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GEODESICS AND KILLING VECTOR FIELDS ON THE TANGENT SPHERE BUNDLE

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Abstract. We show that any Killing vector field on the unit tangent sphere bundle with Sasaki metric of a space of constant curvature $k \neq 1$ is fiber preserving by studying some property of geodesics on the bundle. As a consequence, any Killing vector field on the unit tangent sphere bundle of a space of constant curvature $k \neq 1$ can be extended to a Killing vector field on the tangent bundle.

§1. Introduction

Let (T_1M, g^S) be the unit tangent sphere bundle with Sasaki metric g^S of a Riemannian manifold (M, g). A Killing vector field Z on (T_1M, g^S) is called fiber preserving if each (local) isometry ϕ_t generated by Z maps each fiber into a fiber. If $(M(k), g)$ is a space of constant curvature k, then geodesic flow vector ξ of $(T_1M(k), g^S)$ is a Killing vector field if and only if $k = 1$ (cf. Tanno [6]). It is important to notice that ξ is not fiber preserving.

In this paper we obtain the following:

THEOREM A. *Let $(T_1M(k), g^S)$ be the unit tangent sphere bundle of a space $(M(k), g)$ of constant curvature k. If $k \neq 1$, then any Killing vector field on $(T_1M(k), g^S)$ is fiber preserving.*

One of the authors [3] proved that any Killing vector field Z on the unit tangent sphere bundle (T_1M, g^S) of a Riemannian manifold (M, g) can be extended to a Killing vector field on the tangent bundle (TM, g^S) if Z is fiber preserving. So, by Theorem A we have the following:

THEOREM B. *Any Killing vector field on the unit tangent sphere bundle $(T_1M(k), g^S)$ of a space $(M(k), g)$ of constant curvature $k \neq 1$ can be extended to a Killing vector field on the tangent bundle $(TM(k), g^S)$.*

To prove Theorem A we study some property of geodesics on $(T_1M(k), g^S)$ in §2. As for basic geometry on the tangent bundle or tangent sphere bundle, cf. Dombrowski [1], Sasaki [5], etc.

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§2. Geodesics in $(T_1M(k), g^S)$

Let (T_1M, g^S) be the unit tangent sphere bundle with Sasaki metric g^S of a Riemannian manifold (M, g). By π we denote the projection from T_1M to M. Let $\tilde{C} = \{(x(\sigma), y(\sigma)); 0 \le \sigma \le l\}$ be a curve in (T_1M, g^S) with arc-length parameter σ, where $y(\sigma) \in T_{x(\sigma)}M, g(y(\sigma), y(\sigma)) = 1$. It is a geodesic if and only if

$$(2.1) \qquad \begin{aligned} &(\nabla_{x'}x')(\sigma) = -R(y(\sigma), (\nabla_{x'}y)(\sigma))x'(\sigma), \\ &(\nabla_{x'}\nabla_{x'}y)(\sigma) = \rho(\sigma)y(\sigma) \end{aligned}$$

hold for some function $\rho(\sigma)$, where ∇ (R, resp.) denotes the Riemannian connection (Riemannian curvature tensor, resp.) of (M, g), and $x'(\sigma) = dx(\sigma)/d\sigma$ (cf. Sasaki [5], II, p.152).

Geodesics on the unit tangent sphere bundle $(T_1S^2(1), g^S)$ of a unit 2-sphere $(S^2(1), g)$ were studied by Klingenberg and Sasaki [2]. For a unit m-sphere $(S^m(k), g)$, $m \ge 3$, and for more general $(M(k), g)$, we have the following (cf. Nagy [4]):

THEOREM C. *Let $(T_1M(k), g^S)$ be the unit tangent sphere bundle of a space $(M(k), g)$ of constant curvature k, and let $\tilde{C} = \{(x(\sigma), y(\sigma)); 0 \le \sigma \le l\}$ be a geodesic with arc-length parameter σ in $(T_1M(k), g^S)$. By $C = \{x(\sigma)\}$ we denote the projection $\pi\tilde{C}$ of $\tilde{C}$. Then $\|x'\|^2 = 1 - c^2$ is constant, where $0 \le |c| \le 1$.*

(i) *If $|c| = 1$,* i.e., *C reduces to a point, then $\tilde{C}$ is a (piece of) great circle in a fiber and y is rotated in a 2-plane at $x(0)$.*

(ii) *If $0 < |c| < 1$, then we have the following:*

(ii-a-1) *The geodesic curvature κ of C is constant.*

(ii-a-2) *C satisfies*

$$(2.2) \qquad \nabla_{x'}\nabla_{x'}\nabla_{x'}x' = -k^2c^2\nabla_{x'}x'.$$

(ii-b-1) *If $k = 0$, then $\kappa = 0$ and we have parallel orthonormal vector fields $\{E_1, E_2\}$ along C such that*

$$(2.3) \qquad y(\sigma) = \cos c\sigma \cdot E_1(\sigma) + \sin c\sigma \cdot E_2(\sigma).$$

(ii-b-2) *If $k \ne 0$ and $\kappa = 0$, then we have parallel orthonormal vector fields $\{E_1, E_2, x'/\|x'\|\}$ along C such that y is of the form (2.3).*

(ii-b-3) *If $k \neq 0$ and $\kappa \neq 0$, then we have parallel orthonormal vector fields $\{E_1, E_2\}$ along C such that y is of the form* (2.3).

(ii-c) *The angle $\theta(\sigma)$ between $y(\sigma)$ and $x'(\sigma)$ is given by*

$$\cos\theta(\sigma) = [\alpha/\sqrt{1-c^2}]\sin[(1-k)c\sigma + \beta] \tag{2.4}$$

where α and β are constant. α^2 is given by (2.9) *and $\alpha = 0$ for* (ii-b-2).

(iii) *If $c = 0$, then $C = \{x(\sigma)\}$ is a geodesic with arc-length parameter σ, and y is a parallel vector field along C.*

Proof. By (2.1) the equations of geodesic in $(T_1M(k), g^S)$ is given by

$$\nabla_{x'}x' = -kby + ka\nabla_{x'}y, \qquad \nabla_{x'}\nabla_{x'}y = \rho y, \tag{2.5}$$

where we have put

$$a = a(\sigma) = g(x', y), \qquad b = b(\sigma) = g(x', \nabla_{x'}y), \tag{2.6}$$

and we sometimes omit the parameter σ from the expression for simplicity. We put

$$c^2 = c^2(\sigma) = g(\nabla_{x'}y, \nabla_{x'}y). \tag{2.7}$$

By $g(y,y) = 1$, we have $g(y, \nabla_{x'}y) = 0$ and

$$g(\nabla_{x'}y, \nabla_{x'}y) + g(y, \nabla_{x'}\nabla_{x'}y) = 0, \qquad \text{i.e.,} \quad c^2 + \rho = 0.$$

Differentiating (2.7) and using $(2.5)_2$ we see that c is constant.

Let X be a tangent vector at a point of $(M(k), g)$. By X^H or X^V we denote the horizontal lift or vertical lift of X to $(TM(k), g^S)$ or $(T_1M(k), g^S)$. Since the tangent vector field T of $\tilde{C}$ is expressed as

$$T = \left(\frac{dx}{d\sigma}, \frac{dy}{d\sigma}\right) = x'^H + (\nabla_{x'}y)^V,$$

we have $1 = \|T\|^2 = \|x'\|^2 + c^2$. Therefore $\|x'\|^2 = 1 - c^2$ is constant, and the parameter σ of $C = \{x(\sigma)\}$ is proportional to the arc-length.

If $|c| = 1$, i.e., $\|x'\| = 0$, then $\tilde{C}$ is a geodesic in a fiber. Since each fiber is totally geodesic and isometric to a unit $(m-1)$-sphere, it is a (piece of) great circle. So, y is expressed as $y(\sigma) = \cos\sigma \cdot e_1 + \sin\sigma \cdot e_2$ for some orthonormal vectors $\{e_1, e_2\}$ at $x(0)$.

Next we assume $0 < |c| < 1$. Calculating $a' = \nabla_{x'} a$ and $b' = \nabla_{x'} b$, we obtain

$$a' = (1-k)b, \qquad b' = -(1-k)ac^2. \tag{2.8}$$

Operating $\nabla_{x'}$ to $(2.5)_1$ twice, we obtain (2.2). By $(2.5)_1$ we obtain

$$\|\nabla_{x'} x'\|^2 = k^2(c^2a^2 + b^2).$$

By (2.8) we can show that $c^2a^2 + b^2$ is constant, and the geodesic curvature κ;

$$\kappa^2 = k^2(c^2a^2 + b^2)/(1-c^2)^2.$$

of C is constant. This proves (ii-a-1) and the first part of (ii-b-1). By (2.8) again, we have

$$a(\sigma) = \alpha \sin[(1-k)c\sigma + \beta], \qquad b(\sigma) = c\alpha \cos[(1-k)c\sigma + \beta],$$

where α and β are constant. Here, α^2 is expressed as

$$\alpha^2 = (c^2a^2 + b^2)/c^2 \qquad (\alpha^2 = (1-c^2)^2\kappa^2/c^2k^2, \ \text{if} \ k \neq 0). \tag{2.9}$$

The angle $\theta(\sigma)$ between $y(\sigma)$ and $x'(\sigma)$ is given by $\cos\theta(\sigma) = g(x'/\|x'\|, y) = a(\sigma)/\sqrt{1-c^2}$ and we obtain (2.4).

Now we define vector fields E_1 and E_2 along C by

$$E_1(\sigma) = \cos c\sigma \cdot y(\sigma) - \sin c\sigma \cdot (\nabla_{x'} y)(\sigma)/c,$$

$$E_2(\sigma) = \sin c\sigma \cdot y(\sigma) + \cos c\sigma \cdot (\nabla_{x'} y)(\sigma)/c.$$

Then E_1 and E_2 are parallel orthonormal vector fields along C, and define a parallel 2-plane field Π along C. y is rotated in Π as

$$y(\sigma) = \cos c\sigma \cdot E_1(\sigma) + \sin c\sigma \cdot E_2(\sigma).$$

This proves (ii-b-1) and (ii-b-3). If $k \neq 0$ and $\kappa = 0$, then $a = b = 0$. So, $\{E_1, E_2, x'/\|x'\|\}$ are orthonormal and we have (ii-b-2).

Finally, if $c = 0$, then we have $\nabla_{x'} y = 0$, $\nabla_{x'} x' = 0$ and (iii). □

§3. The converse of Theorem C

THEOREM D. *Let* $(M(k), g)$ *be a space of constant curvature* k. *The converse of two cases* (i) *and* (iii) *of Theorem C is trivial. So, let* $C = \{x(\sigma); 0 \le \sigma \le l\}$ *be a curve of constant geodesic curvature* κ *with* $\|x'\|^2 = 1 - c^2, 0 < |c| < 1$. *Assume that* C *satisfies*

$$\nabla_{x'}\nabla_{x'}\nabla_{x'}x' = -k^2c^2\nabla_{x'}x'. \tag{3.1}$$

(ii*-1) *If* $k = 0$, *then we assume* $\kappa = 0$. *Let* $\{E_1, E_2\}$ *be parallel orthonormal vector fields along* C *and define a vector field* y *along* C *by*

$$y(\sigma) = \cos c\sigma \cdot E_1(\sigma) + \sin c\sigma \cdot E_2(\sigma). \tag{3.2}$$

Then $\tilde{C} = \{(x(\sigma), y(\sigma))\}$ *is a geodesic in* $(T_1M(k), g^S)$.

(ii*-2) *If* $k \ne 0$ *and* $\kappa = 0$, *then let* $\{E_1, E_2, x'/\|x'\|\}$ *be parallel orthonormal vector fields along* C. *Define* y *by* (3.2). *Then* $\tilde{C} = \{(x(\sigma), y(\sigma))\}$ *is a geodesic in* $(T_1M(k), g^S)$.

(ii*-3) *If* $k \ne 0$ *and* $\kappa \ne 0$, *then let* $e_1 = (\nabla_{x'}x')(0)/(1-c^2)\kappa$ *and*

$$e_2 = (\nabla_{x'}\nabla_{x'}x')(0)/kc(1-c^2)\kappa.$$

Define $\{E_1, E_2\}$ *along* C *by parallel translation of* e_1 *and* e_2. *Next we define* y *by*

$$y(\sigma) = \cos(c\sigma + \gamma) \cdot E_1(\sigma) + \sin(c\sigma + \gamma) \cdot E_2(\sigma) \tag{3.3}$$

for constant γ. *Then* $\tilde{C} = \{(x(\sigma), y(\sigma))\}$ *is a geodesic in* $(T_1M(k), g^S)$.

Proof. First we prove (ii*-1) and (ii*-2). By $\kappa = 0$ we have $\nabla_{x'}x' = 0$. By (3.2) we obtain $\nabla_{x'}\nabla_{x'}y = -c^2y$ and, using $g(x', y) = g(x', \nabla_{x'}y) = 0$ for (ii*-2), we have (2.5).

Next, we show (ii*-3). Since $\nabla_{x'}x'/(1-c^2)\kappa$ is a unit vector field along C, we see that $\nabla_{x'}x'$ and $\nabla_{x'}\nabla_{x'}x'$ are orthogonal. Using (3.1), we obtain

$$g(\nabla_{x'}\nabla_{x'}x', \nabla_{x'}\nabla_{x'}x') = k^2c^2g(\nabla_{x'}x', \nabla_{x'}x')$$

and $\nabla_{x'}\nabla_{x'}x'/kc(1-c^2)\kappa$ is a unit vector field along C. Therefore, $\{e_1, e_2\}$ and hence parallel vector fields $\{E_1, E_2\}$ are orthonormal. Then $\{(x(\sigma), y(\sigma))\}$ defined by (3.3) satisfies $(2.5)_2$. By the differential equation (3.1) we have

$$(\nabla_{x'}x')(\sigma)/(1-c^2)\kappa = \cos kc\sigma \cdot E_1(\sigma) + \sin kc\sigma \cdot E_2(\sigma).$$

By (3.3) and $\nabla_{x'}y/c = -\sin(c\sigma+\gamma)E_1 + \cos(c\sigma+\gamma)E_2$, we obtain

$$\text{(3.4)} \qquad \nabla_{x'}x' = (1-c^2)\kappa\cos[(1-k)c\sigma+\gamma]y - (1-c^2)\kappa\sin[(1-k)c\sigma+\gamma]\nabla_{x'}y/c.$$

We put $a = g(x', y)$ and $b = g(x', \nabla_{x'}y)$. By (3.4) and $\nabla_{x'}\nabla_{x'}y = -c^2y$, we obtain

$$a' = (1-c^2)\kappa\cos[(1-k)c\sigma+\gamma] + b,$$
$$b' = -c(1-c^2)\kappa\sin[(1-k)c\sigma+\gamma] - c^2a.$$

Solving the above with $a(0) = g(x'(0), y(0)) = -[(1-c^2)\kappa/kc]\sin\gamma$. we get

$$a = -[(1-c^2)\kappa/kc]\sin[(1-k)c\sigma+\gamma],$$
$$b = -[(1-c^2)\kappa/k]\cos[(1-k)c\sigma+\gamma].$$

So, (3.4) is rewritten as $\nabla_{x'}x' = -kby + ka\nabla_{x'}y$. □

§4. Proof of Theorem A

Assume that a Killing vector field Z on $(T_1M(k), g^S)$ is not fiber preserving. Let $\{\phi_t\}$ be a (local) 1-parameter group of local isometries generated by Z. Since each fiber is totally geodesic in $(T_1M(k), g^S)$ and isometric to a unit $(m-1)$-sphere, we can choose a great circle $\tilde{C} = \{(x(\sigma), y(\sigma)); 0 \le \sigma \le 2\pi\}$ of length 2π in a fiber, such that $\phi_t\tilde{C}$ is not contained in a fiber for t with $0 < t < \varepsilon$. Here we can assume that the domain of definition of ϕ_t contains the fiber containing $\tilde{C}$. In this case, for small t with $0 < t < \varepsilon$, $C_t = \pi\phi_t\tilde{C}$ is a small closed curve, and it can not be a geodesic in $(M(k), g)$. We have $0 < |c_t| < 1$. By Theorem C, (ii-b-1), and (2.9) we see that $k \neq 0$ and $\alpha_t \neq 0$. By (ii-c) we see that the angle $\theta_t(\sigma)$ between $y_t(\sigma)$ and C_t is given by

$$\cos\theta_t(\sigma) = [\alpha_t/\sqrt{1-c_t^2}]\sin[(1-k)c_t\sigma + \beta_t].$$

As $t \to 0$, we have $|c_t| \to 1$. So we have many small t such that $(1-k)c_t$ is not an integer. This means $\theta_t(\sigma) \neq \theta_t(\sigma + 2\pi)$ for such t. This is a contradiction to the fact that $\tilde{C}_t$ is a closed geodesic for any t. □

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