamma_{E^\nu} \le \mathrm{Diff}_{E^\nu}(\Delta')$ is a codimension 2 statement and it is a consequence of Lemma 2.32.

For the last claim, if $\omega_X(D)$ is (S_3), then $\mathcal{Q}$ is (S_2) by [Kol13, Lemma 2.60], thus concluding that r is surjective. $\qquad\square$

2.6 Partial resolutions of demi-normal excellent surfaces

In this subsection, we fix an excellent base ring T such that $\frac{1}{2} \in \mathcal{O}_T$. We start by defining the notion of a pinch point for excellent local rings.

DEFINITION 2.34. Let $(R, \mathfrak{m})$ be a 2-dimensional excellent local ring. We say it is a pinch point if there exists a finite étale morphism $R \to S$ such that $S \simeq R'/(x^2 - zy^2)$, where R' is a 3-dimensional regular local ring and (x, y, z) is a regular system of parameters for R'.

We recall the definition of semi-regular surfaces.

DEFINITION 2.35 (cf. [KSB88], Definition 4.2). A surface S is called semi-regular if for every closed point $s \in S$, the local ring $\mathcal{O}_{S,s}$ is regular, a node (cf. Definition 2.6) or a pinch point.

Motivated by [Kol13, Theorem 10.56], we introduce the notion of semi-regularity for surface pairs.

DEFINITION 2.36. Let (S, H) be a pair, where $S = \mathrm{Spec}(R)$ and where R is an excellent local ring of dimension 2 and H is a Weil $\mathbb{Z}$-divisor. We say that (S, H) is a semi-regular pair if S is semi-regular and if one of the following holds:

(i) the pair (S, H) is snc;
(ii) S is nodal, and there exists an étale morphism $\mathrm{Spec} R/(x^2 - uy^2) \to S$, where R is 3-dimensional regular ring with $u \in R^*$, local parameters x, y, z and $H = (z = 0)$;
(iii) S is a pinch point, and there exists an étale morphism $\mathrm{Spec} R/(x^2 - zy^2) \to S$, where R is a 3-dimensional regular ring with local parameters x, y, z and $H = (x = z = 0)$.

Remark 2.37. As in the characteristic 0 case, the conductor $D \subset S$ of a semi-regular surface is a regular curve. In case iii, if $f \colon T \to S$ is the blow-up of S along D, then the local computations in [Kol13, Definition 1.43] show that the pair $(T, D_T + f^*H)$ is snc.

DEFINITION 2.38 (cf. [KSB88], Definitions 4.3, 4.4). Let (S, H) be a demi-normal surface pair. We say that $\pi \colon T \to (S, H)$ is a semi-regular resolution if

(i) π is a proper morphism;
(ii) π is an isomorphism over the nc locus of (S, H);
(iii) (T, π^*H) is a semi-regular pair;
(iv) no component of the non-normal locus D_T of T is π-exceptional.

We say π is good if additionally

(e) $\mathrm{Ex}(\pi) \cup \pi^*H \cup D_T$ has regular components and transverse intersections.

If (S, H) is a pair, we say that a semi-regular resolution π is thrifty if $a(E, S, F) > -1$ for all π-exceptional divisors E.

Note that the assumption (S, H) is demi-normal implies that π is an isomorphism over a big open set of S.

To show the existence of semi-regular resolutions of excellent surfaces, we follow the strategy of [Pos21a, Section 3.6]. We start with a description of involutions for complete DVR in characteristic $\neq 2$.

LEMMA 2.39. *Let $(R, \mathfrak{m})$ be a complete DVR with residue field $k := R/\mathfrak{m}$ of characteristic $p \neq 2$. Let τ be a non-trivial involution of R such that $\tau(\mathfrak{m}) = \mathfrak{m}$. Then there exists a uniformizer $\pi \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $\tau(\pi) = -\pi$.*

Proof. The proof is similar to [Pos21a, Lemma 3.6.5]. We fix $t \in \mathfrak{m}$ to be a uniformiser.

$\boxed{Suppose\ t - \tau(t) \notin \mathfrak{m}^2.}$ In this case, we set $\pi := t - \tau(t)$. Note that $\tau(\pi) = \tau(t) - \tau^2(t) = \tau(t) - t = -\pi$, and we conclude.

$\boxed{Suppose\ t - \tau(t) \in \mathfrak{m}^2.}$ Then there exists $f \in \mathfrak{m}$ such that $\tau(t) = (1 + f)t$. Moreover, it is easy to see that $\tau(t^k) - t^k \in \mathfrak{m}^{k+1}$. We distinguish two cases.

$\boxed{1.\ \tau\ acts\ non-trivially\ on\ k.}$ Let $\alpha \in k \setminus k^\tau$. Let $\widetilde{\alpha}$ be a lifting of α to R. Note that $\widetilde{\alpha} - \tau(\widetilde{\alpha})(1 + f)$ is invertible. Indeed, as R is local, it is sufficient to note that $\widetilde{\alpha} - \tau(\widetilde{\alpha})(1 + f) \equiv \alpha - \tau(\alpha) \neq 0 \mod \mathfrak{m}$ by choice of α. Consider $s := \widetilde{\alpha}t$. Note that

$$s - \tau(s) \equiv (\widetilde{\alpha} - \tau(\widetilde{\alpha}))t \neq 0 \mod \mathfrak{m}^2,$$

as $\alpha - \tau(\alpha) \neq 0$. Thus $s - \tau(s) \notin \mathfrak{m}^2$, and thus, we can conclude by the previous step.

$\boxed{2.\ \tau\ acts\ trivially\ on\ k.}$ We verify this contradicts the non-triviality of τ. We construct a recursive sequence t_k such that $t_k - \tau(t_k) \in \mathfrak{m}^{k+1}$. Fix $t_0 = 0$ and $t_1 := t$. Suppose t_k is defined. We have $t_k - \tau(t_k) = at^{k+1} \mod \mathfrak{m}^{k+2}$ for some $a \in R$. As 2 is invertible in R, we can define

$$t_{k+1} := t_k - \frac{a}{2}t^{k+1}.$$

Note that $\tau(\frac{a}{2}) \cong \frac{a}{2} \mod \mathfrak{m}$ by hypothesis. Therefore,

$$\tau(t_{k+1}) - t_{k+1} \equiv -at^{k+1} + \frac{a}{2}t^{k+1} - \tau\left(\frac{a}{2}\right)t^{k+1} \mod \mathfrak{m}^{k+2} \equiv 0,$$

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as $\tau(a/2) \equiv a/2 \mod \mathfrak{m}$. Now the sequence (t_k) is a Cauchy sequence and thus converges to $s \in \mathfrak{m}$. Note that by construction, $\tau(s) = s$, reaching a contradiction (as τ is non-trivial). $\qquad \square$

LEMMA 2.40. *Let (S, H) be a quasi-projective snc surface pair over T, and let D be a regular divisor intersecting transversally H. Let $\tau \colon D \to D$ be a non-trivial involution. Then $(S/R(\tau), H/R(\tau))$ is a semi-regular pair.*

Proof. The relation $R(\tau)$ is finite; thus the quotient $p \colon S \to U := S/R(\tau)$ exists, and it is demi-normal by [Pos21a, Lemma 2.3.13]. Moreover, by [Kol13, 9.13], the diagram

$$
\begin{array}{ccc}
D & \longrightarrow & S \\
\downarrow & & \downarrow \\
D/R(\tau) & \longrightarrow & U,
\end{array}
$$

is a push-out square. We may assume that $U = \mathrm{Spec}(R)$ is the spectrum of a local ring with maximal ideal $\mathfrak{m}_R$. Therefore, S is an affine regular scheme $\mathrm{Spec}(A)$, and there exists a Cartier divisor $f \in A$ (resp. $h \in A$) such that $D = (f = 0)$ (resp. $H = (h = 0)$).

If $D \to D/R(\tau)$ is a $(\mathbb{Z}/2\mathbb{Z})$-quotient we have two cases:

(i) A has exactly two maximal ideals. In this case, up to an étale base change, we may assume $A = A_1 \oplus A_2$, where A_1 and A_2 are local rings. Let $f_i \in A_i$ (resp. $h_i \in A_i$) be the local equations of $D|_{\mathrm{Spec}(A_i)}$ (resp. $H|_{\mathrm{Spec}(A_i)}$) for $i = 1, 2$. Note that the transversality hypothesis on H and D implies that $(f_i, h_i) = \mathfrak{m}_{A_i}$. Then the push-out property implies that $\mathfrak{m}_U = R \cap (\mathfrak{m}_{A_1} \oplus \mathfrak{m}_{A_2})$. Let $\tau \colon A_1/f_1 \to A_2/f_2$ be the involution; therefore, $x := (f_1, 0), y = (0, f_2)$ and $z := (h_1, h_2)$ generate $\mathfrak{m}_U$. If τ is trivial, then we have the relations $x = y$ and $H/R(\tau) = (z = 0)$, and we are in case i of Definition 2.38. If τ is not-trivial, then we have the relation $xy = 0$, and therefore, $\mathrm{Spec}(R)$ is nodal and $H/R(\tau) = (z = 0)$, thus ending in case ii of Definition 2.38.

(ii) A is a local ring such that $\tau(\mathfrak{m}_A) = \mathfrak{m}_A$, and let $\tau \colon A/(f) \to A/(f)$ be an involution. The involution τ extends to the completion of A and the completion of R is the preimage of the τ-invariant elements of $A/(f)$. As τ fixes $\mathfrak{m}_A$, the residue field of R is isomorphic to k^τ. The completion $\hat{A}/(f)$ is a complete DVR, and thus there exists a uniformiser $\pi \in \hat{A}/(f)$ such that $\tau(\pi) = -\pi$ by Lemma 2.39. Let $\widetilde{\pi}$ be a lifting of π to $\hat{A}$ such that $H = (\widetilde{\pi} = 0)$ (note that $h/\widetilde{\pi}$ is invertible). We distinguish two cases:

  (a) Suppose $k = k^\tau$. Then $R \subset A$ is the subalgebra generated by $f, \widetilde{\pi}$ and $\widetilde{\pi} f$. Moreover, $H/R(\tau)$ is given by the equations $(\widetilde{\pi} = \widetilde{\pi} f = 0)$, and thus we are in case iii of Definition 2.38.

  (b) Suppose $k^\tau \subsetneq k$. Then there exists $\alpha \in k$ such that $k = k^\tau(\alpha)$ and $\tau(\alpha) = -\alpha$. Let A' to be the preimage of k^τ under the projection $A \to k$. In this case, consider the subalgebra of A' generated by $x := \alpha\widetilde{\pi}, y := f, z := \alpha f$. Therefore, we have the relation $\alpha^2 y^2 = z^2$, showing that R is a nodal singularities and that $H/R(\tau)$ is described by $(x = 0)$, showing we end up in case ii of Definition 2.38. $\qquad \square$

We now show the existence of semi-resolution (in characteristic 0, this is [Kol13, Theorem 10.54]).

COROLLARY 2.41. *Let (S, H) be a quasi-projective demi-normal surface pair over T. Then there exists a good semi-regular resolution $\pi \colon V \to (S, H)$. If, moreover, (S, H) is semi-dlt, then we can choose π to be thrifty.*

Proof. Let $\nu \colon (S^\nu, D_{S^\nu}) \to S$ be the normalisation morphism where D_{S^ν} is the conductor sub-scheme of S^ν. Let $f \colon X \to (S^\nu, D_{S^\nu} + \nu^* H)$ be a log resolution of $(S^\nu, D_{S^\nu} + \nu^* H)$ and let $D_X := f_*^{-1} D_{S^\nu}$. The involution τ lifts to an involution of D_X, and we can apply Lemma 2.40 to construct a projective birational contraction $q \colon Y \to S$ fitting in the commutative diagram

$$
\begin{array}{ccc}
(X, D_X + f_*^{-1}H + \mathrm{Ex}(f)) & \longrightarrow & (Y := X/R(\tau), (f_*^{-1}H + \mathrm{Ex}(f))/\tau) \\
f \downarrow & & q \downarrow \\
(S^\nu, D_{S^\nu} + \nu^* H) & \longrightarrow & (S, H),
\end{array}
$$

such that $q \colon Y \to (S, H)$ is a semi-regular resolution of (S, H).

If (S, H) is semi-dlt, then $(S^\nu, D_{S^n} + \nu^* H)$ is a dlt pair. In this case, we can take $f \colon X \to (S^\nu, D_{S^n} + \nu^* H)$ to be a thrifty log resolution of (S^n, D_{S^n}), and end-product $T \to (S, H)$ is clearly a thrifty semi-regular resolution. $\square$

We show how we can slightly improve the resolution algorithm (see [Kol13, Corollary 10.55] for an analogue in characteristic 0).

DEFINITION 2.42. Let (S, H) be a demi-normal surface pair, and let $S^0 \subset S$ be the largest open set such that $(S^0, H|_{S^0})$ is semi-snc. We say that $\pi \colon T \to (S, F)$ is a semi-log resolution if

(i) π is projective and birational;

(ii) $(T, D_T := \pi_*^{-1} \mathrm{Supp}(F) + \mathrm{Ex}(\pi))$ is a semi-snc pair;

(iii) π is an isomorphism over the generic point of every lc centre of (S, H);

(iv) π is an isomorphism at the generic point of every lc centre of (T, D_T).

THEOREM 2.43. *Let (S, H) be a quasi-projective demi-normal surface pair over T. Then there exists a semi-log resolution $\pi \colon V \to (S, H)$. If (S, H) is semi-dlt, we can choose π to be thrifty.*

Proof. Consider $q \colon Y \to (S, H)$ be the semi-regular resolution constructed in Proposition 2.41. The only problem is around the pinch points of Y, which are isolated by dimension reasons. Therefore, we can localise to a neighbourhood of $y \in Y$, where y is a pinch point, and we let D_y be the local component of the non-normal locus D_T. By blowing-up D_y we obtain our desired semi-log resolution as explained in Remark 2.37. $\square$

2.7 Vanishing theorems for slc surfaces

In this section we generalise the vanishing theorems of Kawamata–Viehweg type for klt surfaces due to Tanaka [Tan18, Theorem 3.3] to the slc case using the method developed by Kollár in [Kol13, Section 10.3]. For an overview on vanishing theorems for slc pairs in characteristic 0, we refer to [Fuj15]. The most general result we prove is Theorem 2.51, which is the fundamental vanishing theorem we will use in Section 3. We start with the case of semi-snc surface pairs.

PROPOSITION 2.43. *Let (S, Δ) be a semi-snc surface pair with Δ a reduced $\mathbb{Z}$-divisor, and let $f\colon S \to T$ be a surjective projective morphism onto a normal scheme of dimension $\dim(T) \geq 1$. Let M be a Cartier divisor on S. Suppose that*

(i) *$M \sim_{\mathbb{Q},f} K_S + \Delta' + L$, where*

   (a) *L is a f-nef $\mathbb{Q}$-divisor, and*
   (b) *$0 \leq \Delta' \leq \Delta$ is a $\mathbb{Q}$-divisor.*

(ii) *if Z is a log canonical centre of (S, Δ), including the irreducible components of Z as well, then*

   (a) *$\dim\left(f(Z)\right) \geq 1$;*
   (b) *if F_Z is the generic fibre of $Z \to f(Z)$, then $\dim F_Z = \nu(L|_{F_Z})$.*

   Then $R^1 f_* \mathcal{O}_S(M) = 0$.

Proof. Using the relative Kawamata–Viehweg vanishing theorem for surfaces [Tan18, Theorem 3.3], we can repeat the same steps of the proof of [Kol13, Corollary 10.34]. □

The following result is useful to reduce various statements to the case of semi-snc pairs.

LEMMA 2.45. *Let (S, Δ) be a semi-dlt surface pair such that $\frac{1}{2} \in \mathcal{O}_S$. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor such that $D \sim_{\mathbb{Q}} K_S + \Delta + M$ for some $\mathbb{Q}$-Cartier divisor M. Then there is a proper birational morphism $g\colon Y \to S$, a Cartier divisor D_Y and a $\mathbb{Q}$-divisor $\Delta_{D,Y}$ such that*

(i) *$(Y, \Delta_{D,Y})$ is semi-snc;*
(ii) *$D_Y \sim_{\mathbb{Q}} K_Y + \Delta_{D,Y} + g^* M$;*
(iii) *if Z is a log canonical centre of $(Y, \Delta_{D,Y})$, then the restriction $g|_Z$ is birational;*
(iv) *$g_* \mathcal{O}_Y(D_Y) = \mathcal{O}_S(D)$;*
(v) *$R^i g_* \mathcal{O}_Y(D_Y) = 0$ for $i > 0$.*

Proof. The same proof of [Kol13, Proposition 10.36] applies as thrifty semi-log resolutions by Theorem 2.43 exist for excellent surfaces and the necessary vanishing theorems hold by Proposition 2.44. □

We generalise Proposition 2.44 to the case of semi-dlt surface pairs.

PROPOSITION 2.46. *Let (S, Δ) be a semi-dlt surface pair such that $\frac{1}{2} \in \mathcal{O}_S$, and let $f\colon S \to T$ be a projective morphism onto a normal scheme of $\dim(T) \geq 1$. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor on S. Suppose that*

(i) *$D \sim_{\mathbb{Q},f} K_S + \Delta + L$, where L is a f-nef $\mathbb{Q}$-divisor;*
(ii) *if Z is a log canonical centre of (S, Δ), including the irreducible components of Z as well, then*

   (a) *$\dim(f(Z)) \geq 1$;*
   (b) *if F_Z is the generic fibre of $Z \to f(Z)$, then $\dim F_Z = \nu(L|_{F_Z})$.*

   Then $R^1 f_* \mathcal{O}_S(D) = 0$.

Proof. Let $g \colon Y \to S$ be a proper birational morphism such that the pair $(Y, \Delta_{D,Y})$ and the Cartier divisor D_Y on Y satisfy the conditions of Lemma 2.45, and denote by $h \colon Y \to T$ the natural composition. By Proposition 2.44, we conclude that $R^1 h_* \mathcal{O}_Y(D_Y) = 0$. As $g_* \mathcal{O}_Y(D_Y) = \mathcal{O}_S(D)$ and $R^i g_* \mathcal{O}_Y(D_Y) = 0$ for $i > 0$ by Proposition 2.44 we deduce $R^1 f_* \mathcal{O}_S(D) = 0$ by the Leray spectral sequence. $\square$

In order to generalise to the slc case, the following is a useful observation:

LEMMA 2.47. *Let (S, Δ) be an slc surface pair such that $\frac{1}{2} \in \mathcal{O}_S$, and let $f \colon S \to T$ be a projective morphism onto a normal scheme of $\dim(T) \geq 1$. Suppose that*

(i) *every irreducible component of S is (R_1);*

(ii) *if Z is a log canonical centre of (S, Δ), then $\dim(f(Z)) \geq 1$.*

Then (S, Δ) is semi-dlt.

Proof. We argue by contradiction. Let E be an exceptional divisor such that $a(E, S, \Delta) = -1$ and $\mathrm{cent}_S(E) \subset \mathrm{nsnc}(S, \Delta)$. The hypothesis (a) guarantees that (S, Δ) is snc at codimension 1 points. Therefore $\mathrm{cent}_S(E)$ is a closed point, contradicting (b). $\square$

COROLLARY 2.48. *Let (S, Δ) be an slc surface pair such that $\frac{1}{2} \in \mathcal{O}_S$, and let $f \colon S \to T$ be a projective morphism onto a normal scheme of $\dim(T) \geq 1$. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor on S. Suppose that*

(i) $D \sim_{\mathbb{Q}, f} K_S + \Delta + L$, *where L is f-nef;*

(ii) *if Z is a log canonical centre of (S, Δ), including the irreducible components of Z as well, then*

    (a) $\dim(f(Z)) \geq 1$;

    (b) *if F_Z is the generic fibre of $Z \to f(Z)$, then $\dim F_Z = \nu(L|_{F_Z})$.*

Then $R^1 f_* \mathcal{O}_S(D) = 0$.

Proof. Let $p \colon \widetilde{S} \to S$ be the double cover of Proposition 2.11. As 2 is invertible, $\mathcal{O}_S(D)$ is a direct summand of $p_* \mathcal{O}_{\widetilde{S}}(p^* D)$. We can thus assume that the irreducible components of S are regular in codimension 1. In this case, (S, Δ) is semi-dlt by Lemma 2.47 and we conclude by Proposition 2.46. $\square$

We can prove a further generalisation of Kawamata–Viehweg vanishing for slc surfaces over curves.

PROPOSITION 2.49. *Let (S, Δ) be a semi-snc surface pair with Δ a reduced divisor, and let $f \colon S \to C$ be a projective surjective contraction onto a normal curve C. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor on S. Suppose that*

(i) *every log canonical centre Z of (S, Δ), including the components of S, dominates C.*

(ii) A *is an f-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on S;*

(iii) $D \sim_{f, \mathbb{Q}} K_S + \Delta' + A$, *where $0 \leq \Delta' \leq \Delta$ is a $\mathbb{Q}$-divisor;*

(iv) *on every connected component S' of S there exists an irreducible component E of S' such that $A|_E$ is $f|_E$-big.*

Then $R^1 f_* \mathcal{O}_S(D) = 0$.

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Proof. We prove the result by induction on the number n of irreducible components of S. If $n = 1$, we conclude by Proposition 2.44.

We prove the induction step. Let E be an irreducible component of S such that $A|_E$ is big over C. Let T be the union of the irreducible components of S except E. Denote $B := E \cap T$, and we consider the short exact sequence:

$$0 \to \mathcal{O}_E(D|_E - B) \to \mathcal{O}_S(D) \to \mathcal{O}_T(D|_T) \to 0.$$

Taking the long exact sequence in cohomology, it is sufficient to show that $R^1 g_* \mathcal{O}_E(D - B) = R^1 g_* \mathcal{O}_T(D|_T) = 0$ to conclude that $R^1 g_* \mathcal{O}_S(D) = 0$.

Since $K_S|_E = K_E + B$, we have $D|_E - B \sim_{g,\mathbb{Q}} K_E + \Delta'|_E + A|_E$; so by Proposition 2.44, we conclude that $R^1 g_* \mathcal{O}_E(D - B) = 0$.

Since $K_S|_T = K_T + B$, we have that $D_T \sim_{\mathbb{Q},g} K_T + \Delta'|_T + B + A|_T$, and that B is not trivial on some irreducible component of every connected component of T, as g has connected fibres. By hypothesis, B must be a non-empty horizontal divisor; thus it is nef over C, and for every connected component of T, there exists an irreducible component F such that $B|_F$ is $g|_F$-big. Therefore, we apply the induction hypothesis to deduce $R^1 g_* \mathcal{O}_T(D|_T) = 0$. $\square$

PROPOSITION 2.50. *Let (S, Δ) be a semi-dlt surface pair such that $\frac{1}{2} \in \mathcal{O}_S$, and let $g \colon S \to C$ be a projective morphism onto a normal curve C. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor on S. Suppose that:*

(i) *every log canonical centre Z of (S, Δ), including the components of S, dominates C;*

(ii) *A is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on S, which is g-nef;*

(iii) *$D \sim_{g,\mathbb{Q}} K_S + \Delta' + A$, where $0 \leq \Delta' \leq \Delta$;*

(iv) *on every connected component of S, if there exists an irreducible component E such that $A|_E$ is $g|_E$-big.*

Then $R^1 g_* \mathcal{O}_S(D) = 0$.

Proof. As in the proof of Proposition 2.46, it sufficient to combine Lemma 2.45 and Proposition 2.49 with the Leray spectral sequence. $\square$

THEOREM 2.51. *Let (S, Δ) be a slc surface pair such that $\frac{1}{2} \in \mathcal{O}_S$, and let $g \colon S \to C$ be a projective morphism onto a normal curve C. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor on S. Suppose that*

(i) *every log canonical centre Z of (S, Δ), including the components of Z, dominates C;*

(ii) *A is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on S, which is g-nef;*

(iii) *$D \sim_{g,\mathbb{Q}} K_S + \Delta' + A$, where $0 \leq \Delta' \leq \Delta.$;*

(iv) *on every connected component of S, there exists an irreducible component E such that $A|_E$ is $g|_E$-big;*

Then $R^1 g_* \mathcal{O}_S(D) = 0$.

Proof. We can repeat the same proof of Corollary 2.48 using Proposition 2.50. $\square$

2.8 Grauert–Riemenschneider theorem for dlt 3-folds

We recall the Grauert–Riemenschneider (GR) vanishing theorem for excellent dlt 3-folds proven by Kollár and the second author in [BK23].

THEOREM 2.52. [BK23, Theorem 2]. *Let (X, Δ) be a 3-dimensional dlt pair whose residue fields of closed points are perfect with characteristic $p \neq 2, 3, 5$. Then G–R vanishing holds on (X, Δ).*

Precisely, let $g \colon Y \to (X, \Delta)$ be a log resolution, and let D be a Weil $\mathbb{Z}$-divisor on Y such that $D \sim_{g,\mathbb{R}} K_Y + \Delta'$ for an effective $\mathbb{R}$-divisor Δ' on Y such that $g_* \Delta' \leq \Delta$ and $\lfloor \mathrm{Ex}(\Delta') \rfloor = 0$. Then, $R^i g_* \mathcal{O}_Y(D) = 0$ for $i > 0$.

The main techniques are the vanishing theorem for surfaces of del Pezzo type over perfect fields proven in [ABL22] and the MMP for 3-folds [BMP+23]. From the G–R vanishing theorem, one can deduce various rationality and Cohen–Macaulay properties for dlt 3-fold singularities, a result we will frequently use to study depths of log canonical 3-fold singularities in terms of a dlt modifications.

COROLLARY 2.53 [BK23, Theorem 17]. *Let (X, Δ) be a 3-dimensional dlt pair whose residue fields of closed points are perfect with characteristic $p \neq 2, 3, 5$. Then*

(i) *X is Cohen–Macaulay, and has rational singularities;*
(ii) *every irreducible component of $\lfloor \Delta \rfloor$ is normal;*
(iii) *if D is a $\mathbb{Z}$-divisor such that $D + \Delta'$ is $\mathbb{Q}$-Cartier for some $\mathbb{Q}$-divisor $0 \leq \Delta' \leq \Delta$, then $\mathcal{O}_X(D)$ is C–M.*

3. Depth of log canonical 3-fold singularities

Setting 3.1. Throughout this section, we suppose $(R, \mathfrak{m})$ is a local ring whose residue field is perfect of characteristic $p \neq 2, 3$, and 5. Let $(X = \mathrm{Spec}(R), x)$ be the associated local scheme, and suppose that there exists a $\mathbb{Q}$-divisor $\Delta \geq 0$ such that (X, Δ) is a log canonical 3-dimensional pair.

We are interested in computing the local cohomology group $H^2_x(X, \mathcal{O}_X)$ when the closed point x is *not* a minimal log canonical centre of (X, Δ). We first show that we can reduce to the case where the minimal log canonical centre is 1-dimensional.

LEMMA 3.2. *Let C be the minimal log canonical centre of (X, Δ) passing through x. Suppose one of the following conditions hold:*

(i) *$x \notin \mathrm{nklt}(X, \Delta)$ (that is, C is empty);*
(ii) *$x \in \mathrm{nklt}(X, \Delta)$ and $\dim(C) = 2$.*

Then X satisfies Serre's condition (S_3).

Proof. Case (a) is proven in Corollary 2.53. In case (b), we deduce (X, Δ) is plt by 2.26 and we conclude by Corollary 2.53. $\square$

By Lemma 3.2, the case of interest, when studying the behavior of local cohomology of $\mathcal{O}_X$, is when $\dim C = 1$. In this case, our main technical result relates the non-vanishing of local cohomology $H^2_x(X, \mathcal{O}_X)$ to the torsion of $R^1 g_* \mathcal{O}_E$, where E is an exceptional divisor over X. More precisely:

THEOREM 3.3. *Let $C \subset X$ be a 1-dimensional minimal log canonical centre for (X, Δ) passing through x. Then there exists a projective birational morphism $g \colon Z \to (X, \Delta)$ with reduced exceptional divisor E such that*

(i) Z is $\mathbb{Q}$-factorial klt with $K_Z + g_*^{-1}\Delta + E \sim_{\mathbb{Q}} g^*(K_X + \Delta)$;

(ii) E is (S_2);

(iii) $H_x^2(X, \mathcal{O}_X) \simeq H_x^0(C, R^1 g_* \mathcal{O}_E)$.

As the rest of the section is devoted to showing Theorem 3.3, from now on we assume the following:

Setting 3.4. Besides the assumptions and the notation of Setting 3.1, let us also fix a 1-dimensional minimal log canonical centre C of (X, Δ) passing through x. Moreover, C is irreducible by Lemma 2.26, and we denote by η its generic point.

3.1 Construction of minimal étale-dlt modifications

The hypothesis of minimality on C allows us to prove the following technical results, which we will use repeatedly:

LEMMA 3.5. *Let $\pi \colon (Y, \Delta_Y) \to (X, \Delta)$ be a crepant proper birational morphism of normal log pairs where (X, Δ) is as in Setting 3.4. Suppose Y is $\mathbb{Q}$-factorial, and let $0 \le \Gamma \le \Delta_Y$. If (Y_η, Γ_η) is dlt (resp. étale-dlt), then (Y, Γ) is dlt (resp. étale-dlt).*

Proof. Suppose that (Y_η, Γ_η) is dlt. If (Y, Γ) is not dlt, then there exists an exceptional divisor E with discrepancy $a(E, Y, \Gamma) = -1$ such that $\mathrm{cent}_Y(E) \subset Y \setminus \mathrm{snc}(Y, \Gamma)$. Since (Y_η, Γ_η) is dlt, we deduce that $\mathrm{cent}_Y(E)$ must be disjoint from Y_η. In particular, $\mathrm{cent}_X(E)$ is a closed point c in C. As $a(E, Y, \Delta_Y) \le a(E, Y, \Gamma) = -1$, this contradicts the minimality of C among the log canonical centres of (X, Δ).

The same proof works in the étale-dlt case by replacing $Y \setminus \mathrm{snc}(Y, \Gamma)$ with the closed subset $Y \setminus \mathrm{etsnc}(Y, \Gamma)$. $\square$

LEMMA 3.6. *Let (X, Δ) as in Setting 3.4. For every exceptional log canonical place E over X, we have $\mathrm{cent}_X(E) = C$.*

Proof. If $\mathrm{cent}_X(E) \ne C$, we have $\mathrm{cent}_X(E) \cap C = \{x\}$, as E is exceptional. Therefore, x is a log canonical centre by Lemma 2.26, contradicting the minimality of C. $\square$

In the next propositions, as in the article in general, $\mathrm{Ex}(\pi)$ denotes the divisorial part of the exceptional set of a proper birational morphism π, not the entire exceptional set.

LEMMA 3.7. *Let $\pi \colon (Y, \Delta_Y) \to (X, \Delta)$ be a crepant proper birational morphism of normal log pairs, where (X, Δ) is as in Setting 3.4. Suppose that Y is $\mathbb{Q}$-factorial and that $\Delta_Y \ge E := \mathrm{Ex}(\pi)$. If $E \ne 0$, then the pair $(Y, \Delta_Y - \varepsilon E)$ is plt for every rational number $\varepsilon \in (0, 1]$.*

Proof. Note that as $\pi_\eta \colon Y_\eta \to \mathrm{Spec}(\mathcal{O}_{X, \eta})$ is a proper birational morphism of normal surfaces, the support of E_η coincides with $\mathrm{Ex}(\pi)_\eta$ also set-theoretically.

Write $\Delta_Y - \varepsilon E = \Delta_Y' + (1 - \varepsilon)E \le \Delta_Y$, where Δ_Y' and E are effective $\mathbb{Q}$-divisors and they have no irreducible components in common. Suppose by contradiction that $(Y, \Delta_Y' + (1 - \varepsilon)E)$ is not plt. By definition, there exists a proper birational modification $f \colon Z \to Y$ extracting an exceptional divisor F with discrepancy $a(F, Y, \Delta_Y' + (1 - \varepsilon)E) = -1$. By the monotonicity of discrepancies [KM98, Lemma 2.27],

$$a(F, Y, \Delta_Y) \le a(F, Y, \Delta_Y' + (1 - \varepsilon)E) = -1.$$

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As (Y, Δ_Y) is log canonical, we conclude $a(F, Y, \Delta_Y) = -1$. As F is an exceptional log canonical place over X, then $\text{cent}_X(F) = C$ by Lemma 3.6. As $\text{cent}_Y(F)$ dominates C, we deduce that $\text{cent}_Y(F) \subset E$ (as E_η coincides set-theoretically with $\text{Ex}(\pi)_\eta$). This last containment implies by [Kol13, Lemma 2.5] that

$$a(F, Y, \Delta'_Y + (1 - \varepsilon)E) = a(F, Y, \Delta_Y - \varepsilon E) > a(F, Y, \Delta_Y) = -1,$$

contradicting the starting assumption $a(F, Y, \Delta_Y - \varepsilon E) = -1$. $\qquad\square$

PROPOSITION 3.8. *Let (X, Δ) be as in Section 3.4. Then there exists a projective birational morphism $g \colon Z \to X$ such that*

(i) *$(Z, g_*^{-1}\Delta + \text{Ex}(g))$ is a $\mathbb{Q}$-factorial log canonical pair such that $K_Z + g_*^{-1}\Delta + \text{Ex}(g) = g^*(K_X + \Delta)$;*

(ii) *the pair $(Z, g_*^{-1}\lfloor\Delta\rfloor + \text{Ex}(g))$ is étale-dlt;*

(iii) *for every $\varepsilon > 0$, the pair $(Z, g_*^{-1}\Delta + (1 - \varepsilon)\text{Ex}(g))$ is plt;*

(iv) *$-\text{Ex}(g)$ is a g-nef $\mathbb{Q}$-Cartier divisor;*

(v) *$g(F) = C$ for every irreducible component F of $\text{Ex}(g)$.*

Proof. Let $\varphi \colon W \to X$ be a log resolution of (X, Δ) such that $\varphi_*^{-1}\Delta$ is regular. In particular, the pair $(W, \varphi_*^{-1}\Delta)$ is plt. Let $\pi \colon Y \to X$ be a log minimal model of this pair over X, which is $\mathbb{Q}$-factorial by the plt assumption. Write

$$K_Y + \pi_*^{-1}\Delta + E + B = \pi^*(K_X + \Delta),$$

where E is an effective $\mathbb{Z}$-divisor and $\lfloor B \rfloor = 0$. In particular, then $-(E + B)$ is a $\mathbb{Q}$-Cartier nef divisor over X. We denote by Y_η the base change of Y over $X_\eta := \text{Spec}(\mathcal{O}_{X,\eta})$, and for a divisor D on Y, we will denote by D_η the localisation $D|_{Y_\eta}$. In particular, Y_η is NOT the fibre over η.

By Remark 2.15, Y_η is a log minimal resolution of the surface $(\text{Spec}\,\mathcal{O}_{X,\eta}, \Delta_\eta)$. Let G be the $\mathbb{Z}$-divisor $\text{Ex}(\pi) - E$, which is supported on the exceptional divisors, which are not log canonical places. Note that $\text{Supp}(B) \subseteq \text{Supp}(G)$.

Next, we define a birational model $h \colon V \to X$ that satisfies the following properties, where the sub-index V denotes the strict transform of the corresponding divisor:

- V is $\mathbb{Q}$-factorial;
- $\text{Ex}(h) = E_V$;
- $-E_{V,\eta}$ is nef;
- $(V, E_V + h_*^{-1}\lfloor\Delta\rfloor)_\eta$ is étale-dlt.

In particular, as $\text{Ex}(h) = E_V$, we conclude $(V, h_*^{-1}\Delta + E_V)$ is crepant birational to (X, Δ). We construct V separately in the two cases corresponding to the two points of Proposition 2.20, when Proposition 2.20 is applied to to the minimal resolution $\pi_\eta \colon Y_\eta \to (X_\eta, \Delta_\eta)$.

$\boxed{\text{Case i of Proposition 2.20:}}$ in this case we have that $\Delta_\eta = G_\eta = 0$ and that E_η is equal to $\text{Ex}(\pi_\eta)$, it is irreducible, and it is anti-nef. Note that the pair $(Y, \pi_*^{-1}\Delta + E + B)$ is $\mathbb{Q}$-factorial log canonical, crepant to (X, Δ) and by Lemma 3.7 the pair $(Y, \pi_*^{-1}\Delta + (1 - \varepsilon)E + B)$ is plt. Additionally, as G is exceptional, it does not have any of the codimension 1 components of $\pi_*^{-1}\Delta$ in its support. This implies that we may find another rational number $\varepsilon' > 0$, such that

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$(Y, \pi_*^{-1}\Delta + (1-\varepsilon)E + B + \varepsilon'G)$ is still plt. Let V be a log minimal model over X of the latter pair. Note that we have

$$K_Y + \pi_*^{-1}\Delta + (1-\varepsilon)E + B + \varepsilon'G \equiv_X \varepsilon'G - \varepsilon E. \tag{6}$$

In particular, this MMP is the identity on Y_η, as

$$(\varepsilon G - \varepsilon E)_\eta = -\varepsilon E_\eta,$$

is nef. This also yields that $-E_{V,\eta}$ is nef. It even implies $(V, E_V + h_*^{-1}\lfloor\Delta\rfloor)_\eta$ is étale-dlt, as $(Y, E + \pi_*^{-1}\lfloor\Delta\rfloor)_\eta$ is étale-dlt by point i of Proposition 2.20. Additionally, by the negativity lemma and by Proposition 6, this MMP turns G anti-effective, which means that it contracts it. Hence, $\mathrm{Ex}(h) = E_V$. Finally, V is $\mathbb{Q}$-factorial as it is a result of a plt MMP.

$\boxed{Case\ ii\ of\ Proposition\ 2.20\text{:}}$ by point ii of Proposition 2.20, we know that $(Y, \pi_*^{-1}\lfloor\Delta\rfloor + E + B)_\eta$ is dlt. Hence, by Lemma 3.5, $(Y, \pi_*^{-1}\lfloor\Delta\rfloor + E + B)$ is also dlt. As the coefficients of B are smaller than 1, we may choose a rational number $\varepsilon > 0$ such that $(Y, \pi_*^{-1}\lfloor\Delta\rfloor + E + B + \varepsilon G)$ is dlt. Let $h: V \to X$ be a log minimal model of this latter pair over X, where V is $\mathbb{Q}$-factorial as we run a dlt MMP on a $\mathbb{Q}$-factorial variety. Note that we have

$$K_Y + \pi_*^{-1}\lfloor\Delta\rfloor + E + B + \varepsilon G \equiv_X \varepsilon G - \pi_*^{-1}\{\Delta\}.$$

Therefore, by the negativity lemma [BMP$^+$23, Lemma 2.16] this MMP turns G anti-effective, which means that it contracts it, and hence it also contracts B. Hence $E_V = \mathrm{Ex}(h)$, and $(V, h_*^{-1}\lfloor\Delta\rfloor + E_V)$ is dlt. The last property we need to show about V is that $-E_{V,\eta}$ is nef. By Lemma 2.20, it is enough to show for this that $-E_\eta - B_\eta$ is nef. However, that is immediate as $-E - B \equiv_X K_Y + \pi_*^{-1}\Delta$, which is nef by the construction of Y as a log minimal model.

Having finished the construction and the verification of the properties of V in both cases, by Lemma 3.7 the pair $(V, h_*^{-1}\Delta + (1-\varepsilon)E_V)$ is plt for every small rational number $\varepsilon > 0$. Let $g: Z \to X$ be the log minimal model of this latter pair, which is $\mathbb{Q}$-factorial. This yields point i and iii. As

$$K_V + h_*^{-1}\Delta + (1-\varepsilon)E_V \equiv_X -\varepsilon E_V,$$

we see that this MMP is the identity in a neighborhood of V_η since $-E_{V,\eta}$ is nef, and that $-E_Z = -\mathrm{Ex}(g)$ is nef. This yields 4. The MMP being identity over η also implies that the exceptional divisor of Z and V are the same, and that $(Z, g_*^{-1}\lfloor\Delta\rfloor + E_Z)_\eta$ is étale-dlt. The former yields point 5 by Lemma 3.6, and the latter together with Lemma 3.5 yields point 2. □

3.2 Computing local cohomology

Setting 3.9. For this subsection, let (X, Δ) be as in Setting 3.4 and let $g: Z \to X$ be the birational modification constructed in Proposition 3.8. Set $E := \mathrm{Ex}(g)$ and $\Delta_Z := g_*^{-1}\Delta + E$.

We note that, by Proposition 3.8 Z is $\mathbb{Q}$-factorial and klt. Hence, by Corollary 2.53, for every $\mathbb{Z}$-divisor D on Z, the divisorial sheaf $\mathcal{O}_Z(D)$ is Cohen–Macaulay. This also implies that any divisor on Z is (S_2) by [Kol13, Corollary 2.61].

The following is the fundamental tool to relate the local cohomology group $H_x^2(X, \mathcal{O}_X)$ to cohomological properties of $g: Z \to X$.

Lemma 3.10. *There is an exact sequence as follows:*

$$0 \to H^0_x(X, R^1 g_* \mathcal{O}_Z) \to H^2_x(X, \mathcal{O}_X) \to H^2_{g^{-1}(x)}(Z, \mathcal{O}_Z).$$

Proof. Note that the composition of derived functors $R\Gamma_x$ and Rg_* satisfies $R\Gamma_x \circ Rg_* = R\Gamma_{g^{-1}x}$. Then we have the usual five-term short exact sequence:

$$0 \to R^1\Gamma_x R^0 g_* \mathcal{O}_Z \to R^1\Gamma_{g^{-1}(x)} \mathcal{O}_Z \to R^0\Gamma_x R^1 g_* \mathcal{O}_Z \to R^2\Gamma_x R^0 g_* \mathcal{O}_Z \to R^2\Gamma_{g^{-1}(x)} \mathcal{O}_Z.$$

To conclude, it is thus sufficient to show that $R^1\Gamma_{g^{-1}(x)} \mathcal{O}_Z$ vanishes. By duality for C–M sheaves (cf. [Kol13, Theorem 10.44] and [KM98, Theorem 5.71]), it is sufficient to show that $R^2 g_* \mathcal{O}_Z(K_Z) = 0$. As every irreducible component F of $\mathrm{Ex}(g)$ surjects onto C, the fibres of g are at most 1-dimensional and we deduce $R^2 g_* \mathcal{O}_Z(K_Z) = 0$ by dimension reasons. $\qquad\square$

We now prove a Grauert–Riemenschneider vanishing theorem for the birational contraction g.

Proposition 3.11. *Let D be a $\mathbb{Z}$-divisor on Z such that $D \sim_{\mathbb{Q},X} K_Z + \Delta'$, where $0 \leq \Delta' \leq g_*^{-1}\Delta$. Then $R^i g_* \mathcal{O}_Z(D) = 0$ for $i > 0$.*

Proof. For $i = 2$, it is immediate as the dimension of the fibres of is at most 1 by i of Proposition 3.8. For the case $i = 1$, as $(Z, g_*^{-1}\lfloor \Delta \rfloor + E)$ is étale-dlt we can apply Proposition 2.33 to the divisor $(D - K_Z - (m+1)E)$ for every $m \geq 0$ to obtain the short exact sequence of $\mathcal{O}_Z$-modules:

$$0 \to \mathcal{O}_Z(D - (m+1)E) \to \mathcal{O}_Z(D - mE) \to \mathcal{O}_E(G_m) \to 0,$$

where G_m is a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor on E. Moreover, there exists a $\mathbb{Q}$-divisor Γ_m such that $0 \leq \Gamma_m \leq \mathrm{Diff}_E(0)$ and

$$\begin{aligned}
G_m &\sim_{\mathbb{Q}} K_E + (D - K_Z - (m+1)E)|_E + \Gamma_m \\
&\sim_{\mathbb{Q}} (D - mE)|_E - (K_Z + E)|_E + K_E + \Gamma_m \\
&\sim_{\mathbb{Q}} (D - mE)|_E - \Gamma'_m,
\end{aligned} \tag{7}$$

where $\Gamma'_m := \mathrm{Diff}_E(0) - \Gamma_m \geq 0$. Passing to cohomology, we obtain the short exact sequence

$$R^1 g_* \mathcal{O}_Z(D - (m+1)E) \to R^1 g_* \mathcal{O}_Z(D - mE) \to R^1 g_* \mathcal{O}_E(G_m) \to 0. \tag{8}$$

We now claim that $R^1 g_* \mathcal{O}_E(G_m) = 0$ for all $m \geq 0$. By applying adjunction, we deduce

$$(D - mE)|_E \sim_{\mathbb{Q}} (K_Z + \Delta' - mE)|_E \sim_{\mathbb{Q}} K_{E^\nu} + \mathrm{Diff}_E(\Delta') - (m+1)E|_E. \tag{9}$$

Combining (7) and (9) we conclude that

$$G_m \sim_{\mathbb{Q}} K_E + \Delta_m - (m+1)E|_E, \tag{10}$$

where $\Delta_m := \mathrm{Diff}_E(\Delta') - \Gamma'_m \leq \mathrm{Diff}_E(\Delta')$. Note that $\Delta_m = (\mathrm{Diff}_E(\Delta') - \mathrm{Diff}_E(0)) + \Gamma_m$ and therefore $\Delta_m \geq 0$. We verify we can apply Theorem 2.51 to $g\colon E \to C$ to show $R^1 g_* \mathcal{O}_E(G_m) = 0$ because

- every log canonical centre of $(E, \mathrm{Diff}_E(\Delta'))$ dominates C by v of 3.8;
- $G_m \sim_{\mathbb{Q}} K_E + \Delta_m + A$, where $0 \leq \Delta_m \leq \mathrm{Diff}(\Delta')$ and $A := (-(m+1)E)|_E$ is g-nef by assumption;
- there is an irreducible component F of E such that $A|_F$ is g-big. If this is not the case then, as the fibres of $g|_E$ are 1-dimensional, $-E_\eta$ is g_η-trivial and thus $E_\eta = 0$ by the negativity lemma.

Combining the vanishing $R^1 g_* \mathcal{O}_E(G_m) = 0$ with the sequence (8), it is sufficient to show $R^1 g_* \mathcal{O}_Z(D - nE) = 0$ for n sufficiently large to conclude that $R^1 g_* \mathcal{O}_Z(D) = 0$ by descending induction. As the pair $(Z, g_*^{-1}\Delta + (1-\varepsilon)E)$ is plt and

$$K_Z + g_*^{-1}\Delta + (1-\varepsilon)E \equiv_X -\varepsilon E,$$

is g-nef, we can consider the birational contraction $p \colon Z \to T$ to its canonical model $h \colon T \to X$. By construction, the pair $(T, h_*^{-1}\Delta + (1-\varepsilon)E_T)$ is plt and $-E \sim_{\mathbb{Q}} p^*(-E_T)$ where $-E_T$ is ample over X. In particular, as

$$D - nE \sim_{\mathbb{Q}} K_Z + \Delta' - nE \sim_{\mathbb{Q},T} K_Z + \Delta',$$

by Theorem 2.52 we deduce $R^i p_* \mathcal{O}_Z(D - nE) = 0$ for $i > 0$. For n sufficiently large and divisible, $nE \sim p^* nE_T$ and thus by the Leray spectral sequence and the projection formula we have

$$R^1 g_* \mathcal{O}_Z(D - nE) \simeq R^1 h_*(p_* \mathcal{O}_Z(D - nE)) \simeq R^1 h_*(p_* \mathcal{O}_Z(D) \otimes \mathcal{O}_T(-nE_T)),$$

which is zero for n sufficiently large by Serre vanishing. $\qquad\square$

As an application of the G–R vanishing, we can finally compute the second local cohomology group at x.

PROPOSITION 3.12. *The following equalities hold:*

(i) $H^2_{g^{-1}(x)}(Z, \mathcal{O}_Z) = 0$;

(ii) $H^0_x(X, R^1 g_* \mathcal{O}_Z) \simeq H^0_x(C, R^1 g_* \mathcal{O}_E)$.

Proof. To prove i, we note that $H^2_{g^{-1}(x)}(Z, \mathcal{O}_Z) = R^1 g_* \mathcal{O}_Z(K_Z)_x$ by local duality for Cohen–Macaulay sheaves and $R^1 g_* \mathcal{O}_Z(K_Z)_x$ vanishes by Proposition 3.11.

To prove ii, as $-E \sim_{\mathbb{Q},X} K_Z + g_*^{-1}\Delta$ we can apply Proposition 3.11 to deduce $R^i g_* \mathcal{O}_Z(-E) = 0$ for $i > 0$. Then the long exact sequence of cohomology associated to $0 \to \mathcal{O}_Z(-E) \to \mathcal{O}_Z \to \mathcal{O}_E \to 0$ implies that $R^1 g_* \mathcal{O}_Z \cong R^1 g_* \mathcal{O}_E$. If $i \colon C \to X$ denotes the closed immersion, the equality $\Gamma_{C,x} = \Gamma_{X,x} \circ i_*$ holds, which implies $H^0_x(X, R^1 g_* \mathcal{O}_E) \simeq H^0_x(C, R^1 g_* \mathcal{O}_E)$. $\qquad\square$

Proof of Theorem 3.3 This is a consequence of Proposition 3.8 and Proposition 3.12. $\qquad\square$

4. (S_2)-condition for locally stable families of surfaces

In this section, we prove the (S_2)-conjecture for locally stable families of surfaces in characteristic $p \neq 2, 3$ and 5. An alternative proof of this theorem also appears in [Arv23, Corollary 23]. In Subsection 4.2, we use this result to show the properness of $\overline{\mathcal{M}}_{2,v}$ over $\mathbb{Z}[1/30]$, contingent upon the existence of semi-stable reduction for family of stable surfaces in positive and mixed characteristic.

4.1 Wild fibres

Setting 4.1. In this section, $(R, \mathfrak{m})$ is a DVR of perfect residue field $k = R/\mathfrak{m}$. We denote by C the spectrum of R and x is its closed point. Given a morphism $f \colon S \to C$, we denote by S_x the fibre over x.

The terminology of wild fibres was introduced by Bombieri and Mumford in [BM77] to study elliptic surface fibrations. We present a more general definition for fibrations of surfaces over curves and we collect some foundational results proven by Raynaud in [Ray70].

DEFINITION 4.2. Let S be a reduced connected surface and let $f\colon S \to C$ be a proper flat morphism such that $f_*\mathcal{O}_S = \mathcal{O}_C$. Consider the decomposition

$$R^1 f_* \mathcal{O}_S = \mathcal{M} \oplus \mathcal{T},$$

where $\mathcal{M}$ is a locally free sheaf of rank $\dim_{k(C)} H^1(S_{k(C)}, \mathcal{O}_{S_{k(C)}})$ and $\mathcal{T}$ is torsion sheaf supported at x. If $\mathcal{T}_x \neq 0$, we say that the schematic fibre $f^{-1}(x)$ is a wild fibre of f.

Given a proper flat morphism $f\colon X \to Y$, we say it is *cohomologically flat in degree 0* if for any morphism $g\colon Y' \to Y$ inducing the base change $f'\colon X' := X \times_Y Y' \to Y'$, then the canonical homomorphism of $\mathcal{O}_{Y'}$-modules $g^* f_* \mathcal{O}_X \to f'_* \mathcal{O}_{X'}$ is an isomorphism (see [FGI+05, 8.3.10]).

LEMMA 4.3. *Let S be a reduced connected surface and let $f\colon S \to C$ be a proper flat morphism such that $f_*\mathcal{O}_S = \mathcal{O}_C$. If x is a closed point of C, then*

$$\mathcal{T}_x \neq 0 \Leftrightarrow \dim_{k(x)} H^0(f^{-1}(x), \mathcal{O}_{f^{-1}(x)}) \geq 2.$$

In particular, a wild fibre is not reduced. Moreover, $\mathcal{T} \neq 0$ if and only if f is not cohomologically flat in degree 0.

Proof. For each $i \geq 0$, consider the natural homomorphism of $k(x)$-vector spaces obtained from the base change $\mathrm{Spec}(k(x)) \to C$:

$$\alpha^i(x)\colon R^i f_* \mathcal{O}_S \otimes k(x) \to H^i(f^{-1}(x), \mathcal{O}_{f^{-1}(x)}).$$

As the fibres of f have dimension 1, clearly $\alpha^2(x)$ is surjective and $R^2 f_* \mathcal{O}_S = 0$. Thus we deduce that $\alpha^1(x)$ is surjective by cohomology and base change for proper morphism [FGI+05, Corollary 8.3.11.b]. Applying once more [FGI+05, Corollary 8.3.11.b], we deduce that $\alpha^0(x)$ is an isomorphism if and only if $\mathcal{T} = 0$. To conclude, by hypothesis $f_* \mathcal{O}_S \otimes_{\mathcal{O}_C} k(x) \simeq k(x)$ and therefore $\alpha^0(x)$ is an isomorphism if and only if $\dim_{k(x)} H^0(f^{-1}(x), \mathcal{O}_{f^{-1}(x)}) = 1$.

Note that if $f^{-1}(x)$ is wild, then $\dim_{k(x)} H^0(f^{-1}(x), \mathcal{O}_{f^{-1}(x)}) \geq 2$. As $f^{-1}(x)$ is geometrically connected, we conclude $f^{-1}(x)$ is not reduced. The final assertion is shown in [FGI+05, Corollary 8.3.11.a]. $\square$

A more precise characterisation of wild fibres for (S_2)-surfaces was proven by Raynaud [Ray70] while investiganting representability criteria for Picard schemes of proper schemes over a DVR.

PROPOSITION 4.4. *Let $f\colon S \to C$ be a proper flat morphism such that $f_* \mathcal{O}_S = \mathcal{O}_C$. Suppose that*

(i) *S is an (S_2)-surface such that its non-normal locus dominates C;*

(ii) *the greatest common denominator of the multiplicities of the geometric special fibre $S_{\overline{k}} := S_k \times_k \overline{k}$ is equal to 1.*

Then f is cohomologically flat in degree 0 and $\mathcal{T} = 0$.

Proof. To verify the statement we can pass to a strict henselianisation A^{sh} of A. The hypothesis guarantee that S satisfies assumption $(N)^*$ of [Ray70, Definition 6.1.4]. Indeed, S_x is S_1 as it is a Cartier divisor on the (S_2)-surface S. Moreover, at every generic point η of an irreducible component of S_x we have that $\mathcal{O}_{S,\eta}$ is regular. Then the statement is proven in the implication $(i) \Rightarrow (iv)$ of [Ray70, Theorem 8.2.1]. $\square$

We recall that no wild fibres appear when the generic fibre of $f\colon S \to C$ is a tree of conic curves.

PROPOSITION 4.5. *Let $f\colon S \to C$ be a proper morphism onto a regular curve such that $f_*\mathcal{O}_S = \mathcal{O}_C$. Suppose that S is an (S_2)-surface such that each of its irreducible components S_i dominates C and the non-normal locus of each S_i dominates C. If $H^1(S_{k(C)}, \mathcal{O}_{S_{k(C)}}) = 0$, then $\mathcal{T} = 0$.*

Proof. As every irreducible component of S dominates C, the morphism f is equi-dimensional. As S has dimension 2, it is Cohen–Macaulay and, as C is regular, f is flat by miracle flatness [Sta, Tag 00R4]. We can thus apply [Ray70, Proposition 9.3.1]. $\square$

Remark 4.6. The conditions imposed on S in Proposition 4.4 and Proposition 4.5 are optimal as shown in the examples of [Ray70, Section 9].

If the generic fibre has arithmetic genus at least 1, then wild fibres can appear when the residue field has characteristic $p > 0$.

Example 4.7. Suppose k is algebraically closed of characteristic $p > 0$. We recall the construction of wild fibres explained by Raynaud [Ray70]. Let $E_{k(C)}$ be an elliptic curve (ordinary if $\mathrm{char}(k(C)) > 0$) such that the special fibre of its Néron model is either a supersingular elliptic curve E_k or the multiplicative group $\mathbb{G}_{m,k}$. Let $S_{k(C)}$ be a regular torsor over $E_{k(C)}$ of order p^n for $n > 0$ and let $f\colon S \to C$ be its minimal model. Then $\mathcal{T} \neq 0$ by [Ray70, Théoréme 9.4.1.b].

Another set of examples, based on Artin–Schreier coverings, is discussed in [KU85, Section 8].

4.2 Cohen–Macaulay criteria for log canonical 3-fold singularities

Throughout this section, we suppose (X, Δ) is 3-dimensional log canonical singularity as in Setting 3.1.

Note that, as proved in [AP23, Theorem 1], the 1-dimensional scheme C is normal and thus regular. Using Theorem 3.3 the failure of Cohen–Macaulay-ness is explained by the presence of a wild fibre on a proper birational modification. We use this to show that if the surface singularity at the generic point of C is rational, then X is Cohen–Macaulay.

PROPOSITION 4.8. *Let $C \subset (X, \Delta)$ as in Setting 3.4. If $\mathcal{O}_{X,\eta}$ is a rational surface singularity, then X is (S_3) at x. In particular, if $C \subset \mathrm{Supp}(\Delta)$, then X is (S_3) at x.*

Proof. Let $g\colon (Z, g_*^{-1}\Delta + E) \to (X, \Delta)$ be the crepant proper birational morphism constructed in Theorem 3.3. It is sufficient to show that $H_x^0(E, R^1(g|_E)_*\mathcal{O}_E)$ vanishes to conclude. Note that E is (S_2). Moreover, the irreducible components of E and their non-normal loci dominate C as it is the minimal log canonical centre passing through x. As $\mathcal{O}_{X,\eta}$ is a rational singularity and C is regular, we apply Proposition 4.5 to conclude $H_x^0(C, R^1 g_*\mathcal{O}_E) = 0$. For the last assertion, we just observe that $\mathcal{O}_{X,\eta}$ is a rational singularity by [Kol13, Proposition 2.28]. $\square$

COROLLARY 4.9. *Let $C \subset (X, \Delta)$ as in Section 3.4. Let $g\colon (Z, g_*^{-1}\Delta + E) \to (X, \Delta)$ be the modification constructed in Theorem 3.3. If $H_x^2(X, \mathcal{O}_X) \neq 0$, then*

(i) $\mathcal{O}_{X,\eta}$ is not rational and C is not contained in $\mathrm{Supp}(\Delta)$,

(ii) the fibre E_x is wild.

Proof. (a) is proven in Proposition 4.8. As $h^1(E_\eta, \mathcal{O}_{E_\eta}) \neq 0$ and $\mathcal{O}_{X,\eta}$ is log canonical, then $\deg_\eta \omega_{E_\eta} = 0$. By Theorem 3.3.iii we have $H_x^0(C, R^1 g_*\mathcal{O}_E) \simeq H_x^2(X, \mathcal{O}_X) \neq 0$, and thus E_x is a wild fibre for $g\colon E \to C$. $\square$

As a byproducts of the results of Section 3 and the properties of wild fibres Proposition 4.4.1, we can prove the (S_3)-condition of log canonical 3-folds pairs in the case where a Cartier divisor is an addendum of the boundary divisor. This answers [Kol23a, Question 8] affirmatively if the characteristic of the residue field is different from $2, 3$ and 5.

THEOREM 4.10. *Let $(X, X_0 + \Delta)$ be a 3-dimensional log canonical pair and let x be a closed point of X. Suppose X_0 is a non-zero effective Cartier divisor such that $x \in X_0$. Then X is (S_3) at x.*

Proof. We can localise at the closed point x. As X_0 is effective, x cannot be a minimal log canonical centre for (X, Δ). By Lemma 3.2 and Corollary 4.9, we can suppose that

(i) the minimal log canonical centre C of $(X, X_0 + \Delta)$ passing through x has dimension 1;

(ii) if η is the generic point of C, then $\mathcal{O}_{X,\eta}$ is a non-rational surface singularity and $\eta \notin \mathrm{Supp}(X_0 + \Delta)$.

If $g \colon (Z, g_*^{-1}\Delta + E) \to (X, \Delta)$ is the crepant proper birational morphism constructed in Theorem 3.3, it is sufficient to show that $H_x^0(E, R^1(g|_E)_* \mathcal{O}_E)$ vanishes to conclude. For this we argue by contradiction.

First we note that the pair $(Z, g_*^{-1}\Delta + E + g^*X_0)$ is log canonical and E is (S_2). By adjunction [Kol13, Lemma 4.8] the pair $(E, \mathrm{Diff}_E(g^*X_0 + g_*^{-1}\Delta))$ is slc and, as X_0 is Cartier, we have $\mathrm{Diff}_E(g^*X_0 + g_*^{-1}\Delta) = (g^*X_0)|_E + \mathrm{Diff}_E(g_*^{-1}\Delta)$ by [Kol13, Lemma 2.5]. As E_x is a wild fibre, by Proposition 4.4 each of its irreducible components is non-reduced. As X_0 is an effective Cartier divisor not containing C, then $(g^*X_0)|_E$ must have coefficients strictly larger than 1, contradicting that $(E, \mathrm{Diff}_E(g^*X_0 + g_*^{-1}\Delta))$ is slc. $\qquad\square$

4.3 Properness of the moduli space of stable surfaces

We briefly recall the natural set-up for the study of stable and locally stable families and we refer to [Kol23b, Chapter 2] for a thorough discussion.

Let $C = \mathrm{Spec}(R)$, where $(R, \mathfrak{m})$ is a DVR with perfect residue field $k := R/\mathfrak{m}$ of characteristic $p > 0$, and fraction field $K := \mathrm{Frac}(R)$. We say that a morphism $f \colon X \to C$ is *family of varieties* is f is a flat morphism of finite type such that for every $c \in C$ the fiber X_c is pure dimensional, geometrically reduced and geometrically connected. We denote the special (resp. generic) fibre of f by X_k (resp. X_K). A *family of pairs* is $f \colon (X, \Delta) \to C$ is a family of varieties $f \colon X \to C$ together with an effective Mumford $\mathbb{Q}$-divisor Δ on X such that $\mathrm{Supp}(\Delta)$ does not contain any irreducible components of X_k and none of the irreducible components of $X_k \cap \mathrm{Supp}(\Delta)$ is contained in $\mathrm{Sing}(X_k)$.

DEFINITION 4.11. We say $f \colon (X, \Delta) \to C$ is a locally stable (or slc) family if f is a family of pairs and $(X, \Delta + X_k)$ has slc singularities. We say f is a stable family if f is a projective locally stable family such that $K_X + \Delta$ is ample over C.

In [BMP+23, Corollary 10.2], $\overline{\mathcal{M}}_{2,v}$ is shown to exist as a separated Artin stack of finite type over $\mathbb{Z}[1/30]$ with finite diagonal. The main open question on $\overline{\mathcal{M}}_{2,v}$ is whether it is a proper stack over $\mathbb{Z}[1/30]$ (some cases are discussed in [BMP+23, Theorem 10.6]).

To prove properness, one has to prove the valutative criterion for families of stable surfaces. As explained in [Pos21b, Section 6], this can be reduced to two problems on locally stable

families: the existence of a locally-stable reduction and the (S_2)-condition on stable limits. We recall their precise formulation.

(LSR) Let $X \to C$ be a flat projective morphism where X is a regular 3-fold. Let E be a reduced effective divisor on X such that $(X, E + (X_k)_{\mathrm{red}})$ is snc for every closed point $c \in C$. Then there exists a finite morphism $C' \to C$ such that: if Y is the normalization of $X \times_C C'$ and E_Y is the pull-back divisor, then every closed fiber $Y_{k'}$ is reduced and every $(Y, E_Y + Y_{k'})$ is log canonical.

(S_2) Let $(X, \Delta) \to C$ be a stable family of surface pairs. Then X_k is (S_2).

In equicharacteristic 0, existence of semi-stable reduction is proven in [KKM$^+$73] (see also [KM98, Theorem 7.17]) and the (S_2)-property is proven in [Ale08] (see also [Kol23b, Definition-Theorem 2.3]). While semi-stable reduction of surfaces is still an open conjecture, the results of [BK23] can be used to prove the (S_2)-condition for the closure of the locus of klt stable varieties (see the last lines of the proof of [BMP$^+$23, Theorem 10.6]). We now settle the general semi-log canonical case.

THEOREM 4.12. *Suppose $p \neq 2, 3$ and 5. If $f \colon (X, \Delta) \to C$ is a stable family of surfaces, then X_k is (S_2) and $(X_k, \mathrm{Diff}_{X_k}(\Delta))$ is slc.*

Proof. If X is normal, then $(X, \Delta + X_k)$ is a log canonical pair. For every closed point $p \in X_k$, the local ring $\mathcal{O}_{X,p}$ is (S_3) by Theorem 4.2.3. As X_k is a Cartier divisor, we deduce X_k is (S_2) by [Kol13, Corollary 2.61]. By performing adjunction [Kol13, Definition 4.2], we deduce that the normalisation $(X_k^\nu, \mathrm{Diff}_{X_k^\nu}(\Delta))$ is log canonical by [Kol13, Lemma 4.8]. Therefore $(X_k, \mathrm{Diff}_{X_k}(\Delta))$ is semi-log canonical by definition.

Suppose X is demi-normal and let $\pi \colon Y \to X$ be its normalisation. We write $K_Y + D + \pi^*\Delta = \pi^*(K_X + \Delta)$, where D is the divisorial part of the conductor. Then $(Y, D + \pi^*\Delta) \to C$ is a stable family of pairs, where Y is normal. By the previous step, Y_k is (S_2) and the pair $(Y_k, \mathrm{Diff}_{Y_k}(\pi^*\Delta))$ is slc. We conclude X_k is S_2 and $(X_k, \mathrm{Diff}_{X_k}(\Delta))$ is slc by [Pos21a, Proposition 4.2.6]. $\square$

We now have all the ingredients to prove the main result of this article.

THEOREM 4.13. *Assume (LSR). Then the moduli stack $\overline{\mathcal{M}}_{2,v}$ is proper over $\mathbb{Z}[1/30]$ and the coarse moduli space $\overline{M}_{2,v}$ is projective over $\mathbb{Z}[1/30]$.*

Proof. The proof of [Pos21b, Theorem 6.0.5] works also in mixed characteristic and thus the (LSR) hypothesis together with Theorem 4.12 conclude the properness of $\overline{\mathcal{M}}_{2,v}$. The projectivity of $\overline{M}_{2,v}$ is then shown in [Pat17, Theorem 1.2]. $\square$

We conclude by giving an application to the asymptotic invariance of plurigenera for log canonical minimal surface pairs of general type. This generalises the klt case proven in [BBS24, Theorem 4.1].

COROLLARY 4.14. *Suppose $p > 5$. Let (X, Δ) be a 3-dimensional pair and let $\pi \colon (X, \Delta) \to C$ be a projective contraction such that $(X_k, \mathrm{Diff}_{X_k}(\Delta))$ is log canonical. If $K_X + \Delta$ is nef and big over C, then there exists $m_0 > 0$ such that*

$$\dim_k H^0(X_k, m(K_{X_k} + \Delta_k)) = \dim_K H^0(X_K, m(K_{X_K} + \Delta_K)) \ \ \textit{for all} \ \ m \in m_0 \mathbb{N}$$

Proof. By inversion of adjunction [BMP$^+$23, Corollary 10.1], we conclude $(X, X_k + \Delta)$ is a log canonical pair. We first claim that $K_X + \Delta$ is semi-ample over C. Since the characteristic of

the residue field of R is $p > 0$, by [Wit21, Theorem 2.2] it is sufficient to check semiampleness fiber-wise. For this, note that $K_{X_K} + \Delta_K$ and $(K_X + \Delta)|_{X_k} \sim_\mathbb{Q} K_{X_k} + \mathrm{Diff}_{X_k}(\Delta)$ are semiample by the abundance theorem for log canonical surfaces [Tan20]. Let $f : (X, \Delta) \to (Z, \Delta_Z = f_*\Delta)$ be the semiample birational contraction associated to $K_X + \Delta$. Note that $f : (X, X_k + \Delta) \to (Z, Z_k + \Delta)$ is also crepant.

By [BBS24, Lemma 2.17], it is sufficient to check that $(f_k)_*\mathcal{O}_{X_k} = \mathcal{O}_{Z_k}$ to conclude. By considering the Stein factorisation $f_k : X_k \to Y \to Z_k$, we are left to show $g : Y \to Z_k$ is an isomorphism. As the morphism $g : Y \to Z_k$ is a birational morphism and Y is normal, it is sufficient to verify that Z_k is normal to conclude. By construction $(Z, Z_k + \mathrm{Diff}_Z(\Delta_Z))$ is log canonical and thus Z_k satisfies the (S_2) condition by Theorem 4.12. By Serre's criterion for normality, we are thus left to show that Z_k is (R_1), and we argue by contradiction. Suppose there exists a codimension 1 point η of Z_k such that Z_k is not normal. Then by inversion of adjunction η is the generic point of a log canonical centre of $(Z, Z_k + \Delta)$ and thus it is nodal. By [Bri22, Lemma 2.7] the normalisation of Z_k is a universal homeomorphism and thus we conclude. $\square$

Remark 4.15. In [Kol23a, Theorem 1], Kollár proves that the moduli space of stable 3-folds is not proper over any field of characteristic $p > 0$. In particular, [Kol23a, Example 4] show that Theorem 4.12, Corollary 4.13 and Corollary 4.14 do not generalise to dimension 3, even for large p.

Question 4.16. We leave open the question whether Theorem 4.3.2 hold in characteristic $p \leq 5$. Note that the examples of non-normal plt centres constructed [CT19] are not Cartier.

5. Counterexamples to local Kawamata–Viehweg vanishing

We conclude by constructing a counterexample to the local Kawamata–Viehweg vanishing theorem for log canonical 3-dimensional singularities in positive and mixed characteristic (Theorem 1.3). The counterexample is obtained by taking the relative cone over an elliptic surface fibration with a wild fibre.

5.1 Relative cone construction

In this section we develop the theory of relative cones, expanding on [Kol13, Section 3.2]. Let $f : X \to T$ be a projective flat morphism of normal integral schemes with $f_*\mathcal{O}_X = \mathcal{O}_T$ and let L be an f-ample invertible sheaf. We define the affine T-scheme:

$$C_a(X, f, L) = \mathrm{Spec}_T \bigoplus_{m \geq 0} f_*\mathcal{O}_X(mL) \to T.$$

The scheme $C_a(X, f, L)$ is the *relative cone* of f with respect to L. The natural subscheme $V_T \subset C_a(X, f, L)$ defined by the ideal sheaf $\bigoplus_{m \geq 1} f_*\mathcal{O}_X(mL)$ is isomorphic to T and it is called the *relative vertex*. The variety $C_a^*(X, f, L) = C_a(X, f, L) \setminus V_T \simeq BC_a^*(X, f, L) \setminus E$ is called the *relative punctured cone*.

We have the following commutative diagram

$$
\begin{array}{ccc}
BC_a(X, L) := \mathrm{Spec}_X \bigoplus_{m \geq 0} L^m & \xrightarrow{\ p\ } & C_a(X, f, L) \\
{\scriptstyle \pi} \downarrow & & \downarrow \\
X & \xrightarrow{\quad f \quad} & T,
\end{array}
$$

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where p is a birational projective morphism with exceptional divisor $E \simeq X$ onto V_T such that

$$\mathcal{O}_{BC_a(X,f,L)}(E)|_E = L^{\vee}.$$

The following is a generalisation of [Kol13, Proposition 3.14] to the relative setting.

PROPOSITION 5.1. *With the same setting as above, we have*

(i) $\operatorname{Pic}(C_a(X, f, L)) \simeq \operatorname{Pic}(T)$;

(ii) $\operatorname{Cl}(C_a(X, f, L)) \simeq \operatorname{Cl}(X)/\langle L \rangle$.

Let Δ_X be a $\mathbb{Q}$-divisor on X, and assume $K_X + \Delta_X$ is $\mathbb{Q}$-Cartier. We define $\Delta_{BC_a(X,L)} = \pi^* \Delta$ and $\Delta_{C_a(X,f,L)} = p_* \Delta_{BC_a(X,L)}$. We have the following

(iii) $K_{BC_a(X,L)} + \Delta_{BC_a(X,L)} + E \sim_{\mathbb{Q}} \pi^*(K_X + \Delta)$,

(iv) $m(K_{C_a(X,f,L)} + \Delta_{C_a(X,f,L)})$ *is Cartier iff $m(K_X + \Delta) \sim_f L^{rm}$ for some $r \in \mathbb{Q}$. In this case we have*

$$K_{BC_a(X,L)} + \Delta_{BC_a(X,L)} + (1+r)E \sim_{\mathbb{Q}} p^*(K_{C_a(X,f,L)} + \Delta_{C_a(X,f,L)}).$$

Proof. Since $BC_a(X, L)$ is an $\mathbb{A}^1$-bundle over X, we have $\operatorname{Cl}(BC_a(X, L)) \simeq \operatorname{Cl}(X)$ and $\operatorname{Pic}(BC_a(X, L)) \simeq \operatorname{Pic}(X)$. Let us note that we have the following commutative diagram:

$$
\begin{array}{ccccc}
\operatorname{Pic}(X) & \xrightarrow{\pi^*} & \operatorname{Pic}(BC_a(X, f, L)) & \longrightarrow & \operatorname{Pic}(E) \\
\big\uparrow{\scriptstyle f^*} & & \big\uparrow{\scriptstyle p^*} & & \big\uparrow \\
\operatorname{Pic}(T) & \longrightarrow & \operatorname{Pic}(C_a(X, f, L)) & \longrightarrow & \operatorname{Pic}(V_T),
\end{array}
$$

where the top arrows are all isomorphisms. We prove (i). Let D be an invertible sheaf on $C_a(X, f, L)$; then $p^* D|_E$ is the pull-back of a line bundle on V_T, thus proving (i). Items (ii) and (iii) are proven in [Kol13, Proposition 3.14]. Recall that since $\operatorname{Pic}(C_a^*(X, f, L)) \hookrightarrow \operatorname{Cl}(C_a^*(X, f, L)) \simeq \operatorname{Cl}(X)/\langle L \rangle$, the kernel of the morphism $\pi|_{C_a^*(X,f,L)}^*\colon \operatorname{Pic}(X) \to \operatorname{Pic}(C_a^*(X, f, L))$ is the subgroup generated by L.

As for (iv) $m(K_{C_a^*(X,f,L)} + \Delta_{C_a^*(X,f,L)})$ is Cartier if and only if the Weil divisor $m(K_{C_a^*(X,f,L)} + \Delta_{C_a^*(X,f,L)})$ is the pull-back of a Cartier divisor on T by (i). In turn, this is equivalent to asking whether $\pi|_{C_a^*(X,f,L)}^*(mK_X + m\Delta) = \pi|_{C_a^*(X,f,L)}^* f^* D$ for some Cartier divisor D on T. This is equivalent to $m(K_X + \Delta) - f^* D \sim L^{rm}$ for some $r \in \mathbb{Q}$, thus concluding the first part. As for the last equality, let us write

$$K_{BC_a(X,L)} + \Delta_{BC_a(X,L)} + (1+a)E \sim_{\mathbb{Q}} p^*(K_{C_a(X,f,L)} + \Delta_{C_a(X,f,L)}).$$

By restricting to E, we have $K_X + \Delta + aE|_E \sim_{\mathbb{Q},f} 0$, which becomes $rL - aL = 0$; thus $r = a$. $\square$

As a corollary we have the following result on the singularities of relative cones (to compare with [Kol13, Lemma 3.1]).

PROPOSITION 5.2. *In the previous setting, assume $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $K_X + \Delta \sim_{f,\mathbb{Q}} rL$ for some $r \in \mathbb{Q}$. Then $(C_a(X, f, L), \Delta_{C_a(X,f,L)})$ is*

(i) *klt if $r < 0$ and (X, Δ) is klt;*

(ii) *log canonical if $r \leq 0$ and (X, Δ) is log canonical.*

Proof. By Proposition 5.1, we have

$$\mathrm{discrep}((C_a(X, f, L), \Delta_{C_a(X,f,L)})) = \min\left\{-(1+r), \mathrm{discrep}(BC_a(X, L), \Delta_{BC_a(X,L)} + (1+r)E)\right\}.$$

Since π is a smooth morphism and E is a section for π, we conclude by [Kol13, 2.14, Equation (4)] that

$$\mathrm{discrep}(BC_a(X, L), \Delta_{BC_a(X,L)} + E) = \mathrm{totdiscrep}(X, \Delta).$$

Thus (a) and (b) are automatic. $\qquad\square$

5.2 Failure of the (S_3)-condition at a non-minimal lc centre

We construct an example showing that Theorem 1.2 is not valid in general in positive and mixed characteristic, thus showing that the statement of Theorem 4.10 is sharp. We fix C to be the spectrum of a DVR whose closed point is perfect of characteristic $p > 0$.

PROPOSITION 5.3. *Let S be a regular surface, and let $f: S \to C$ be a minimal elliptic fibration together with a relatively f-ample invertible sheaf L. Then $X := C_a(S, f, L)$ is a 3-dimensional log canonical singularity, the map $p: Y := BC_a(S, L) \to X$ is a log resolution, and the vertex V_C is the unique log canonical centre of X.*

Proof. By Proposition 5.1, we have $K_Y + E \sim_{\mathbb{Q}} p^* K_X$. Since (Y, E) is log smooth and E is irreducible, we conclude that E is the unique log canonical place of X. Thus V_C is the unique log canonical centre of X. $\qquad\square$

In the next lemmas we compute the local cohomology at a closed point in a log canonical place.

LEMMA 5.4. *Let D be a $\mathbb{Z}$-divisor on Y. If $D \sim_{p,\mathbb{Q}} K_Y$, then $R^1 p_* \mathcal{O}_Y(D) = 0$.*

Proof. By [Tan18, Theorem 3.3], the relative Kawamata–Viehweg vanishing theorem holds for the morphism $p|_E: E \to V_C$. Since $-E$ is p-ample and $D \sim_{p,\mathbb{Q}} (K_Y + E) - E$, we conclude by [BK23, Proposition 22]. $\qquad\square$

PROPOSITION 5.5. *Let $x \in V_C \simeq C$. Then $H_x^2(X, \mathcal{O}_X) \simeq H_x^0(C, \mathcal{T})$.*

Proof. By the Leray spectral sequence for local cohomology $H_x^i(X, R^j p_* \mathcal{O}_Y) \Rightarrow H_{p^{-1}(x)}^{i+j}(Y, \mathcal{O}_Y)$, we have the exact sequence

$$H_x^0(X, R^1 p_* \mathcal{O}_Y) \to H_x^2(X, \mathcal{O}_X) \to H_{p^{-1}(x)}^2(Y, \mathcal{O}_Y).$$

Claim 5.6. The following isomorphisms hold:

(i) $H_{p^{-1}(x)}^2(Y, \mathcal{O}_Y) = 0$,

(ii) $R^1 p_* \mathcal{O}_Y \simeq R^1 (p|_E)_* \mathcal{O}_E$.

Proof. To prove (i), because Y is regular we can apply duality [Kol13, 10.44] to deduce $H_{p^{-1}(x)}^2(Y, \mathcal{O}_Y) \simeq (R^1 p_* \mathcal{O}_Y(K_Y))_x$, and we conclude by Lemma 5.4.

To prove (ii), it is enough to show that $R^1 p_* \mathcal{O}_Y(-E) = 0$ as $R^2 \pi_* \mathcal{O}_Y(-E) = 0$ since the fibres of p are at most 1-dimensional. Since $-E \sim_{p,\mathbb{Q}} K_Y$, we conclude again by Lemma 5.4. $\qquad\square$

Using Corollary 5.6, we have

$$H_x^2(X, \mathcal{O}_X) \simeq H_x^0(X, R^1 p_* \mathcal{O}_Y) \simeq H_x^0(X, R^1(p|_E)_* \mathcal{O}_E).$$

We denote by $i\colon V_C \to X$ the natural injection. Then we have the following isomorphism $H_x^0(X, R^1(p|_E)_* \mathcal{O}_E) \simeq H_x^0(X, i_*(\mathcal{M} \oplus \mathcal{T}))$. Using again the Leray spectral sequence of local cohomology for i we have $H_x^0(X, i_*(\mathcal{M} \oplus \mathcal{T})) \simeq H_x^0(C, \mathcal{M} \oplus \mathcal{T}) = H_x^0(C, \mathcal{T})$, since $\mathcal{M}$ is locally free thus concluding. $\square$

Proof of Theorem 1.3. Let $f\colon S \to C$ be a minimal elliptic fibration such that $\mathcal{T} \neq 0$ (such a surface exists by Example 4.7). By Proposition 5.3, $X = C_a(S, f, L)$ is a 3-dimensional log canonical variety, where the relative vertex V_C is the unique minimal log canonical centre. By Proposition 5.5, we deduce that the local cohomology $H_x^2(X, \mathcal{O}_X) \neq 0$, thus proving (c). $\square$

Question 5.7. In Theorem 1.3 we construct a log canonical 3-fold singularity X with a minimal 1-dimensional log canonical centre and not C–M, showing optimality of Proposition 4.2.1 and Theorem 4.2.3 in the case where the exceptional divisor E_η is a regular curve of genus 1. We do not know if the failure of Cohen–Macaualy-ness can appear in the case where E_η is a nodal curve. $\square$

Remark 5.8. A guiding principle in birational geometry in characteristic p says that properties of klt and dlt singularities should behave similarly to characteristic 0 if p is sufficiently large compared to the dimension [Tot19, Section 6]. This principle does not apply to log canonical singularities. For example, in [Kol23a, Corollary 6], Kollár shows examples of 4-dimensional log canonical pairs with non-weakly normal lc centres in every characteristic $p > 0$. Theorem 1.3 shows that pathological phenomena already appear in dimension 3 for every prime number.

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