Annals of Actuarial Science (2025), **19**, pp. 159–192
doi:10.1017/S1748499524000204

ORIGINAL RESEARCH PAPER

One-year and ultimate correlations in dependent claims run-off triangles

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(Received 15 September 2022; revised 05 June 2024; accepted 07 June 2024; first published online 14 November 2024)

Abstract

We investigate bottom-up risk aggregation applied by insurance companies facing reserve risk from multiple lines of business. Since risk capitals should be calculated in different time horizons and calendar years, depending on the regulatory or reporting regime (Solvency II vs IFRS 17), we study correlations of ultimate losses and correlations of one-year losses in future calendar years in lines of business. We consider a multivariate version of a Hertig's lognormal model and we derive analytical formulas for the ultimate correlation and the one-year correlations in future calendar years. Our main conclusion is that the correlation coefficients that should be used in a bottom-up aggregation formula depend on the time horizon and the future calendar year where the risk emerges. We investigate analytically and numerically properties of the ultimate and the one-year correlations, their possible values observed in practice, and the impact of misspecified correlations on the diversified risk capital.

Keywords: ultimate correlation; one-year correlations; claims reserving; Hertig's lognormal model; Solvency II; IFRS 17

1. Introduction

In this paper, we investigate bottom-up risk aggregation applied by insurance companies facing reserve risk from multiple lines of business. In the bottom-up approach, an insurance company first determines risk capitals for each line of business and next aggregates the risk capitals by applying the variance-covariance formula in order to determine a diversified risk capital at the level of a company. Such an approach to establish a diversified risk capital for an insurance company is specified in Solvency II Standard Formula and is often used in actuarial practice. Clearly, correlations play an important role in bottom-up risk aggregation. Depending on the regulatory or reporting regime, the required risk capitals should be calculated in different time horizons and calendar years. In Solvency II regulatory regime, insurance companies calculate solvency capital requirements and risk margins (risk capitals for future calendar years in one-year time horizons). In IFRS 17 reporting standard, insurance companies calculate risk adjustments (risk capitals in ultimate time horizon). The goal of this paper is to study correlation coefficients of ultimate losses and correlations of one-year losses in future calendar years in lines of business which one should use in ultimate horizon and in future calendar years in one-year horizon when applying a bottom-up risk aggregation formula. Questions about correlations in different time horizons and calendar years have recently gained more attention among actuaries, as insurance companies now have to quantify, at the same time, their risks in one-year and ultimate time horizons under Solvency II and IFRS 17.

In the actuarial literature, one can find many flexible multivariate claims development models based on Gaussian (lognormal) distributions, Tweedie GLMs, common shocks, and copulas, see among others Braun (2004), Shi and Frees (2011), Avanzi et al. (2016a), Avanzi et al. (2018), Iturria et al. (2021). In this paper, we consider a multivariate version of Hertig's lognormal model from Merz et al. (2012) and Chapter 5 in Wüthrich (2015). We focus on three types of dependence between claims developments in loss triangles, which include cell-wise, calendar year, and AR(1) trend correlation between and within loss triangles. We define ultimate correlation and one-year correlations in future calendar years between lines of business. By their definitions, these correlations serve as the inputs to the bottom-up risk aggregation formula which should be applied to determine a diversified risk capital from stand-alone risk capitals in ultimate and one-year time horizons. We derive analytical formulas for the ultimate correlation and the one-year correlations in future calendar years in our multivariate Hertig's lognormal model. The formulas allow us to study the values of the correlation coefficients for bottom-up risk aggregation in different time horizons and in different calendar years. They also allow us to switch (calibrate) the one-year correlation from the ultimate correlation, and vice verse. For special cases of our claims development model, we derive explicit (and simpler) formulas for the ultimate correlation and the one-year correlations in future calendar years. We analytically investigate the properties of the ultimate and the one-year correlations. Finally, we consider eleven lines of business under the Solvency II regulation from the Polish market and numerically study the properties of the ultimate and the one-year correlations, their possible values observed in practice, and the impact of misspecified correlations on diversified risk capital.

Compared to Merz et al. (2012) and Wüthrich (2015), the main contributions of this paper are the following:

- We derive new formulas for the ultimate and the one-year correlations, and prove their new properties,
- We derive new relations between the ultimate and the one-year correlations,
- We discuss in detail the role of information when we define predictions, mean squared errors of predictions, and, finally, correlations for losses for multiple lines of business,
- We present an extensive numerical study of the ultimate and the one-year correlations and their impact on capital based on real data.

Our first conclusion is that the correlation coefficients that should be used for bottom-up risk aggregation depend on the time horizon and the calendar year of the risk measurement period (at least in the Hertig's model). The three most important correlation coefficients that should be used for deriving the Solvency II capital requirement, the Solvency II risk margin, and the IFRS 17 risk adjustment are different. Our second conclusion is that we identify practically relevant cases when the ultimate correlation is smaller than the one-year correlation in the next calendar year. In these cases, if an insurance company uses one-year correlations from Solvency II for ultimate risk aggregation in IFRS 17, it tends to overestimate the diversified risk capital in the considered model (the risk adjustment at the level of a company). At the same time, our numerical study with real data shows that the ultimate correlation can be larger than the one-year correlation in the next calendar year. Our third conclusion is that even though we observe (in some cases substantial) differences in the ultimate and the one-year correlations, the impact of a misused correlation on capital is rather small and reaches 4% in our numerical study. However, we point out that we can easily construct a synthetic example where the use of an improper correlation leads to the misestimation of capital by 7.5%. The key message from this paper to actuaries is that the ultimate and the one-year correlations are different and differences in their values should be investigated as they may have an impact on calculations performed in Solvency II and IFRS 17.

The topic of correlations in claims reserving has already gained attention in the actuarial literature. We would like to refer to Avanzi et al. (2016b) and Taylor (2018) where practical aspects of

https://doi.org/10.1017/S1748499524000204 Published online by Cambridge University Press

defining correlation matrices and possible values of ultimate correlations observed in practice are discussed. However, the topic concerning the relation between the ultimate and the one-year correlations is new. The notions of the ultimate correlation and the one-year correlation have been also recently introduced by El Alami et al. (2022). El Alami et al. (2022) consider a different actuarial model with additive cash flows from elliptical distribution with special dependence structures. The authors only investigate the one-year correlation in the next calendar year, and they do not discuss the one-year correlations in future calendar years. Interestingly, El Alami et al. (2022) also show that, in their model, the one-year correlation in the next calendar year is higher than the ultimate correlation, and the IFRS 17 capital can be misestimated by 8% in their semi-synthetic example.

This paper is structured as follows. In Section 2, we introduce a multivariate Hertig's lognormal model. The key ideas and first conclusions are presented in Section 3. In Section 4, we define the ultimate correlation and the one-year correlations in future calendar years. The two key relations between the correlations are derived in Section 5. Numerical examples are presented in Section 6. All proofs can be found in Appendix.

2. The multivariate model of claims development

We study a multivariate version of Hertig's lognormal model of claims development from Chapter 5 in Wüthrich (2015) and Merz et al. (2012). In the sections below, we present key results on the multivariate Hertig's lognormal model, which we need in this paper.

Let us consider N lines of business, which are labeled by $n = 1, \ldots, N$. We denote accident years by $i \in \{1, \ldots, I\}$ and development years by $j \in \{0, \ldots, J\}$. As always, we assume that $I \geq J + 1$ and all claims are settled within $J + 1$ development years. To define cell-wise correlations between claims development processes in the lines of business, we assume that all lines of business have the same number of historical accident years and development years, hence I and J do not depend on n. The cumulative payments from accident year i after development year j for line of business n are denoted by $C_{i,j,n}$.

In our multivariate Hertig's lognormal model, we assume that

- The development of claims follows the process:

$$C_{i,j,n} = C_{i,j-1,n} e^{\xi_{i,j,n}}, \quad (i, j, n) \in \big\{1, \ldots, I\big\} \times \big\{0, \ldots, J\big\} \times \big\{1, \ldots, N\big\}, \tag{2.1}$$

  and we set w.l.o.g. $C_{i,-1,n} = 1$.

We define the vectors:

$$\boldsymbol{\xi}_{i,j} = \big(\xi_{i,j,1}, \ldots, \xi_{i,j,N}\big)^T \in \mathbb{R}^N, \quad \boldsymbol{\xi}_i = \big(\boldsymbol{\xi}_{i,0}^T, \ldots, \boldsymbol{\xi}_{i,J}^T\big)^T \in \mathbb{R}^a, \quad \boldsymbol{\xi} = \big(\boldsymbol{\xi}_1^T, \ldots, \boldsymbol{\xi}_I^T\big)^T \in \mathbb{R}^d,$$

where $a = N(J + 1)$ and $d = aI$, and we assume that

- Conditionally, given $\boldsymbol{\Theta} \in \mathbb{R}^a$, the random vector $\boldsymbol{\xi}$ has a multivariate Gaussian distribution with fixed positive-definite covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{d\times d}$ and conditional expected values $\mathbb{E}\big[\boldsymbol{\xi}_i|\boldsymbol{\Theta}\big] = \boldsymbol{\Theta}$ for all $i \in \{1, \ldots, I\}$,
- The parameter $\boldsymbol{\Theta}$ has a multivariate Gaussian distribution with prior mean $\boldsymbol{\mu} \in \mathbb{R}^a$ and positive-definite prior covariance matrix $\mathbf{T} \in \mathbb{R}^{a\times a}$.

We are interested in the following dependence structures between the claims run-offs:

- Dependence A: Cell-wise correlation:

$$Cov\big[\xi_{i,j,n},\xi_{l,z,m}|\mathbf{\Theta}\big]=\begin{cases}\sigma_{j,n}\sigma_{z,m}\rho, & (i,j)=(l,z),\ (i,j,n)\neq(l,z,m),\\ \sigma_{i,n}^2, & (i,j,n)=(l,z,m),\\ 0, & otherwise.\end{cases}$$

- Dependence B: Cell-wise and calendar year correlation:

$$Cov\big[\xi_{i,j,n},\xi_{l,z,m}|\mathbf{\Theta}\big]=\begin{cases}\sigma_{j,n}\sigma_{z,m}\rho, & i+j=l+z, (i,j,n)\neq(l,z,m),\\ \sigma_{i,n}^2, & (i,j,n)=(l,z,m),\\ 0, & otherwise\end{cases}$$

- Dependence C: Cell-wise, calendar year, and trend AR(1) correlation:

$$Cov\big[\xi_{i,j,n},\xi_{l,z,m}|\mathbf{\Theta}\big]=\begin{cases}\sigma_{j,n}\sigma_{z,m}\rho^h, & i+j-l-z=h-1,\ (i,j,n)\neq(l,z,m),\ h=1,2,\ldots,\\ \sigma_{i,n}^2, & (i,j,n)=(l,z,m),\\ 0, & otherwise\end{cases}$$

In dependence structure A, we assume that there is only a cell-wise correlation between the claims development noises $\boldsymbol{\xi}$ in the loss triangles of lines of business. In dependence structure B, we additionally assume that there is a correlation between the claims development noises $\boldsymbol{\xi}$ arising in the same calendar year within and between the loss triangles. Finally, in dependence structure C, we assume, in addition to A and B, that there is a correlation between the claims development noises $\boldsymbol{\xi}$ in all calendar years, within and between the loss triangles, yet the correlation decreases for more distant calendar years. For details on the three dependence structures, we refer to Chapter 5.2.5 in Wüthrich (2015). The dependence structures A, B, and C are controlled with one correlation parameter ρ. We can also allow for different correlation coefficients between and within loss triangles, as well as for different cell-wise and calendar-year correlation coefficients. We do not need these extensions in the paper.

As far as the parameters' uncertainty is concerned, we focus on the following cases:

- Parameters' uncertainty: The uncertainties of the a priori parameters' estimates for the lines of business and the development years are independent of each other, and we set:

$$\mathbf{T}=\tau^2(\boldsymbol{\mu}^2)^T\mathbf{1}_{a\times a}, \tag{2.2}$$

  where $\tau>0$, $\boldsymbol{\mu}^2$ denotes a vector with squared elements of $\boldsymbol{\mu}$, and $\mathbf{1}_{a\times a}$ denotes an identity matrix of dimension $a\times a$. The parameter τ plays the role of a coefficient of variation,
- No parameters' uncertainty: we set $\tau=0$ in Eq. (2.2).

Again, the choice of $\mathbf{T}$ is motivated in Chapter 5.2.5 in Wüthrich (2015). In Bayesian setting, the diagonal structure of $\mathbf{T}$ implies that we make independent decisions about the parameters' estimates based on their a priori knowledge. We remark that we only consider the parameters' uncertainty related to the expected value of $\boldsymbol{\xi}$. If we would like to measure uncertainty related to the specification of the covariance matrix of $\boldsymbol{\xi}$, then a full simulation model has to be run in the spirit of Shi et al. (2012).

Let $t=1,2,\ldots$ denote a calendar year. We introduce $\mathcal{D}_t=\big\{(i,j,n)\in\{1,\ldots,I\}\times\{0,\ldots,J\}\times\{1,\ldots,N\},\ i+j\leq t\big\}$. The set $\mathcal{D}_t$ contains indices of the cumulative payments $C_{i,j,n}$, equivalently, the indices of the claims development noises $\xi_{i,j,n}$, which have been observed at the end of calendar year t for all lines of business $n=1,\ldots,N$. We also introduce the filtration:

$$\mathcal{F}_t=\sigma\big\{C_{i,j,n}:(i,j,n)\in\mathcal{D}_t\big\},$$

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which describes the information available after t calendar years from all lines of business $n = 1, \ldots, N$.

Next, we define matrices $\mathcal{P}_{\mathcal{D}_t} : \mathbb{R}^d \mapsto \mathbb{R}^{|\mathcal{D}_t|}$ and $\mathcal{P}_{\mathcal{D}_t^c} : \mathbb{R}^d \mapsto \mathbb{R}^{|\mathcal{D}_t^c|}$ such that

$$\boldsymbol{\xi} \mapsto \boldsymbol{\xi}^{\mathcal{D}_t} = \mathcal{P}_{\mathcal{D}_t}\boldsymbol{\xi}, \quad \boldsymbol{\xi} \mapsto \boldsymbol{\xi}^{\mathcal{D}_t^c} = \mathcal{P}_{\mathcal{D}_t^c}\boldsymbol{\xi}. \tag{2.3}$$

The vector $\boldsymbol{\xi}$ contains the Gaussian noises describing the whole claims development process for all lines of business. The vector $\boldsymbol{\xi}^{\mathcal{D}_t}$ contains the Gaussian noises from the claims development process, which have been observed at the end of the calendar year t, and the vector $\boldsymbol{\xi}^{\mathcal{D}_t^c}$ contains the Gaussian noises from the claims development process, which will be observed after the calendar year t.

The goal in claims reserving is to derive the conditional distribution of $\boldsymbol{\xi}^{\mathcal{D}_t^c}$ given $\boldsymbol{\xi}^{\mathcal{D}_t}$. This distribution can be derived from Theorems A.1–A.2, see Corollary 5.3 in Wüthrich (2015) and Theorem 3.4 in Merz et al. (2012).

Theorem 2.1. *The conditional distribution of* $\boldsymbol{\xi}^{\mathcal{D}_t^c}$ *given* $\boldsymbol{\xi}^{\mathcal{D}_t}$ *is multivariate Gaussian with the conditional mean*

$$\boldsymbol{\mu}^{post}_{\mathcal{D}_t^c} = \mathbb{E}\big[\boldsymbol{\xi}^{\mathcal{D}_t^c}\big|\boldsymbol{\xi}^{\mathcal{D}_t}\big] = \mathcal{P}_{\mathcal{D}_t^c}\mathcal{A}\boldsymbol{\mu} + \mathcal{Q}_{\mathcal{D}_t,\mathcal{D}_t^c}\Big(\boldsymbol{\xi}^{\mathcal{D}_t} - \mathcal{P}_{\mathcal{D}_t}\mathcal{A}\boldsymbol{\mu}\Big),$$

and conditional covariance matrix

$$\mathcal{S}^{post}_{\mathcal{D}_t^c} = cov\big[\boldsymbol{\xi}^{\mathcal{D}_t^c}\big|\boldsymbol{\xi}^{\mathcal{D}_t}\big] = \mathcal{P}_{\mathcal{D}_t^c}\mathcal{S}\mathcal{P}^T_{\mathcal{D}_t^c} - \mathcal{Q}_{\mathcal{D}_t,\mathcal{D}_t^c}\mathcal{P}_{\mathcal{D}_t}\mathcal{S}\mathcal{P}^T_{\mathcal{D}_t^c}$$

where

$$\mathcal{A} = (\mathbf{1}_{a\times a}, \ldots, \mathbf{1}_{a\times a})^T \in \mathbb{R}^{d\times a},$$

$$\mathcal{Q}_{\mathcal{D}_t,\mathcal{D}_t^c} = \mathcal{P}_{\mathcal{D}_t^c}\mathcal{S}\mathcal{P}^T_{\mathcal{D}_t}\Big(\mathcal{P}_{\mathcal{D}_t}\mathcal{S}\mathcal{P}^T_{\mathcal{D}_t}\Big)^{-1},$$

$$\mathcal{S} = cov\big[\boldsymbol{\xi}\big] = \boldsymbol{\Sigma} + \mathcal{A}\mathbf{T}\mathcal{A}^T.$$

Let us recall that $\mathbf{1}_{a\times a} \in \mathbb{R}^{a\times a}$ denotes an identity matrix of dimension $a \times a$.

Corollary 2.1. *Under dependence A and B and without parameters' uncertainty* ($\tau = 0$), $\boldsymbol{\xi}^{\mathcal{D}_t^c}$ *is independent of* $\boldsymbol{\xi}^{\mathcal{D}_t}$.

Under the assumptions of Corollary 2.1, we will derive explicit results for ultimate and one-year correlations. As we will illustrate in Section 3.3, the two cases highlighted in Corollary 2.1 are very special cases of the general claims reserving problem we investigate.

In the sequel, we will select elements of the vector $\boldsymbol{\xi}^{\mathcal{D}_t^c}$. Let us define a vector $\mathbf{e}_{t|i,j\in\mathcal{J},n}$ of dimension $|\mathcal{D}_t^c| \times 1$, which contains zeros and ones, such that

$$\mathbf{e}^T_{t|i,j\in\mathcal{J},n}\boldsymbol{\xi}^{\mathcal{D}_t^c} = \sum_{j=t-i+1}^{J} \xi_{i,j,n}\mathbf{1}\{j \in \mathcal{J}\}.$$

We will use the notation:

$$\mathbf{e}^T_{t|i,j,n}\boldsymbol{\xi}^{\mathcal{D}_t^c} = \xi_{i,j,n}\mathbf{1}\{t-i+1 \le j \le J\}, \quad \mathbf{e}^T_{t|i,j\le M,n}\boldsymbol{\xi}^{\mathcal{D}_t^c} = \sum_{j=t-i+1}^{M} \xi_{i,j,n}\mathbf{1}\{t-i+1 \le M\},$$

where the indicators guarantee that we choose indices $j \in \mathcal{J}$ such that $j \in [t-i+1, M]$, otherwise $\mathbf{e}_{t|i,j\in\mathcal{J},n}$ only contains zeros.

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3. The first outlook

Before we present the general results, we focus on special (simple) cases in order to better understand one-year and ultimate correlations, misestimation of capital resulting from misused (misestimated) correlation, and the technique we use to derive the results.

3.1 Ultimate and one-year correlations

We consider two lines of business ($n = 1, 2$) with one accident year ($I = 1$). We study dependence A without parameters' uncertainty. We investigate the claims development processes after the first calendar year (we set $t = 1$ and the filtration $\mathcal{F}_1$ is known to the actuary). From (2.1), we define the ultimate payment

$$C_{J,n} = C_0 e^{\sum_{j=1}^{J} \xi_{j,n}}.$$

We define the valuation of the ultimate payment after the first calendar year

$$\hat{C}^1_{J,n} = \mathbb{E}[C_{J,n}|\mathcal{F}_1] = C_{0,n} e^{\sum_{j=1}^{J} \left(\mu_{j,n} + \frac{1}{2}\sigma^2_{j,n}\right)}, \tag{3.1}$$

as well as the sequence of valuations of the ultimate payment in the future calendar years, after $1 + k$ calendar years,

$$\hat{C}^{1+k}_{J,n} = \mathbb{E}[C_{J,n}|\mathcal{F}_{1+k}] = C_{0,n} e^{\sum_{j=1}^{k} \xi_{j,n}} e^{\sum_{j=k+1}^{J} \left(\mu_{j,n} + \frac{1}{2}\sigma^2_{j,n}\right)}, \quad k = 0, \ldots, J. \tag{3.2}$$

Clearly, $\hat{C}^{1+J}_{J,n} = C_{J,n}$. The valuations (3.1)–(3.2) are used to define the ultimate loss and the one-year loss in calendar year $2 + k$

$$L^{Ult}_n = C_{J,n} - \hat{C}^1_{J,n}, \quad L^{1YR,2+k}_n = \hat{C}^{2+k}_{J,n} - \hat{C}^{1+k}_{J,n}, \quad k = 0, \ldots, J - 1.$$

For the definition of the ultimate loss and the one-year loss in actuarial claims reserving, we refer, for example, to Chapter 1 in Wüthrich (2015) and Wüthrich and Merz (2015).

By direct calculations for lognormal distributions, we can derive the Pearson correlation coefficient between the ultimate losses in the two lines of business (ultimate correlation)

$$\begin{aligned}
\rho^{Ult} &:= corr\left[L^{Ult}_1, L^{Ult}_2|\mathcal{F}_1\right] = \frac{cov\left[\hat{C}^{1+J}_{J,1}, \hat{C}^{1+J}_{J,2}|\mathcal{F}_1\right]}{\sqrt{Var\left[\hat{C}^{1+J}_{J,1}|\mathcal{F}_1\right]}\sqrt{Var\left[\hat{C}^{1+J}_{J,2}|\mathcal{F}_1\right]}} \\
&= \frac{\hat{C}^1_{J,1}\hat{C}^1_{J,2} e^{\frac{1}{2}\sum_{j=1}^{J} 2\sigma_{j,1}\sigma_{j,2}\rho} - \hat{C}^1_{J,1}\hat{C}^1_{J,2}}{\hat{C}^1_{J,1}\hat{C}^1_{J,2}\sqrt{e^{\sum_{j=1}^{J}\sigma^2_{j,1}} - 1}\sqrt{e^{\sum_{j=1}^{J}\sigma^2_{j,2}} - 1}} \\
&= \frac{e^{\sum_{j=1}^{J}\sigma_{j,1}\sigma_{j,2}\rho} - 1}{\sqrt{e^{\sum_{j=1}^{J}\sigma^2_{j,1}} - 1}\sqrt{e^{\sum_{j=1}^{J}\sigma^2_{j,2}} - 1}} \approx \frac{\sum_{j=1}^{J}\sigma_{j,1}\sigma_{j,2}}{\sqrt{\sum_{j=1}^{J}\sigma^2_{j,1}}\sqrt{\sum_{j=1}^{J}\sigma^2_{j,2}}}\rho,
\end{aligned} \tag{3.3}$$

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as well as, the Pearson correlation coefficients between the one-year losses in the two lines of business in calendar year $2+k$ (one-year correlations)

$$\begin{aligned}\rho_{2+k}^{1YR} &:= corr\left[L_1^{1YR,2+k}, L_2^{1YR,2+k}|\mathcal{F}_1\right]\\ &= \frac{cov\left[\hat{C}_{J,1}^{2+k}, \hat{C}_{J,2}^{2+k}|\mathcal{F}_1\right] - cov\left[\hat{C}_{J,1}^{1+k}, \hat{C}_{J,2}^{1+k}|\mathcal{F}_1\right]}{\sqrt{Var\left[\hat{C}_{J,1}^{2+k}|\mathcal{F}_1\right] - Var\left[\hat{C}_{J,1}^{1+k}|\mathcal{F}_1\right]}\sqrt{Var\left[\hat{C}_{J,2}^{2+k}|\mathcal{F}_1\right] - Var\left[\hat{C}_{J,2}^{1+k}|\mathcal{F}_1\right]}}\\ &= \frac{e^{\sum_{j=1}^{k+1}\sigma_{j,1}\sigma_{j,2}\rho} - e^{\sum_{j=1}^{k}\sigma_{j,1}\sigma_{j,2}\rho}}{\sqrt{e^{\sum_{j=1}^{k+1}\sigma_{j,1}^2} - e^{\sum_{j=1}^{k}\sigma_{j,1}^2}}\cdot\sqrt{e^{\sum_{j=1}^{k+1}\sigma_{j,2}^2} - e^{\sum_{j=1}^{k}\sigma_{j,2}^2}}}\\ &\approx \frac{\sigma_{k+1,1}\sigma_{k+1,2}}{\sigma_{k+1,1}\sigma_{k+1,2}}\rho = \rho, \quad k = 0, \ldots, J-1. \end{aligned} \tag{3.4}$$

In the calculations above, we use the property that $cov\left[\hat{C}_{J,n}^{2+k}, \hat{C}_{J,m}^{1+k}|\mathcal{F}_1\right] = cov\left[\hat{C}_{J,n}^{1+k}, \hat{C}_{J,m}^{1+k}|\mathcal{F}_1\right]$ for $n, m = 1, 2$. We have two remarks concerning the correlations derived:

- The approximations in (3.3)–(3.4) hold for small $(\sigma_{j,n})_{j=1}^{J}$, which is very often the case in practice (note that $\sigma_{0,n}$ can still be large, which is often the case in practice);
- Even if $\rho = 1$, the correlations (3.3)–(3.4) are not necessarily equal to 1, which is a well-known fact for lognormal distributions.

The correlation coefficients (3.3)–(3.4) are the main object of this paper. Ultimate and one-year correlations are used in practice for different purposes and their role in actuarial capital modeling will be explained in the sequel. We start by presenting the conclusions from this example. The one-year correlations in the future calendar years are not equal, yet they are almost equal if the volatility parameters are small. The ultimate correlation is different from the one-year correlations. Let us assume that the volatility parameters are small so that the approximations hold. By the Cauchy–Schwarz inequality, we get the inequality

$$\rho^{Ult} \leq \rho^{1YR} = \rho, \tag{3.5}$$

where the equality is achieved only if $\sigma_{j,1} = \alpha\sigma_{j,2}$ for all $j = 1, \ldots, J$ and for some $\alpha > 0$, which is unlikely situation in practice. Hence, from Eq. (3.5), we can conclude that the ultimate correlation is lower than the one-year correlation in our example. Let us try to answer how small the ultimate correlation can be compared to the one-year correlation. By the Cassels's inequality, see Eq. (3.2) in Watson (1955), we get the inequality

$$\rho^{Ult} \geq C\rho^{1YR} = C\rho, \tag{3.6}$$

where $C = 2\sqrt{\frac{L}{(1+L)^2}} \leq 1$, and $L = \frac{\max_{j=1,\ldots J}\sigma_{j,2}/\sigma_{j,1}}{\min_{j=1,\ldots,J}\sigma_{j,2}/\sigma_{j,1}} \geq 1$. We observe that $C = 1$ if and only if $L = 1$, and $L = 1$ if and only if $\sigma_{j,1} = \alpha\sigma_{j,2}$ for all $j = 1, \ldots, J$ and for some $\alpha > 0$, which is the case discussed above. If $C < 1$, then the equality in Eq. (3.6) is achieved only if $J = 2$ and $\frac{\sigma_{2,1}}{\sigma_{1,1}} = \frac{\sigma_{1,2}}{\sigma_{2,2}}$ (see Eq. (3.2) in Watson 1955). The case with $J = 2$ is not interesting in practice. If we still consider $J = 2$ and in addition we assume that $\sigma_{1,n} \geq \sigma_{2,n}$, $n = 1, 2$, which is the typical case in practice as volatilities decrease in development periods, then again we have $L = C = 1$. Hence, the lower bound in Eq. (3.6) is not achieved, but we expect that the larger L, the smaller ρ^{Ult} compared to ρ^{1YR}.

Let us consider a special case of the volatility parameters. We assume that

$$\sigma_{j,n} = \sigma_{1,n}e^{-\alpha_n(j-1)}, \quad j = 1, \ldots, J,\ \alpha_1 > \alpha_2. \tag{3.7}$$

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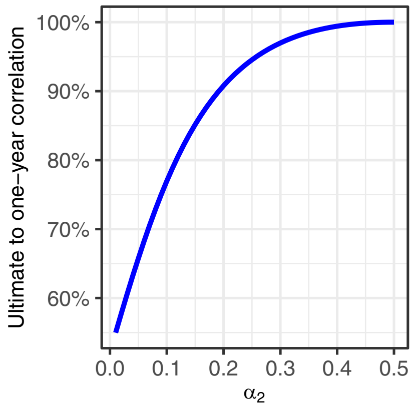

Figure 1. The ratio of the ultimate correlation (3.3) to the one-year correlation (3.4) under the volatility parameters specified with (3.7).

We set $J = 15$ and $\sigma_{1,n} = 0.1$ (the value of $\sigma_{1,n}$ is reasonable based on the estimations we perform in Section 5). The approximations (3.3)–(3.4) can be applied. We have $L = e^{15(\alpha_1 - \alpha_2)}$. We set $\alpha_1 = 0.5$ and we consider $\alpha_2 \in (0, 0.5)$. The ratio of the ultimate correlation (3.3) to the one-year correlation (3.4), calculated with the approximations, is presented in Fig. 1. As expected, the smaller α_2, the larger L, and the larger the difference between ρ^{Ult} and ρ^{1YR}. In our synthetic example, we observe that ρ^{Ult} can be 40% smaller than ρ^{1YR}.

3.2 The impact of correlations on diversified capital

From Fig. 1, we can observe that the difference between the ultimate and the one-year correlation can reach (in our synthetic example) 40%. From practical point of view, we are interested to what extent a misuse (misestimation) of correlation can impact capital measures. In this paper, we investigate the bottom-up (variance-covariance) risk aggregation formula.

Definition 3.1. *Let $x > 0$ and $y > 0$ denote stand-alone (marginal) risk capitals for losses for two lines of business and ρ denote the correlation coefficient between the losses in the lines of business. The bottom-up risk aggregation formula implies that the diversified risk capital is calculated with the rule*

$$\sqrt{x^2 + y^2 + 2xy\rho}. \tag{3.8}$$

The following result is known, both in theory and practice.

Theorem 3.1. *Let us consider the bottom-up aggregation of stand-alone risk capitals $x > 0$ and $y > 0$. If we increase the correlation coefficient between the losses from ρ to p, then the maximal relative increase of the diversified risk capital arises for $x = y$. For $x = y$, the relative increase of the diversified risk capital is equal to $\sqrt{\frac{1+p}{1+\rho}} - 1$.*

Theorem 3.1 shows that there are natural limits on the impact of a misuse of correlation on capital measures if we apply the bottom-up risk aggregation formula. In Fig. 2, we present the maximal changes in the diversified capital for two lines of business (i.e. for lines of business with equal stand-alone capitals) when we increase the correlation between the lines from ρ to p. The change in the diversified capital is not very large unless the change in the correlation is very large. For example, if we consider buckets of correlations $[0, 0.25]$, $[0.25, 0.5]$, $[0.5, 0.75]$, $[0.75, 1]$ and

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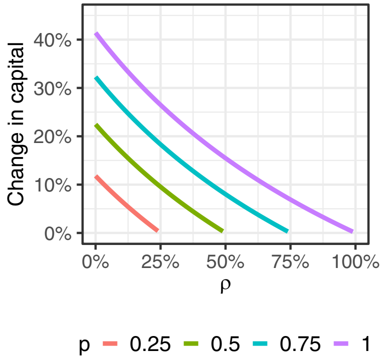

Figure 2. The maximal changes in the diversified capital for two lines of business if we increase the correlation from ρ to p, the changes are only investigated for $\rho \leq p$.

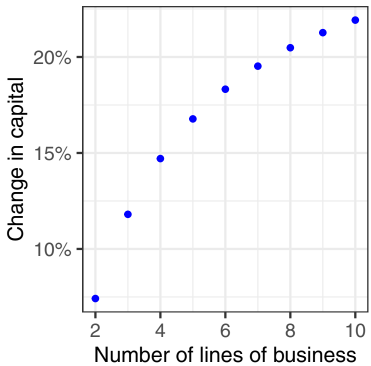

Figure 3. The maximal changes in the diversified capital for multiple lines of business if we increase the correlations from $\rho = 0.3$ to $p = 0.5$ for $n = 2, \ldots, 10$ lines of business.

we increase the correlation from the lower bound to the upper bound for each bucket, then the diversified capital increases by 7%–12%. In other words, if we consider lines of business with equal stand-alone capitals and the true correlation is within a bucket, the maximal overestimation of capital is 7%–12% (assuming that the true correlation is the lower bound and we choose the upper bound). In our synthetic example of Section 3.1, if we use $\rho^{1YR} = 0.5$ instead of the true correlation $\rho^{Ult} = 0.3$ (the ultimate correlation is 40% lower than the one-year correlation), we overestimate the capital at most by 7.5%, and this value can only be reached if the lines of business have equal stand-alone capitals. This misestimation of capital resulting from misused correlation is large, but not very large, especially compared to the difference in correlation.

We also investigate the multivariate extension of the risk aggregation formula (3.8) and Theorem 3.1. In Fig. 3 we observe that the impact of misused (misestimated) correlations on the diversified capital increases if the number of lines of business increases. If we use $\rho^{1YR} = 0.5$ instead of the true correlations $\rho^{Ult} = 0.3$ for all 10 lines of business, we overestimate the diversified capital by 22%, which is now a very large error.

Our synthetic examples should give a clear signal to actuaries that appropriate correlations should be used for capital measures. In Section 5, we investigate values of one-year and ultimate correlations, which can be observed in practice, as well as misestimations of capitals resulting from misused correlations.

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3.3 The role of information in prediction of claims

Let us start with a classical Bayesian estimation in a Gaussian model. We consider

$$(X_1, X_2)|\Theta_1 = \theta_1, \Theta_2 = \theta_2 \sim N\left(\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}, \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\right), \quad (\Theta_1, \Theta_2) \sim N\left(\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}, \begin{bmatrix}\tau^2 & 0\\ 0 & \tau^2\end{bmatrix}\right).$$

By Theorems A.1–A.2, we get the distribution

$$(X_1, X_2, \Theta_1) \sim N\left(\begin{bmatrix}\mu_1\\ \mu_2\\ \mu_3\end{bmatrix}, \begin{bmatrix}1+\tau^2 & \rho & \tau^2\\ \rho & 1+\tau^2 & 0\\ \tau^2 & 0 & \tau^2\end{bmatrix}\right),$$

from which we immediately get the conditional distributions of $\Theta_1|X_1 = x_1$ and $\Theta_1|X_1 = x_1, X_2 = x_2$. We calculate the moments

$$\begin{aligned}
\mathbb{E}[\Theta_1|X_1 = x_1] &= \mu_1 + \frac{\tau^2}{1+\tau^2}(x_1 - \mu_1),\\
\lim_{\tau\to\infty} \mathbb{E}[\Theta_1|X_1 = x_1] &= x_1,\\
\lim_{\tau\to 0} \mathbb{E}[\Theta_1|X_1 = x_1] &= \mu_1,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[\Theta_1|X_1 = x_1, X_2 = x_2] &= \mu_1 + \begin{bmatrix}\tau^2 & 0\end{bmatrix} \cdot \begin{bmatrix}1+\tau^2 & \rho\\ \rho & 1+\tau^2\end{bmatrix}^{-1} \cdot \begin{bmatrix}x_1 - \mu_1\\ x_2 - \mu_2\end{bmatrix}\\
&= \mu_1 + \frac{\tau^2(1+\tau^2)}{(1+\tau^2)^2 - \rho^2}(x_1 - \mu_1) - \frac{\tau^2\rho}{(1+\tau^2)^2 - \rho^2}(x_2 - \mu_2),\\
\lim_{\tau\to\infty} \mathbb{E}[\Theta_1|X_1 = x_1, X_2 = x_2] &= x_1,\\
\lim_{\tau\to 0} \mathbb{E}[\Theta_1|X_1 = x_1, X_2 = x_2] &= \mu_1.
\end{aligned}$$

Hence, if $\tau \in (0, \infty)$, then $\mathbb{E}[\Theta_1|X_1 = x_1] \neq \mathbb{E}[\Theta_1|X_1 = x_1, X_2 = x_2]$. If (X, Y), conditional on (Θ_1, Θ_2), are correlated, then the Bayesian estimate of θ_1 (the marginal expected value of X_1) is different depending on the information we use in the prediction of θ_1: X_1 or (X_1, X_2).

In our framework of dependence structures for claims' run-offs from Section 2, we consider two very simple dependent loss triangles with cumulative payments with the distributions

$$\begin{aligned}
&(X_{1,2}, X_{2,1}, Y_{1,2}, Y_{2,1}, X_{2,2}, Y_{2,2})|\Theta_1 = \theta_1, \Theta_2 = \theta_2, \Phi_1 = \phi_1, \Phi_2 = \phi_2\\
&\sim N\left(\begin{bmatrix}\theta_2 & \theta_1 & \phi_2 & \phi_1 & \theta_2 & \phi_2\end{bmatrix}^T, \begin{bmatrix}\mathbf{P} & \mathbf{Q} & \mathbf{c} & \dots\\ \mathbf{Q} & \mathbf{P} & \mathbf{c} & \dots\\ \mathbf{c} & \mathbf{c} & \mathbf{1}_{2\times 2} & \dots\\ \dots\dots & & \dots & \dots\end{bmatrix}\right),
\end{aligned}$$

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	$X_{1,2}$
$X_{2,1}$	$X_{2,2}$

	$Y_{1,2}$
$Y_{2,1}$	$Y_{2,2}$

Figure 4. The loss triangles in the synthetic example.

where

$$\mathbf{P}=\begin{bmatrix}1 & corr(X_{1,2},X_{2,1})\\ corr(X_{1,2},X_{2,1}) & 1\end{bmatrix}=\begin{bmatrix}1 & corr(Y_{1,2},Y_{2,1})\\ corr(Y_{1,2},Y_{2,1}) & 1\end{bmatrix},$$

$$\mathbf{Q}=\begin{bmatrix}corr(X_{1,2},Y_{1,2}) & corr(X_{1,2},Y_{2,1})\\ corr(X_{2,1},Y_{1,2}) & corr(X_{2,1},Y_{2,1})\end{bmatrix},$$

$$\mathbf{c}=\left[corr(X_{1,2},X_{2,2})\ corr(X_{2,1},X_{2,2})\right]^T=\left[corr(Y_{1,2},X_{2,2})\ corr(Y_{2,1},X_{2,2})\right]^T,$$

and

$$(\Theta_1,\Theta_2,\Phi_1,\Phi_2)\sim N\left(\left[\mu_1\ \mu_2\ \eta_1\ \eta_2\right]^T,\tau^2\cdot\mathbf{1}_{4\times4}\right).$$

The loss triangles are presented in Fig. 4. The matrix $\mathbf{P}$ describes calendar year correlations within the triangles, the matrix $\mathbf{Q}$ – cell-wise and calendar year correlations between the triangles, and the vector $\mathbf{c}$ – trend correlations between the triangles. The goal is to predict $X_{2,2}$ – the ultimate payment from the first lines of business.

We can predict $X_{2,2}$ depending on the information from one line of business or two lines of business. By Theorem A.1, we derive the two estimators

$$\begin{aligned}\mathbb{E}[X_{2,2}|X_{1,2}X_{2,1}]&=\mathbb{E}\Big[\mathbb{E}[X_{2,2}|X_{1,2}X_{2,1},\Theta_1,\Theta_2,\Phi_1,\Phi_2]\Big|X_{1,2},X_{2,1}\Big]\\&=\mathbb{E}[\Theta_2|X_{1,2},X_{2,1}]+\mathbf{c}\cdot\mathbf{P}^{-1}\cdot\begin{bmatrix}X_{1,2}-\mathbb{E}[\Theta_2|X_{1,2},X_{2,1}]\\ X_{2,1}-\mathbb{E}[\Theta_1|X_{1,2},X_{2,1}]\end{bmatrix},\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}[X_{2,2}|X_{1,2},X_{2,1},Y_{1,2},Y_{2,1}]\\&=\mathbb{E}\Big[\mathbb{E}[X_{2,2}|X_{1,2}X_{2,1},Y_{1,2},Y_{2,1},\Theta_1,\Theta_2,\Phi_1,\Phi_2]\Big|X_{1,2},X_{2,1},Y_{1,2},Y_{2,1}\Big]\\&=\mathbb{E}[\Theta_2|X_{1,2},X_{2,1},Y_{1,2},Y_{2,1}]\\&\quad+\left[\mathbf{c}\ \mathbf{c}\right]\cdot\begin{bmatrix}\mathbf{P}&\mathbf{Q}\\ \mathbf{Q}&\mathbf{P}\end{bmatrix}^{-1}\cdot\begin{bmatrix}X_{1,2}-\mathbb{E}[\Theta_2|X_{1,2},X_{2,1},Y_{1,2},Y_{2,1}]\\ X_{2,1}-\mathbb{E}[\Theta_1|X_{1,2},X_{2,1},Y_{1,2},Y_{2,1}]\\ Y_{1,2}-\mathbb{E}[\Phi_2|X_{1,2},X_{2,1},Y_{1,2},Y_{2,1}]\\ Y_{2,1}-\mathbb{E}[\Phi_1|X_{1,2},X_{2,1},Y_{1,2},Y_{2,1}]\end{bmatrix}.\end{aligned}$$

We assume that $\mathbf{Q}$ is a non-zero matrix in order to have a dependence between the loss triangles – the loss triangles are correlated cell-wise and (possibly) calendar year, i.e. $(X_{1,2},X_{2,1})$ and $(Y_{1,2},Y_{2,1})$ are correlated. We can deduce that if we allow for a trend correlation in the loss triangles (the vector $\mathbf{c}$ is non-zero), then $\mathbb{E}[X_{2,2}|X_{1,2},X_{2,1}]\neq\mathbb{E}[X_{2,2}|X_{1,2},X_{2,1},Y_{1,2},Y_{2,1}]$. If there is no trend correlation (the vector $\mathbf{c}$ is zero), then $\mathbb{E}[X_{2,2}|X_{1,2},X_{2,1}]\neq\mathbb{E}[X_{2,2}|X_{1,2},X_{2,1},Y_{1,2},Y_{2,1}]$ since $\mathbb{E}[\theta_2|X_{1,2},X_{2,1}]\neq\mathbb{E}[\theta_2|X_{1,2},X_{2,1},Y_{1,2},Y_{2,1}]$, unless we set $\tau=0$ (see the conclusion from the beginning of this section). The two important cases of dependence structures A and B with

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$\tau = 0$ are pointed out in Corollary 2.1 and will be presented separately in the sequel. Other cases of dependence structures and parameters' uncertainty are clearly more subtle and complicated to consider within the multivariate Hertig's lognormal model. We exclude the limit $\tau \to \infty$ from consideration, as in this case the variance of the Bayesian estimator diverges.

Our calculations above illustrate that the Bayesian prediction of the ultimate payment in a single line of business is in general different depending on the information we use in the prediction. We can expect that not only the prediction of the ultimate payment in a single line of business but also the mean square error of this prediction, hence the marginal evaluation of the reserve risk, are different if we use the information from the single line of business or from multiple lines of business in our portfolio. This property of the multivariate Hertig's lognormal model has been observed in the numerical examples in Chapter 5 in Wüthrich (2015), but we present above more mathematical insight into this property. For Dependencies A, B, and C and our form of parameter's uncertainty, we identify the only two cases for which the use of the information from a single line of business or from multiple lines of business leads to the same evaluation of the reserve risk. In the sequel, we introduce correlations implied by the mean square errors of predictions, and in order to have proper correlation coefficients, we should use the same information in the evaluation of the reserve risk for a single line of business and multiple lines of business.

4. Risk measures and correlations for bottom-up risk aggregation

We measure the reserve risk at the end of calendar year $t = I$. In the next sections, we derive risk measures in ultimate and one-year horizons and define ultimate and one-year correlations between lines of business.

4.1 Ultimate risk and ultimate correlation

For accident year i, such that $i + J > t$, the ultimate liability (the ultimate cumulative payments) for the accident year and line of business n is given by

$$C_{i,J,n} = C_{i,t-i,n} e^{\sum_{j=t-i+1}^{J} \xi_{i,j,n}} = C_{i,t-i,n} e^{\mathbf{e}^T_{t|i,j\leq J,n} \boldsymbol{\xi}^{\mathcal{D}^c_t}}. \tag{4.1}$$

For other accident years, the claims have been fully developed and these accident years are no longer investigated in the claims reserving problem. The total ultimate liability for all accident years and all lines of business is given by

$$C_J = \sum_{i=t-J+1}^{J} \sum_{n=1}^{N} C_{i,J,n}.$$

The best estimate of the ultimate liability at the end of calendar year t is defined with

$$\hat{C}^t_{i,J,n} = \mathbb{E}\big[C_{i,J,n} | \mathcal{F}_t\big], \quad i + J > t.$$

By Theorems 2.1 and A.3, see Chapter 5.2.3 in Wüthrich (2015) and Theorem 4.1 in Merz et al. (2012), we can get the formula

$$\begin{aligned}\hat{C}^t_{i,J,n} &= C_{i,t-i,n} e^{\sum_{j=t-i+1}^{J} \mathbb{E}\left[\xi_{i,j,n}|\mathcal{F}_t\right] + \frac{1}{2}\sum_{j,l=t-i+1}^{J} cov\left[\xi_{i,j,n}, \xi_{i,l,n}|\mathcal{F}_t\right]} \\ &= C_{i,t-i,n} e^{\mathbf{e}^T_{t|i,j\leq J,n} \boldsymbol{\mu}^{post}_{\mathcal{D}^c_t} + \frac{1}{2}\mathbf{e}^T_{t|i,j\leq J,n} \mathcal{S}^{post}_{\mathcal{D}^c_t} \mathbf{e}_{t|i,j\leq J,n}},\end{aligned} \tag{4.2}$$

for accident years such that $i + J > t$.

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We investigate the ultimate loss projected from the end of calendar year t. The ultimate loss for accident year i and line of business n is given by

$$L_{i,n}^{Ult,t} = C_{i,J,n} - \hat{C}_{i,J,n}^{t},$$

for accident years such that $i + J > t$. The total ultimate loss for all accident years and all lines of business is given by

$$L^{Ult,t} = \sum_{n=1}^{N} \sum_{i=t-J+1}^{I} L_{i,n}^{Ult,t}.$$

In claims reserving, the risk of future payments is usually measured with the mean square error of prediction, which coincides with the variance measure in our model. By Theorems 2.1 and A.3, see Chapter 5.2.3 in Wüthrich (2015) and Theorem 4.3 in Merz et al. (2012), we have the following result on the ultimate risk of the ultimate loss.

Theorem 4.1. *We have the formula for the ultimate risk measure*

$$\begin{aligned} Var\left[L^{Ult,t}\middle|\mathcal{F}_t\right] &= \sum_{n,m=1}^{N} \sum_{i,l=t-J+1}^{I} cov\left[L_{i,n}^{Ult,t}, L_{l,m}^{Ult,t}\middle|\mathcal{F}_t\right] = \sum_{n,m=1}^{N} \sum_{i,l=t-J+1}^{I} cov\left[C_{i,J,n}, C_{l,J,m}\middle|\mathcal{F}_t\right] \\ &= \sum_{n,m=1}^{N} \sum_{i,l=t-J+1}^{I} \hat{C}_{i,J,n}^{t}\hat{C}_{l,J,m}^{t}\left(e^{\sum_{j=t-i+1}^{J} \sum_{z=t-l+1}^{J} cov\left[\xi_{i,j,n},\xi_{l,z,m}|\mathcal{F}_t\right]} - 1\right) \\ &= \sum_{n,m=1}^{N} \sum_{i,l=t-J+1}^{I} \hat{C}_{i,J,n}^{t}\hat{C}_{l,J,m}^{t}\left(e^{\mathbf{e}_{t|i,j\le J,n}^{T} \mathcal{S}_{\mathcal{D}_t^c}^{post} \mathbf{e}_{t|l,j\le J,m}} - 1\right). \end{aligned} \tag{4.3}$$

If we measure the ultimate risk for single line of business n, we calculate the above sum for $n = m$. As discussed in Section 3.3, we do not modify the matrix $\mathcal{S}_{\mathcal{D}_t^c}^{post}$, since for single line of business n we still calculate $Var\left[L_n^{Ult,t}\middle|\mathcal{F}_t\right]$, i.e. we calculate the mean square error of prediction given the information from multiple lines of business.

We now introduce the notion of the ultimate correlation. The ultimate correlation represents the correlation that should be used for a bottom-up aggregation of risk capitals in ultimate time horizon. The implied ultimate correlation is implied by the variance risk measures from Theorem 4.1.

Definition 4.1. *For two lines of business, denoted by $n = 1, 2$, the implied ultimate correlation is derived from the relation*

$$\begin{aligned} Var\left[L^{Ult,t}\middle|\mathcal{F}_t\right] &= Var\left[L_1^{Ult,t}\middle|\mathcal{F}_t\right] + Var\left[L_2^{Ult,t}\middle|\mathcal{F}_t\right] \\ &\quad + 2\sqrt{Var\left[L_1^{Ult,t}\middle|\mathcal{F}_t\right]Var\left[L_2^{Ult,t}\middle|\mathcal{F}_t\right]}\,corr\left[L_1^{Ult,t}, L_2^{Ult,t}\middle|\mathcal{F}_t\right], \end{aligned} \tag{4.4}$$

where

$$L_n^{Ult,t} = \sum_{i=t-J+1}^{I} L_{i,n}^{Ult,t}, \quad n = 1, 2.$$

The implied ultimate correlation is just the Pearson correlation between the ultimate losses $L_1^{Ult,t}$ and $L_2^{Ult,t}$ conditional on $\mathcal{F}_t$. We denote $corr\left[L_1^{Ult,t}, L_2^{Ult,t}\middle|\mathcal{F}_t\right]$ by ρ_t^{Ult}.

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Remark 4.1. We could define the implied ultimate correlation as a coefficient that satisfies the relation

$$Var\Big[L^{Ult,t}\big|\mathcal{F}_t\Big] = Var\Big[L_1^{Ult,t}\big|\mathcal{F}_t^1\Big] + Var\Big[L_2^{Ult,t}\big|\mathcal{F}_t^2\Big] \\ +2\sqrt{Var\Big[L_1^{Ult,t}\big|\mathcal{F}_t^1\Big]Var\Big[L_2^{Ult,t}\big|\mathcal{F}_t^2\Big]}corr\Big[L_1^{Ult,t}, L_2^{Ult,t}\big|\mathcal{F}_t\Big], \tag{4.5}$$

where $\mathcal{F}_t^1$ and $\mathcal{F}_t^2$ denote the information available from single lines of business. Such a coefficient would not be a correlation coefficient and its interpretation could be difficult. In our numerical experiments from Section 5, we end up with negative values and values above 100% if we use Eq. (4.5). We believe that the definition (4.4) is much better for a mathematical investigation of ultimate and one-year correlations. The role of the information in prediction of claims in the multivariate Hertig's lognormal model is discussed in Section 3.3. In particular, among Dependencies A, B, and C and our form of parameter's uncertainty, the implied correlations defined with (4.4) and (4.5) coincide only for A and B with $\tau = 0$. For implied ultimate correlation and the role of information in its definition see also a discussion in Chapter 5.2.6 in Wüthrich (2015).

Formula (4.4) is valid for any dependence structure between lines of business and any parameters' uncertainty. We can derive explicit results for the two special cases, which we specify in Corollary 2.1.

Theorem 4.2. *Let the ultimate correlation be given with*

$$\rho_t^{Ult} = \frac{P}{\sqrt{Q_1}\sqrt{Q_2}}.$$

- *For dependence structure A and no parameters' uncertainty, we have*

$$P = \sum_{i=t-J+1}^{I} \hat{C}_{i,J,n}^t \hat{C}_{i,J,m}^t \left(e^{\sum_{j=t-i+1}^{J} \sigma_{j,n}\sigma_{j,m}\rho} - 1\right)$$

$$\approx \sum_{i=t-J+1}^{I} \hat{C}_{i,J,n}^t \hat{C}_{i,J,m}^t \sum_{j=t-i+1}^{J} \sigma_{j,n}\sigma_{j,m}\rho,$$

$$Q_n = \sum_{i=t-J+1}^{I} \left(\hat{C}_{i,J,n}^t\right)^2 \left(e^{\sum_{j=t-i+1}^{J} \sigma_{j,n}^2} - 1\right)$$

$$\approx \sum_{i=t-J+1}^{I} \left(\hat{C}_{i,J,n}^t\right)^2 \sum_{j=t-i+1}^{J} \sigma_{j,n}^2, \quad n = 1, 2,$$

where the approximations hold for small $(\sigma_{j,n})_{j=1}^J$ for $n = 1, 2$. Moreover, we have the upper bound on the ultimate correlation $\rho_t^{Ult} \le \rho$ for small $(\sigma_{j,n})_{j=1}^J$ for $n = 1, 2$.

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- *For dependence structure B and no parameters' uncertainty, we have*

$$P = \sum_{i,l=t-J+1}^{I} \hat{C}^t_{i,J,n}\hat{C}^t_{l,J,m}\left(e^{\sum_{j=t-i+1}^{J\wedge(J+l-i)}\sigma_{j,n}\sigma_{i+j-l,m}\rho} - 1\right)$$

$$\approx \sum_{i,l=t-J+1}^{I} \hat{C}^t_{i,J,n}\hat{C}^t_{l,J,m}\sum_{j=t-i+1}^{J\wedge(J+l-i)}\sigma_{j,n}\sigma_{i+j-l,m}\rho$$

$$Q_n = \sum_{i=t-J+1}^{I}\left(\hat{C}^t_{i,J,n}\right)^2\left(e^{\sum_{j=t-i+1}^{J\wedge(J+l-i)}\sigma^2_{j,n}} - 1\right)$$

$$+ \sum_{i,l=t-J+1,\ i\neq l}^{I} \hat{C}^t_{i,J,n}\hat{C}^t_{l,J,n}\left(e^{\sum_{j=t-i+1}^{J\wedge(J+l-i)}\sigma_{j,n}\sigma_{i+j-l,n}\rho} - 1\right)$$

$$\approx \sum_{i=t-J+1}^{I}\left(\hat{C}^t_{i,J,n}\right)^2\sum_{j=t-i+1}^{J}\sigma^2_{j,n}$$

$$+ \sum_{i,l=t-J+1,\ i\neq l}^{I} \hat{C}^t_{i,J,n}\hat{C}^t_{l,J,n}\sum_{j=t-i+1}^{J\wedge(J+l-i)}\sigma_{j,n}\sigma_{i+j-l,n}\rho, \quad n = 1, 2,$$

where the approximations hold for small $(\sigma_{j,n})_{j=1}^{J}$ for $n = 1, 2$. The ultimate correlation ρ_t^{Ult} can be larger or smaller than ρ.

The above results on ultimate correlations are new. The relations between the ultimate correlations and the correlation ρ have been observed, without any proof, in the numerical examples in Chapter 5.2.6 in Wüthrich (2015).

The main purpose of the ultimate correlation is to use it in a bottom-up aggregation of stand-alone risk capitals in ultimate time horizon. In this approach, an insurance company specifies risk capitals in ultimate time horizon for (two) lines of business $RC_{t,1}^{Ult}$ and $RC_{t,2}^{Ult}$ and applies the variance-covariance aggregation formula to derive the diversified risk capital

$$\sqrt{\left(RC_{t,1}^{Ult}\right)^2 + \left(RC_{t,2}^{Ult}\right)^2 + 2\cdot RC_{t,1}^{Ult}\cdot RC_{t,2}^{Ult}\cdot\rho_t^{Ult}}, \tag{4.6}$$

where ρ_t^{Ult} denotes the calibrated ultimate correlation. In many practical applications, the risk capitals in Eq. (4.6) are related to variance risk measures of ultimate losses. Traditionally, actuaries have been measuring risk in ultimate time horizon. Recently, the new IFRS 17 accounting standard also forces insurance companies to calculate risk adjustments, which should be related to a risk measure in ultimate time horizon.

4.2 One-year risks and one-year correlations

We now study a sequence of the best estimates of the ultimate liability at the end of future calendar years $t, t+1, \ldots$, which are needed to define one-year risks in future calendar years.

Let $k = 0, 1, \ldots, J-1$. We define $\hat{C}^{t+k}_{i,J,n} = \mathbb{E}\left[C_{i,J,n}|\mathcal{F}_{t+k}\right]$ for accident years such that $i + J \geq t + k$. We can derive the formula for the best estimate of the ultimate liability at the end of any

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calendar year $t+k$ viewed from the end of calendar year t, see the proof in Appendix and Chapter 5.2.4 in Wüthrich (2015). We point out that we use a different notation than Wüthrich (2015).

Proposition 4.1. *For $k = 0, 1, \ldots, J-1$, we have the formula for the best estimate of the ultimate liability*

$$\hat{C}^{t+k}_{i,J,n} = C_{i,t-i,n} e^{\sum_{j=t-i+1}^{J} \mathbb{E}\left[\xi_{i,j,n} | \mathcal{F}_{t+k}\right]} \cdot e^{+\frac{1}{2}\sum_{j,l=t-i+1}^{J} cov\left[\xi_{i,j,n}, \xi_{i,l,n} | \mathcal{F}_t\right] - \frac{1}{2}\sum_{j,l=t-i+1}^{J} cov\left[\mathbb{E}[\xi_{i,j,n}|\mathcal{F}_{t+k}], \mathbb{E}[\xi_{i,l,n}|\mathcal{F}_{t+k}] | \mathcal{F}_t\right]}$$

$$= C_{i,t-i,n} e^{\mathbf{p}^T_{t|i,k,n} \boldsymbol{\xi}^{\mathcal{D}^c_t} + r_{t|i,k,n}}, \quad i+J \geq t+k, \tag{4.7}$$

where

$$\mathbf{p}^T_{t|i,k,n} = \mathbf{e}^T_{t|i,j\leq t-i+k,n} + \mathbf{e}^T_{t+k|i,j\leq J,n} \mathcal{Q}_{\mathcal{D}_{t+k}, \mathcal{D}^c_{t+k}} \mathcal{P}_{\mathcal{D}_{t+k}} \mathcal{P}^T_{\mathcal{D}^c_t} \mathbf{1}\{i+J > t+k\},$$

$$r_{t|i,k,n} = \left(\mathbf{e}^T_{t|i,j\leq J,n} - \mathbf{p}^T_{t|i,k,n}\right) \boldsymbol{\mu}^{post}_{\mathcal{D}^c_t} + \frac{1}{2}\mathbf{e}^T_{t|i,j\leq J,n} \mathcal{S}^{post}_{\mathcal{D}^c_t} \mathbf{e}_{t|i,j\leq J,n} - \frac{1}{2}\mathbf{p}^T_{t|i,k,n} \mathcal{S}^{post}_{\mathcal{D}^c_t} \mathbf{p}_{t|i,k,n}.$$

Let us investigate the one-year loss in calendar year $t+k+1$ projected from the end of calendar year $t+k$. The one-year loss in calendar year $t+k+1$ for accident year i and line of business n is given by

$$L^{1YR,t+k+1}_{i,n} = \hat{C}^{t+k+1}_{i,J,n} - \hat{C}^{t+k}_{i,J,n},$$

for accident years such that $i+J > t+k$. The total one-year loss in calendar year $t+k+1$ for all accident years and lines of business is given by

$$L^{1YR,t+k+1} = \sum_{n=1}^{N} \sum_{i=t+k-J+1}^{I} L^{1YR,t+k+1}_{i,n}.$$

The risk of the one-year loss is usually measured, as the risk of the ultimate loss, with the mean squared error of prediction, which again agrees with the variance measure in our model. In order to quantify the one-year risk for the next calendar year, we should calculate the conditional variance

$$Var\left[L^{1YR,t+k+1} \middle| \mathcal{F}_{t+k}\right], \tag{4.8}$$

since the one-year risk in calendar $t+k+1$ is projected from the end of calendar year $t+k$. Since we measure the risk at the end of calendar year t and we can only project the risk from the end of calendar year t, we calculate the conditional expected value of Eq. (4.8) given $\mathcal{F}_t$:

$$\mathbb{E}\left[Var\left[L^{1YR,t+k+1} \middle| \mathcal{F}_{t+k}\right] \middle| \mathcal{F}_t\right],$$

as a projection of the one-year risk in the future calendar year.

There is an obvious relation between the ultimate loss and the one-year losses in future calendar years

$$L^{Ult,t} = \sum_{k=0}^{J-1} L^{1YR,t+k+1}.$$

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From Wüthrich and Merz (2015), we also have the following crucial property for the ultimate risk and the one-year risks

$$Var\big[L^{Ult,t}\big|\mathcal{F}_t\big] = \sum_{k=0}^{J-1} Var\big[L^{1YR,t+k+1}\big|\mathcal{F}_t\big] = \sum_{k=0}^{J-1} \mathbb{E}\Big[Var\big[L^{1YR,t+k+1}\big|\mathcal{F}_{t+k}\big]\mathcal{F}_t\Big], \tag{4.9}$$

which shows how the ultimate risk can be split into the one-year risks in future calendar years under variance as the risk measure. In particular, the one-year losses in future calendar years are not correlated. The decomposition (4.9) does not hold in general, but it holds in Bayesian claims reserving models, and in our claims reserving model.

We can now derive the one-year risk of the one-year loss in all future calendar years. The result can be found in Chapter 5.2.4 in Wüthrich (2015) and is also proved in Appendix.

Theorem 4.3. *For $k = 0, 1, \ldots, J-1$, we have the formula for the one-year risk measure in calendar year $t+k+1$*

$$\begin{aligned}
&\mathbb{E}\Big[Var\big[L^{1YR,t+k+1}\big|\mathcal{F}_{t+k}\big]\mathcal{F}_t\Big] = Var\Big[L^{1YR,t+k+1}\big|\mathcal{F}_t\Big]\\
&= \sum_{n,m=1}^{N} \sum_{i,l=t+k-J+1}^{I} cov\Big[L^{1YR,t+k+1}_{i,n}, L^{1YR,t+k+1}_{l,m}\big|\mathcal{F}_t\Big]\\
&= \sum_{n,m=1}^{N} \sum_{i,l=t+k-J+1}^{I} cov\Big[\hat{C}^{t+k+1}_{i,J,n} - \hat{C}^{t+k}_{i,J,n}, \hat{C}^{t+k+1}_{l,J,m} - \hat{C}^{t+k}_{l,J,m}\big|\mathcal{F}_t\Big]\\
&= \sum_{n,m=1}^{N} \sum_{i,l=t+k-J+1}^{I} \Big(cov\Big[\hat{C}^{t+k+1}_{i,J,n}\hat{C}^{t+k+1}_{l,J,m}\big|\mathcal{F}_t\Big] - cov\Big[\hat{C}^{t+k}_{i,J,n}\hat{C}^{t+k}_{l,J,m}\big|\mathcal{F}_t\Big]\Big)\\
&= \sum_{n,m=1}^{N} \sum_{i,l=t+k-J+1}^{I} \hat{C}^{t}_{i,J,n}\hat{C}^{t}_{l,J,m}\Big(e^{\sum_{j=t-i+1}^{J}\sum_{z=t-l+1}^{J} cov\left[\mathbb{E}[\xi_{i,j,n}|\mathcal{F}_{t+k+1}],\mathbb{E}[\xi_{l,z,n}|\mathcal{F}_{t+k+1}]|\mathcal{F}_t\right]}\\
&\qquad - e^{\sum_{j=t-i+1}^{J}\sum_{z=t-l+1}^{J} cov\left[\mathbb{E}[\xi_{i,j,n}|\mathcal{F}_{t+k}],\mathbb{E}[\xi_{l,z,n}|\mathcal{F}_{t+k}]|\mathcal{F}_t\right]}\Big)\\
&= \sum_{n,m=1}^{N} \sum_{i,l=t+k-J+1}^{I} \hat{C}^{t}_{i,J,n}\hat{C}^{t}_{l,J,m}\Big(e^{\mathbf{p}^T_{t|i,k+1,n}\mathcal{S}^{post}_{\mathcal{D}^c_t}\mathbf{p}_{t|l,k+1,m}} - e^{\mathbf{p}^T_{t|i,k,n}\mathcal{S}^{post}_{\mathcal{D}^c_t}\mathbf{p}_{t|l,k,m}}\Big).
\end{aligned} \tag{4.10}$$

If we want to measure the one-year risk for a single line of business n, we calculate the above sum with $n = m$.

We now define the one-year correlations in future calendar years till the liability's run-off. The one-year correlations represent the correlations that should be used for a bottom-up aggregation of risk capitals in one-year time horizon in future calendar years. The implied one-year correlations are implied from the variance risk measures from Theorem 4.3.

Definition 4.2. *For two lines of business, denoted by $n = 1, 2$, and for $k = 0, 1, \ldots, J-1$, the implied one-year correlation in calendar year $t+k+1$ is derived from the relation*

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$$Var\Big[L^{1YR,t+k+1}\big|\mathcal{F}_t\Big] = Var\Big[L_1^{1YR,t+k+1}\big|\mathcal{F}_t\Big] + Var\Big[L_2^{1YR,t+k+1}\big|\mathcal{F}_t\Big]$$
$$+2\sqrt{Var\Big[L_1^{1YR,t+k+1}\big|\mathcal{F}_t\Big]Var\Big[L_2^{1YR,t+k+1}\big|\mathcal{F}_t\Big]}corr\Big[L_1^{1YR,t+k+1},L_2^{1YR,t+k+1}\big|\mathcal{F}_t\Big], \quad (4.11)$$

where

$$L_n^{1YR,t+k+1} = \sum_{i=t+k-J+1}^{I} L_{i,n}^{1YR,t+k+1}, \quad n = 1, 2.$$

The implied one-year correlation is just the Pearson correlation between the one-year losses $L_1^{1YR,t+k+1}$ and $L_2^{1YR,t+k+1}$ conditional on $\mathcal{F}_t$. We denote $corr\Big[L_1^{1YR,t+k+1},L_2^{1YR,t+k+1}\big|\mathcal{F}_t\Big]$ by ρ_{t+k+1}^{1YR}.

In our two special cases, we can derive explicit results on the one-year correlations. The theorem below presents new results on one-year correlations.

Theorem 4.4. *Let the one-year correlations in future calendar years be given with*

$$\rho_{t+k+1}^{1YR} = \frac{P^k}{\sqrt{Q_1^k}\sqrt{Q_2^k}}, \quad k = 0, 1, \ldots, J-1.$$

- *For dependence structure A and no parameter uncertainty, we have*

$$P^k = \sum_{i=t+k-J+1}^{I} \hat{C}_{i,J,n}^t \hat{C}_{i,J,m}^t \Big(e^{\sum_{j=t-i+1}^{t+k-i+1}\sigma_{j,n}\sigma_{j,m}\rho} - e^{\sum_{j=t-i+1}^{t+k-i}\sigma_{j,n}\sigma_{j,m}\rho}\Big),$$
$$\approx \sum_{i=t+k-J+1}^{I} \hat{C}_{i,J,n}^t \hat{C}_{i,J,m}^t \sigma_{t+k-i+1,n}\sigma_{t+k-i+1,m}\rho,$$
$$Q_n^k = \sum_{i=t+k-J+1}^{I} \big(\hat{C}_{i,J,n}^t\big)^2 \Big(e^{\sum_{j=t-i+1}^{t+k-i+1}\sigma_{j,n}^2} - e^{\sum_{j=t-i+1}^{t+k-i}\sigma_{j,n}^2}\Big)$$
$$\approx \sum_{i=t+k-J+1}^{I} \big(\hat{C}_{i,J,n}^t\big)^2 \sigma_{t+k-i+1,n}^2,$$

  where the approximations hold for small $(\sigma_{j,n})_{j=1}^J$ for $n = 1, 2$. Moreover, we have the upper bound on the one-year correlations $\rho_{t+k+1}^{1YR} \le \rho$ and $\rho_{t+J}^{1YR} = \rho$ for small $(\sigma_{j,n})_{j=1}^J$ for $n = 1, 2$.
- *For dependence structure B and no parameter uncertainty, we have*

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$$P^k = \sum_{i,l=t+k-J+1}^{I} \hat{C}^t_{i,J,n}\hat{C}^t_{l,J,m}\left(e^{\sum_{j=t-i+1}^{(t+k-i+1)\wedge(J+l-i)}\sigma_{j,n}\sigma_{i+j-l,m}\rho} - e^{\sum_{j=t-i+1}^{(t+k-i)\wedge(J+l-i)}\sigma_{j,n}\sigma_{i+j-l,m}\rho}\right)$$

$$\approx \sum_{i,l=t+k-J+1}^{I} \hat{C}^t_{i,J,n}\hat{C}^t_{l,J,m}\sigma_{t+k-i+1,n}\sigma_{t+k+1-l,m}\rho,$$

$$Q^k_n = \sum_{i=t+k-J+1}^{I} \left(\hat{C}^t_{i,J,n}\right)^2\left(e^{\sum_{j=t-i+1}^{t+k-i+1}\sigma^2_{j,n}} - e^{\sum_{j=t-i+1}^{t+k-i}\sigma^2_{j,n}}\right)$$

$$+ \sum_{i,l=t+k-J+1,\ i\neq l}^{I} \hat{C}^t_{i,J,n}\hat{C}^t_{l,J,n}\left(e^{\sum_{j=t-i+1}^{(t+k-i+1)\wedge(J+l-i)}\sigma_{j,n}\sigma_{i+j-l,n}\rho} - e^{\sum_{j=t-i+1}^{(t+k-i)\wedge(J+l-i)}\sigma_{j,n}\sigma_{i+j-l,n}\rho}\right)$$

$$\approx \sum_{i=t+k-J+1}^{I} \left(\hat{C}^t_{i,J,n}\right)^2\sigma^2_{t+k-i+1,n} + \sum_{i,l=t+k-J+1,\ i\neq l}^{I} \hat{C}^t_{i,J,n}\hat{C}^t_{l,J,n}\sigma_{t+k-i+1,n}\sigma_{t+k+1-l,n}\rho,$$

where the approximations hold for small $(\sigma_{j,n})_{j=1}^J$ *for* $n=1,2$. *Moreover, we have the lower bound on the one-year correlations* $\rho^{1YR}_{t+k+1} \geq \rho$ *and* $\rho^{1YR}_{t+J} = \rho$ *for small* $(\sigma_{j,n})_{j=1}^J$ *for* $n=1,2$.

The one-year correlation in the next calendar year (for $k=0$) is mainly used by insurance companies in a bottom-up risk aggregation in Solvency II to derive the regulatory capital. In practice, an insurance company specifies solvency capital requirements in one-year time horizon for the next calendar year for (two) lines of business $SCR^{1YR}_{t+1,1}$ and $SCR^{1YR}_{t+1,2}$ and applies the variance-covariance aggregation formula to derive the diversified solvency capital requirement:

$$\sqrt{\left(SCR^{1YR}_{t+1,1}\right)^2 + \left(SCR^{1YR}_{t+1,2}\right)^2 + 2\cdot SCR^{1YR}_{t+1,1}\cdot SCR^{1YR}_{t+1,2}\cdot\rho^{1YR}_{t+1}}, \tag{4.12}$$

where ρ^{1YR}_{t+1} denotes the calibrated one-year correlation in the next calendar year. The solvency capital requirements in Eq. (4.12) are often related to variance risk measures of one-year losses. The key question is to what extent the ultimate correlation ρ^{Ult}_t used in Eq. (4.6) can differ from the one-year correlation ρ^{1YR}_{t+1} used in Eq. (4.12), and consequently what sizes of misestimation of capitals can we observe if we use an improper correlation for the given time horizon. We investigate this question in the next two sections.

The one-year correlations in future calendar years (for all $k\geq 0$) are important when we calculate risk margins in Solvency II. The risk margin is calculated as

$$\sum_{k=0}^{\infty} CoC\frac{SCR^{1YR}_{t+k+1}}{\left(1+r_{t+k+1}\right)^{k+1}}, \tag{4.13}$$

where CoC is a cost of capital, r_{t+k+1} is the risk-free rate in calendar year $t+k+1$, and SCR^{1YR}_{t+k+1} is the projected solvency capital requirement for calendar year $t+k+1$. The solvency capital requirements can be determined with

$$SCR^{1YR}_{t+k+1} = 3\cdot\mathbb{E}\left[\sqrt{Var\left[L^{1YR,t+k+1}\middle|\mathcal{F}_{t+k}\right]}\middle|\mathcal{F}_t\right]$$

$$\approx 3\cdot\sqrt{\mathbb{E}\left[Var\left[L^{1YR,t+k+1}\middle|\mathcal{F}_{t+k}\right]\middle|\mathcal{F}_t\right]} = 3\cdot\sqrt{Var\left[L^{1YR,t+k+1}\middle|\mathcal{F}_t\right]}, \tag{4.14}$$

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where the approximation is suggested by Wüthrich and Merz (2015) and the factor of 3 is assumed in Solvency II Standard Formula. In practice, many insurance companies project solvency capital requirements for future calendar years for each line of business and aggregate them with the variance-covariance aggregation formula to derive the diversified risk margin for a company with (4.13)–(4.14). In this approach, one should use the one-year correlations in future calendar years for the aggregation of the Solvency Capital Requirements (SCRs) projected for the future calendar years, which are likely to be different from the one-year correlation in the next calendar year. In the next two sections, we investigate patterns of the one-year correlations ρ^{1YR}_{t+k+1} in future calendar years, for $k=0,1,\ldots$, inspect differences in the one-year correlations in future calendar years and misestimation of capitals resulting from the choice of improper correlations.

4.3 Two key relations between the ultimate correlation and the one-year correlations

In this section, we derive two new relations between the ultimate correlation and the one-year correlations. In the next section, we investigate correlations and their impact on capitals in a numerical study with real data.

From Eqs. (4.9), (4.4), and (4.11), we can derive the equality

$$
\begin{aligned}
& Var\Big[L_1^{Ult,t}\big|\mathcal{F}_t\Big]+Var\Big[L_2^{Ult,t}\big|\mathcal{F}_t\Big]+2\sqrt{Var\Big[L_1^{Ult,t}\big|\mathcal{F}_t\Big]Var\Big[L_2^{Ult,t}\big|\mathcal{F}_t\Big]}corr\Big[L_1^{Ult,t},L_2^{Ult,t}\big|\mathcal{F}_t\Big]\\
&=Var\big[L^{Ult,t}\big|\mathcal{F}_t\big]=\sum_{k=0}^{J-1}Var\big[L^{1YR,t+k+1}\big|\mathcal{F}_t\big]\\
&=\sum_{k=0}^{J-1}\Big(Var\Big[L_1^{1YR,t+k+1}\big|\mathcal{F}_t\Big]+Var\Big[L_2^{1YR,t+k+1}\big|\mathcal{F}_t\Big]\\
&\quad+2\sqrt{Var\Big[L_1^{1YR,t+k+1}\big|\mathcal{F}_t\Big]Var\Big[L_2^{1YR,t+k+1}\big|\mathcal{F}_t\Big]}corr\Big[L_1^{1YR,t+k+1},L_2^{1YR,t+k+1}\big|\mathcal{F}_t\Big]\Big).
\end{aligned}
$$

Since (4.9) also holds for $L_1^{Ult,t}$ and $L_2^{Ult,t}$, we end up with the following relation between the risk measures

$$
\begin{aligned}
&\sqrt{Var\Big[L_1^{Ult,t}\big|\mathcal{F}_t\Big]Var\Big[L_2^{Ult,t}\big|\mathcal{F}_t\Big]}corr\Big[L_1^{Ult,t},L_2^{Ult,t}\big|\mathcal{F}_t\Big]\\
&=\sum_{k=0}^{J-1}\sqrt{Var\Big[L_1^{1YR,t+k+1}\big|\mathcal{F}_t\Big]Var\Big[L_2^{1YR,t+k+1}\big|\mathcal{F}_t\Big]}corr\Big[L_1^{1YR,t+k+1},L_2^{1YR,t+k+1}\big|\mathcal{F}_t\Big], \qquad (4.15)
\end{aligned}
$$

which allows us to state our first key relation between the correlations.

Theorem 4.5. *We set $t=I$. For $n=1,2$, let $(R_n^{t+k})_{k=0}^{J-1}$ denote a risk run-off pattern for line of business n measured with*

$$
R_n^{t+k+1}=\frac{\sqrt{Var\Big[L_n^{1YR,t+k+1}\big|\mathcal{F}_t\Big]}}{\sqrt{Var\Big[L_n^{Ult,t}\big|\mathcal{F}_t\Big]}}\in(0,1).
$$

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- *We have the following relation between the ultimate correlation and the one-year correlations*

$$\rho_t^{Ult} = \sum_{k=0}^{J-1} R_1^{t+k+1} R_2^{t+k+1} \rho_{t+k+1}^{1YR}, \tag{4.16}$$

together with the lower and the upper estimates on the ultimate correlation in terms of the one-year correlations and the risk run-off patterns

$$C \cdot \min_{k=0,\ldots,J-1} \left\{\rho_{t+k+1}^{1YR}\right\} \le \rho_t^{Ult} \le \max_{k=0,\ldots,J-1} \left\{\rho_{t+k+1}^{1YR}\right\}, \tag{4.17}$$

with

$$C = 2\sqrt{\frac{L}{1+L^2}} \le 1, \quad L = \frac{\max_{k=0,\ldots,J-1} R_1^{t+k+1}/R_2^{t+k+1}}{\min_{k=0,\ldots,J-1} R_1^{t+k+1}/R_2^{t+k+1}} \ge 1.$$

- *If $R_1^{t+k+1} \neq \alpha R_2^{t+k+1}$ for some $k = 0, 1 \ldots, J-1$ and all constants $\alpha > 0$, then the ultimate and the one-year correlations in future calendar years cannot be all equal.*

Remark 4.2. The assumption that $R_1^{t+k+1} = \alpha R_2^{t+k+1}$ for all $k = 0, \ldots, J-1$ and some $\alpha > 0$ is unrealistic in practice and means that we consider scaled businesses. We exclude this case from considerations. However, if the assumption holds, then potentially we could have $\rho_t^{Ult} = \rho_{t+k+1}^{1YR}$. If we consider dependence A without parameters' uncertainty, then, by direct calculations, one can check that the ultimate and the one-year correlations are all equal to ρ, for small $(\sigma_{j,n})_{j=1}^J$, $n = 1, 2$.

Theorem 4.5 generalizes our preliminary results on the ultimate and the one-year correlations from Section 3.1. The first conclusion from Theorem 4.5 is that the correlation coefficients which should be used for bottom-up aggregation of stand-alone risk capitals depend on the time horizon and the calendar year of the risk measurement period (at least in the Hertig's model). This is a very important conclusion for actuarial practice. The most straightforward approach in actuarial practice would be to take the one-year correlations between lines of business from Solvency II Standard Formula, which were developed to derive the diversified solvency capital requirement in one-year time horizon in the next calendar year (the regulatory capital in Solvency II), and apply the same correlations in all future calendar years to estimate the future diversified solvency capital requirements used for the calculation of the risk margin, as well as to correlate the risk capitals in ultimate time horizon to calculate the risk adjustment for IFRS 17 standard. Theorem 4.5 shows that the three important correlation coefficients that should be used to derive the Solvency II capital requirement, the Solvency II risk margin, and the IFRS 17 risk adjustment are different. The second conclusion is that we expect that the ultimate correlation is low compared to the one-year correlations if the constant L is large (L comes from the Cassels's inequality, see Section 3.1 for an initial discussion and Appendix for the proof). We remark that the constant L is large e.g. if there are large differences in the run-off patterns of R_1^{t+k} vs R_2^{t+k} for $k = 0, \ldots, J-1$.

Let us remark that Theorem 4.5 holds in any claims development model provided that the decomposition formula (4.9) holds. From Wüthrich and Merz (2015) we know that (4.9) holds approximately in Chain Ladder models, hence we expect that the key conclusions from Theorem 4.5 should also hold in Chain Ladder models.

In the last theorem, we investigate more closely the ultimate correlation versus the one-year correlation in the next calendar year. Hence, we compare the correlation coefficients that should be used for bottom-up risk aggregation in Solvency II and IFRS 17. We assume an exponential

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pattern of the volatility parameters which is often observed in practice. We state our second key relation between the correlations.

Theorem 4.6. *We set $t = I$. For $n = 1, 2$, let us assume that $\sigma_{j,n} = \sigma_{0,n} e^{-\alpha_n j}$, $j = 0, \ldots, J$ and the extrapolated volatility $\sigma_{J+1,n}$ vanishes. For dependence structures A and B and no parameters' uncertainty, we have the following relation between the ultimate correlation and the one-year correlation in the next calendar year*

$$\rho_t^{Ult} \approx \frac{\sqrt{1 - e^{-2\alpha_1}}\sqrt{1 - e^{-2\alpha_2}}}{1 - e^{-(\alpha_1+\alpha_2)}} \rho_{t+1}^{1YR} \leq \rho_{t+1}^{1YR}. \tag{4.18}$$

Remark 4.3. In Appendix, we show that this approximation is more crude for dependence B than A, as it requires faster convergence of $\sigma_{J+1,n}$ to zero. The equality in Eq. (4.18) holds only for $\alpha_1 = \alpha_2$.

Our third conclusion from Theorem 4.6 is that we identify practically relevant cases when the ultimate correlation is smaller than the one-year correlation in the next calendar year. In these cases, if an insurance company uses one-year correlations from Solvency II for ultimate risk aggregation in IFRS 17, it tends to overestimate the diversified risk capital in the considered model (the risk adjustment at the level of a company). We note that the reduction in the correlation coefficient when we switch from the one-year correlation to the ultimate correlation is large when the volatility parameters in two lines of business have different tails behavior (α_1 is different from α_2) and is small when the volatility parameters in two lines of business have similar tails behavior (α_1 is close to α_2) – the larger the difference between α_1 and α_2, the smaller the ultimate correlation compared to the one-year correlation. As illustrated in Section 3.1, a larger difference in α_1 and α_2 implies a larger constant L from Theorem 4.5, hence the second conclusion from Theorem 4.5 agrees with the conclusion from Theorem 4.6.

We should point out that the ultimate and the one-year correlations, defined in this paper, depend on the claims development process assumed here. Clearly, the correlations derived in our multivariate Hertig's lognormal model cannot be used in a claims development model different from the one in which we calculate the risk measures (variance measures). Yet, we believe that the formulas presented here could be helpful to infer relations between ultimate and one-year correlations. Interestingly, our conclusion from Theorem 4.6 agrees with the result from El Alami et al. (2022), where the authors also show, in a different actuarial model, that the ultimate correlation is smaller than the one-year correlation in the next calendar year.

5. Numerical examples

In this section, we investigate possible numerical values of the ultimate and one-year correlations, which may be observed in practice, and the impact of misused correlations on capital. We consider historical loss triangles from eleven lines of business under Solvency II regulation from the Polish market. The data set is available from KNF (2020). For each loss triangle, we estimate the parameters of the marginal Hertig's lognormal model and smooth the parameters in late development periods with exponential functions. We do not estimate any particular dependence structures between the lines of business, instead, we just assume dependence structures A, B, and C driven by the correlation parameter ρ. Recall Section 2 for dependence structures.

5.1 Solvency II lines of business 4, 7 and 12

Line 4 is motor vehicle liability insurance, line 7 is fire and other damage to property insurance, and line 12 is miscellaneous financial loss. The volatility parameters $(\sigma_j)_{j=1}^J$ for lines 4 and 7 are small, hence the approximations presented in the previous sections hold. The volatility parameters

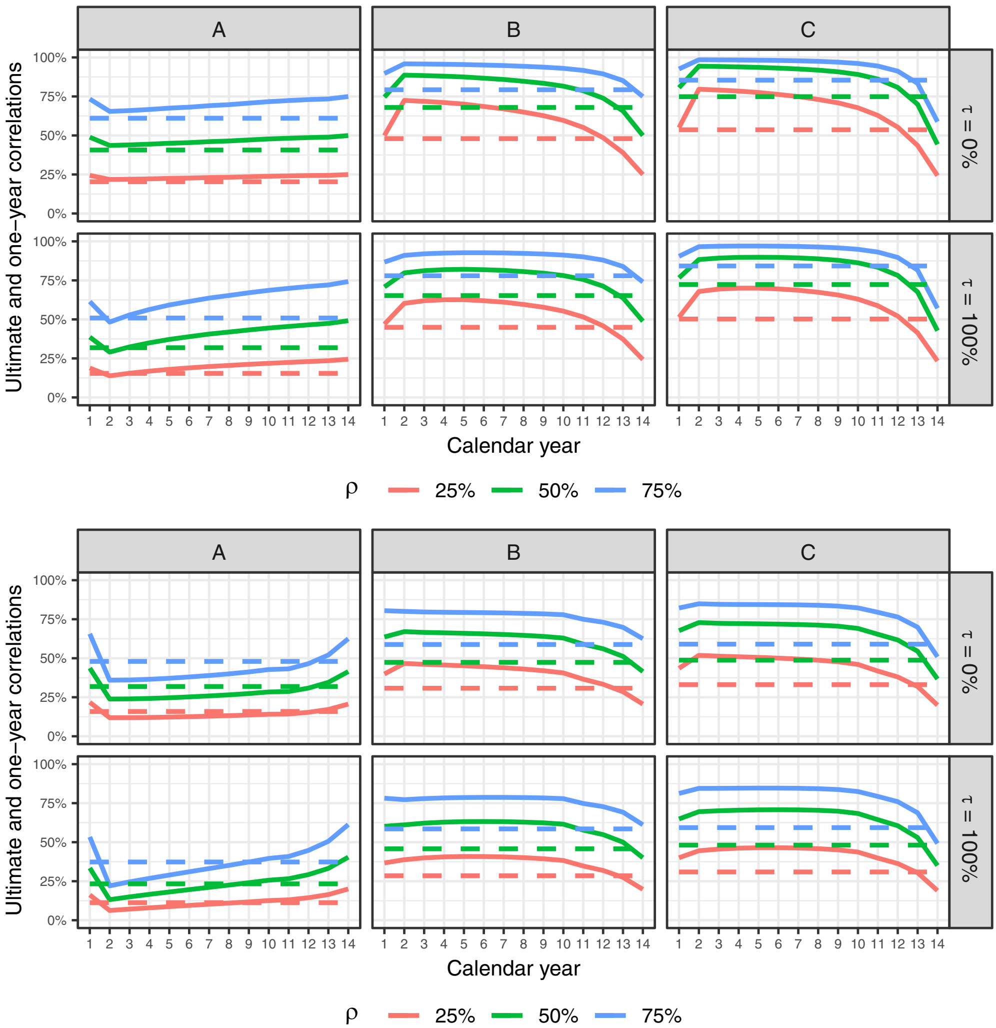

Figure 5. The one-year correlations in future calendar years (solid lines) and the ultimate correlations (dotted lines) – lines 4 and 7 (top) and lines 4 and 12 (bottom).

$(\sigma_j)_{j=1}^J$ for line 12 are large, hence, the approximations fail. Lines 4 and 7 are more regular and homogeneous lines of business, and line 12 is known to be more risky and less homogeneous. The numerical results confirm the analytical results for lines 4 and 7, and present new insights for lines 4 and 12.

The ultimate and the one-year correlations are presented in Fig. 5. We can clearly observe that the correlations depend on the time horizon and the calendar year where the risk emerges. The one-year correlation in the next calendar year (the correlation for Solvency II capital) is larger than the ultimate correlation (the correlation for IFRS 17 capital) in all cases. Under dependence A, the one-year correlations (correlations for Solvency II risk margin) decrease in the first two calendar years and then increase in the calendar years after the calendar year 3. Under dependence B and C, the one-year correlations increase in the first two calendar years and then show a decreasing pattern with respect to the calendar year. We can see that in the one-year correlations can be above and below, and cross, the ultimate correlation. Yet, for all cases except dependence A for lines 4

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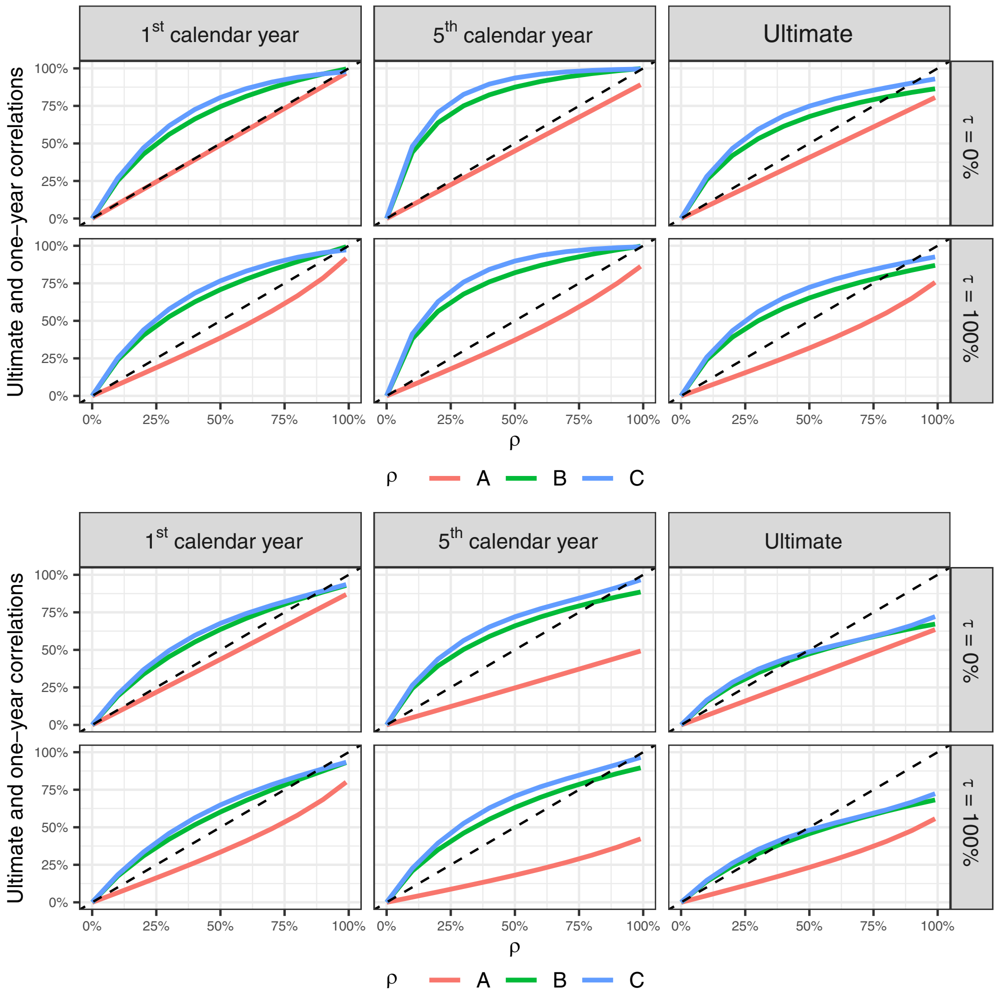

Figure 6. The one-year correlations in future calendar years and the ultimate correlations (solid lines) as a function of ρ, together with the diagonal (dotted lines) – lines 4 and 7 (top) and lines 4 and 12 (bottom).

and 12, the one-year correlations are above the ultimate correlation for most calendar years. Under dependence A and B for lines 4 and 7, the one-year correlation in the last calendar year reaches ρ, whereas in all other cases, the one-year correlation in the last calendar year is below ρ. The impact of τ on correlations is very small; however, τ has an impact on the mean square errors of predictions.

In Fig. 6 we compare the ultimate and the one-year correlations with the driving correlation parameter ρ. Under dependence A, the ultimate and the one-year correlations are always below ρ. Under dependence B and C for lines 4 and 7, the one-year correlations are above ρ, but under dependence B and C for lines 4 and 12, the one-year correlations fall below ρ for large ρ. Under dependence B and C, the ultimate correlations are above ρ for small ρ and fall below ρ for large ρ. As pointed out in Section 3.1, $\rho = 1$ does not necessarily imply the ultimate and the one-year correlations equal to 1.

From the point of practical applications, the most important is the impact of misused correlation on capital. We investigate the Solvency II capital requirement, the Solvency II risk margin,

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Table 1. Risk capitals and their misestimation resulting from misspecified correlations for lines 4 and 7

τ	Dep. A	Dep. B	Dep. C	RA_true	RA_1YR	SCR_true	SCR_ult	RM_true	RM_1YR	RM_ult
0%	25%	0%	0%	512.3	1.3%	1166.5	−1.5%	238.3	0.2%	−0.6%
0%	50%	0%	0%	544.4	2.3%	1265.5	−2.6%	249.1	0.4%	−1.1%
0%	75%	0%	0%	574.7	3.1%	1357.3	−3.3%	259.3	0.5%	−1.5%
0%	0%	25%	0%	801.9	0.4%	1642.8	−0.5%	384.1	−1.6%	−1.9%
0%	0%	50%	0%	1028.3	1.3%	2068.2	−1.6%	491.5	−1.0%	−1.9%
0%	0%	75%	0%	1213.2	1.9%	2420.0	−2.3%	578.9	−0.4%	−1.8%
0%	0%	0%	25%	919.2	0.3%	1798.2	−0.4%	444.7	−1.7%	−1.9%
0%	0%	0%	50%	1430.7	0.9%	2588.1	−1.2%	694.3	−0.9%	−1.6%
0%	0%	0%	75%	2029.9	1.0%	3473.9	−1.3%	973.1	−0.4%	−1.1%
100%	25%	0%	0%	721.7	0.9%	1513.6	−1.1%	340.7	0.2%	−0.4%
100%	50%	0%	0%	732.5	1.6%	1576.5	−2.0%	342.0	0.3%	−0.8%
100%	75%	0%	0%	728.3	2.4%	1611.1	−2.8%	336.2	0.4%	−1.2%
100%	0%	25%	0%	1032.4	0.4%	2066.7	−0.5%	488.9	−1.0%	−1.3%
100%	0%	50%	0%	1230.8	1.0%	2464.7	−1.3%	581.0	−0.7%	−1.4%
100%	0%	75%	0%	1372.9	1.5%	2748.9	−1.9%	647.9	−0.4%	−1.5%
100%	0%	0%	25%	1138.4	0.2%	2207.9	−0.3%	543.0	−1.2%	−1.4%
100%	0%	0%	50%	1581.9	0.7%	2903.4	−0.9%	759.3	−0.9%	−1.4%
100%	0%	0%	75%	2084.8	0.8%	3607.4	−1.1%	995.7	−0.4%	−1.1%

and the IFRS 17 risk adjustment. We assume the cost of capital is equal to 6% and the constant risk-free rate is equal to 3%. For the purpose of calculating the risk margin, we measure the risk using the standard deviation of the one-year loss multiplied by 3 (which agrees with the approach from Solvency II Standard Formula). For the purpose of calculating the risk adjustment, we measure the risk using one standard deviation of the ultimate loss (which is close to the probability of fulfilling the liability at the level of 85% in ultimate time horizon, the confidence level targeted by many insurance companies). The stand-alone risk capitals for the two lines of business are calculated with the variance measures presented in the paper (conditional on the information from the two lines of business) and we aggregate these stand-alone risk capitals with various correlation coefficients. In Tables 1 and 2, we present the following measures:

- RA_true – the risk adjustment obtained by using the ultimate correlation in the risk aggregation in ultimate time horizon,
- RA_1YR – the misestimation caused if the risk adjustment is obtained by using the one-year correlation in the next calendar year in the risk aggregation in ultimate time horizon,
- SCR_true – the solvency capital requirement obtained by using the one-year correlation in the next calendar year in the risk aggregation in one-year time horizon,
- SCR_ult – the misestimation caused if the solvency capital requirement is obtained by using the ultimate correlation in the risk aggregation in one-year time horizon,
- RM_true – the risk margin obtained by using the one-year correlations in the future calendar years in the risk aggregation in one-year time horizon,
- RM_1YR – the misestimation caused if the risk margin is obtained by using the one-year correlation in the next calendar year in the risk aggregation in one-year time horizon,
- RM_ult – the misestimation caused if the risk margin is obtained by using the ultimate correlation in the risk aggregation in one-year time horizon.

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Table 2. Risk capitals and their misestimation resulting from misspecified correlations for lines 4 and 12

τ	Dep. A	Dep. B	Dep. C	RA_true	RA_1YR	SCR_true	SCR_ult	RM_true	RM_1YR	RM_ult
0%	25%	0%	0%	550.1	2.3%	1316.7	−2.4%	245.7	0.8%	−0.5%
0%	50%	0%	0%	583.5	4.0%	1429.0	−4.1%	254.5	1.4%	−0.8%
0%	75%	0%	0%	615.2	5.4%	1533.6	−5.4%	262.8	1.9%	−1.1%
0%	0%	25%	0%	803.2	2.6%	1734.7	−3.1%	376.0	−0.3%	−1.7%
0%	0%	50%	0%	1012.7	3.8%	2143.3	−4.6%	474.5	−0.1%	−2.1%
0%	0%	75%	0%	1186.0	4.5%	2486.1	−5.4%	555.4	0.0%	−2.3%
0%	0%	0%	25%	904.3	2.7%	1861.8	−3.3%	430.1	−0.4%	−1.8%
0%	0%	0%	50%	1364.9	3.3%	2568.2	−4.6%	657.8	−0.2%	−1.9%
0%	0%	0%	75%	1900.7	2.6%	3311.7	−3.9%	911.8	−0.1%	−1.4%
100%	25%	0%	0%	768.0	1.8%	1715.7	−2.1%	350.0	0.7%	−0.3%
100%	50%	0%	0%	780.1	3.4%	1798.3	−3.8%	348.6	1.3%	−0.5%
100%	75%	0%	0%	778.3	5.0%	1854.4	−5.3%	340.1	2.0%	−0.8%
100%	0%	25%	0%	992.0	2.0%	2056.1	−2.6%	464.1	−0.1%	−1.2%
100%	0%	50%	0%	1137.5	2.8%	2333.2	−3.6%	533.6	−0.1%	−1.5%
100%	0%	75%	0%	1244.5	3.3%	2534.5	−4.3%	585.7	0.0%	−1.7%
100%	0%	0%	25%	1067.2	2.0%	2127.4	−2.6%	505.9	−0.2%	−1.2%
100%	0%	0%	50%	1421.2	2.5%	2635.6	−3.5%	683.5	−0.1%	−1.4%
100%	0%	0%	75%	1866.2	2.1%	3228.3	−3.2%	895.3	−0.1%	−1.2%

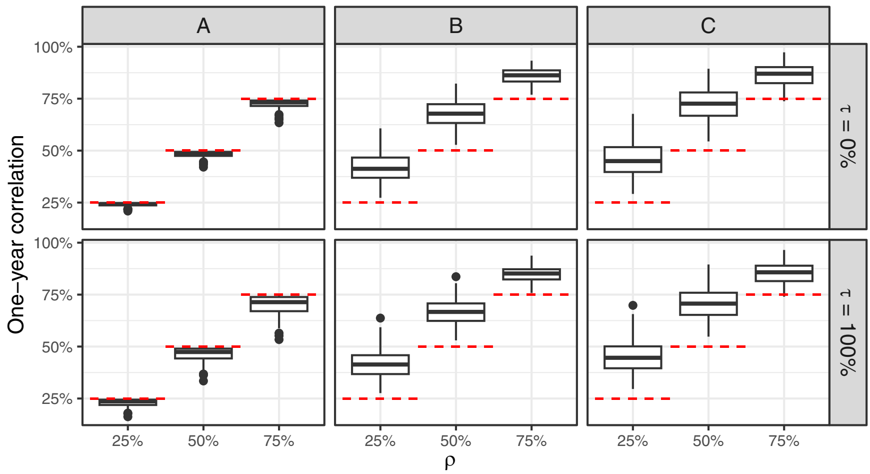

Figure 7. The one-year correlation in the next calendar year (the dotted lines represent the assumed ρ).

Even though we observe (in some cases substantial) differences in the ultimate and the one-year correlations, the impact of misused correlation on the Solvency II and IFRS 17 capitals is rather small in our numerical example with data from the Polish market. We have already discussed the roots of this phenomenon in Section 3.2. For lines 4 and 7, the maximal misestimation of capital is 3.3%, but it is reached for $\rho = 0.75$, which is likely to be too high correlation in practice, see Avanzi et al. (2016b). For $\rho = 0.25,\ 0.5$, the misestimation of capital is below 2.6%. For lines 4 and 12, the maximal misestimation of capital is 5.4% for $\rho = 0.75$, and for $\rho = 0.25,\ 0.5$, the misestimation of capital is below 4.6%.

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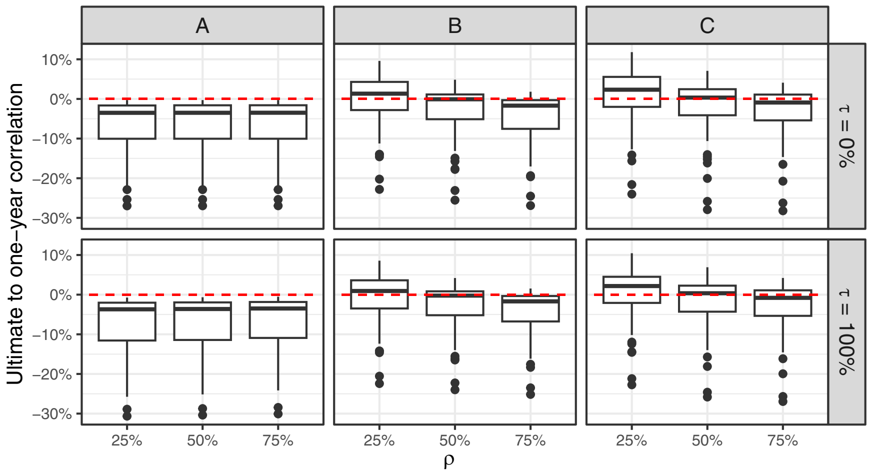

Figure 8. The relative difference of the ultimate correlation compared to the one-year correlation in the next calendar year.

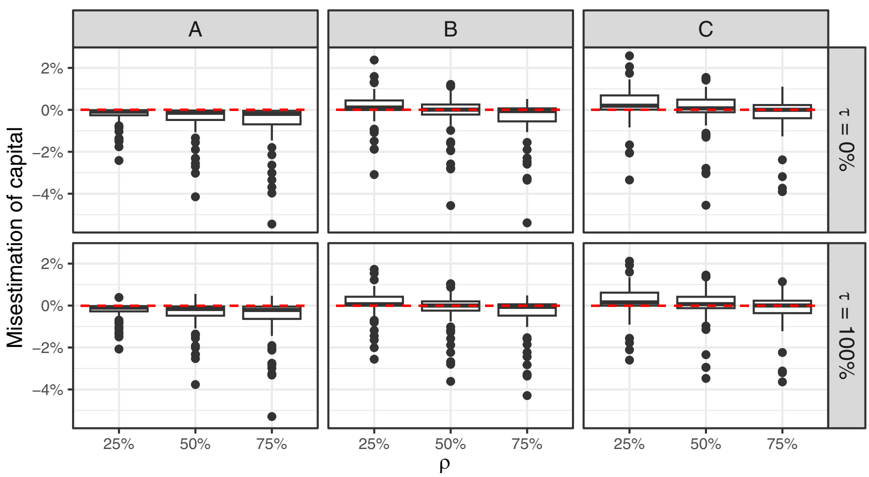

Figure 9. The misestimation of the Solvency II capital requirement caused by using the ultimate correlation instead of the one-year correlation in the next calendar year.

We point out that if we calculate the risk adjustment (the solvency capital requirement) with the one-year correlation in the next calendar year (the ultimate correlation), instead of the ultimate correlation (the one-year correlation in the next calendar year), we overestimate (underestimate) the capital. This result agrees with the observation that the ultimate correlation is always lower than the one-year correlation in the next calendar year in all cases in our example. If we calculate the risk margin with the ultimate correlation, instead of the one-year correlations in the future calendar years, we under-estimate the risk margin. This result is intuitive since in our example the one-year correlations in the first calendar year are always above the ultimate correlations and the capital requirement in the first calendar year has the larger impact on the value of the risk margin due to discounting of the capital requirements and decreasing capital requirements in calendar years (apart from dependence A for lines 4 and 12, the ultimate correlations are even below the one-year correlations in almost all future calendar years). For dependence A, the one-year correlations in the next calendar year are above the one-year correlations in the earliest future

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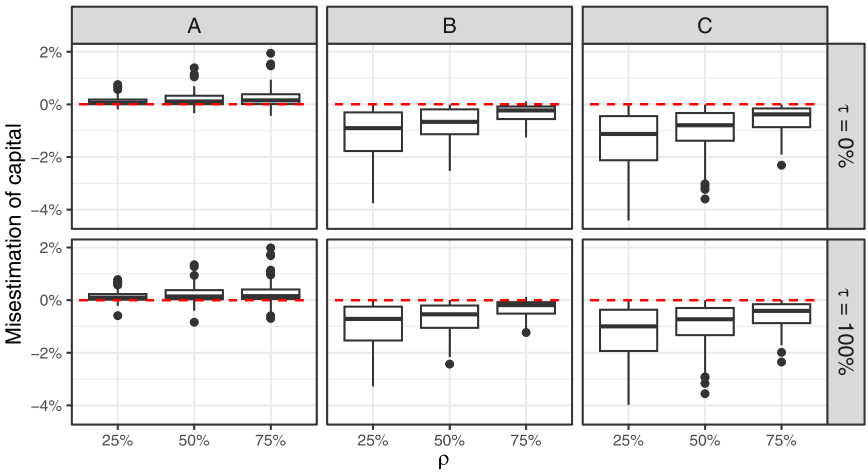

Figure 10. The misestimation of the Solvency II risk margin caused by using the one-year correlation in the next calendar year instead of the one-year correlations in future calendar years.

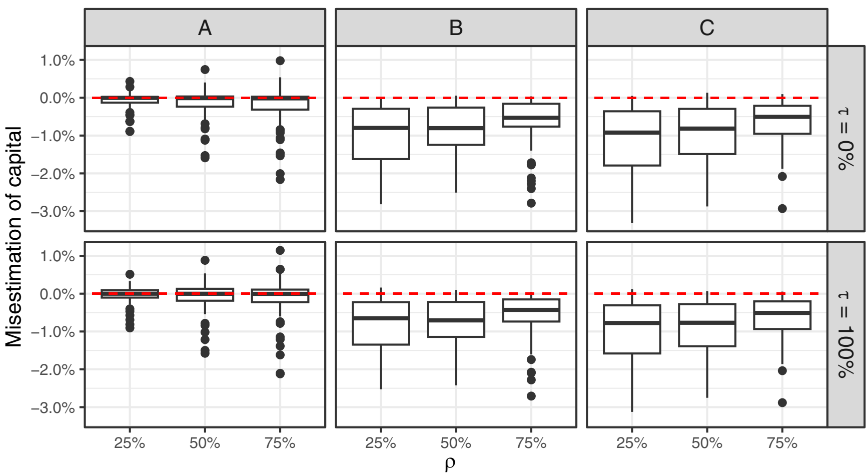

Figure 11. The misestimation of the Solvency II risk margin caused by using the ultimate correlation instead of the one-year correlations in future calendar years.

calendar years, hence we overestimate the risk margin if we use the one-year correlation in the next calendar year instead of the one-year correlations in the future calendar years. For dependence B and C, the one-year correlations in the next calendar year are below the one-year correlations in the earliest future calendar years, hence we underestimate the risk margin if we use the one-year correlation in the next calendar year instead of the one-year correlations in the future calendar years.

5.2 All Solvency II lines of business

In Figs. 7–11 we present the results for all pairs of lines of business from all eleven lines of business. The conclusions are similar to the pairs 4–7 and 4–12. The only crucial point is that for

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Table 3. The misestimation of the Solvency II capital requirement caused by using the ultimate correlation instead of the one-year correlation in the next calendar year for a portfolio with multiple lines of business

τ	Dep. A	Dep. B	Dep. C	SCR_true	SCR_ult
0%	25%	0%	0%	2221.1	−3.1%
0%	50%	0%	0%	2770.9	−4.1%
0%	75%	0%	0%	3231.3	−4.5%
0%	0%	25%	0%	3139.8	−1.7%
0%	0%	50%	0%	4187.0	−3.2%
0%	0%	75%	0%	5022.3	−4.2%
0%	0%	0%	25%	3413.6	−1.5%
0%	0%	0%	50%	4944.5	−3.0%
0%	0%	0%	75%	6171.5	−3.4%
100%	25%	0%	0%	2596.1	−3.0%
100%	50%	0%	0%	3178.2	−4.1%
100%	75%	0%	0%	3696.7	−4.7%
100%	0%	25%	0%	3672.6	−1.8%
100%	0%	50%	0%	4773.9	−3.0%
100%	0%	75%	0%	5649.8	−3.8%
100%	0%	0%	25%	3921.9	−1.4%
100%	0%	0%	50%	5499.6	−2.6%
100%	0%	0%	75%	6633.9	−3.0%

dependence B and C, we can observe cases when the ultimate correlation is above the one-year correlation in the next calendar year. Consequently, the solvency capital requirement calculated with the ultimate correlation can be overestimated. The box-plot for the risk adjustment is a reflection of Fig. 9 relative to 0%.

For most pairs of lines of business, the misestimation of the Solvency II and IFRS 17 capitals resulting from misused correlation is small, but it can reach 3%–6%.

We finally calculate the misestimation of the solvency capital requirement for a portfolio consisting of all eleven lines of business resulting from using pairwise ultimate correlations instead of pairwise one-year correlations in the next calendar year. As a stand-alone risk capital in a line of business, we use standard deviation of the one-year loss in the line of business in the next calendar year. The standard deviation for a line of business is calculated under the information from the single line of business. The results are presented in Table 3. If we restrict our attention to $\rho = 0.25,\ 0.5$, then the Solvency II capital requirement is underestimated by 4.1%. Please note that in Fig. 9 we identify the cases where the ultimate correlation is larger and smaller than the one-year correlation, whereas the results from Table 3 show that the cases when the ultimate correlation is lower than the one-year correlation are dominant if we take into account the volume of the risk of the lines of business. The misestimation of 4.1% is not very large, but it should not be neglected in practice. Let us recall that in Section 3.2, we easily constructed a synthetic example in which we demonstrate that the capital can be underestimated by 7% if we use the ultimate correlation instead of the one-year correlation.

6. Conclusions

We demonstrate with analytical formulas and numerical examples that the ultimate correlation and the one-year correlations in future calendar years are different in a multivariate Hertig's log-normal model of claims developments in multiple lines of business. Our numerical results based

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on real data from the Polish market do not show the ultimate and the one-year correlations can differ to such an extent that they can lead to very large differences in the Solvency II and IFRS 17 risk capitals if an incorrect correlation is used in the bottom-up risk aggregation. However, we believe that our results should give a clear signal to actuaries that the ultimate and the one-year correlations are different and these differences should be investigated in practice as they may have an impact on calculations performed in Solvency II and IFRS 17.

Data availability statement. The data that support the findings of this study are openly available from KNF (2020). The code to generate the results is available from Marcin Szatkowski on request.

Funding statement. The first version of this paper was financially supported with grant NCN 2018/31/B/HS4/02150.

Competing interests. Łukasz Delong and Marcin Szatkowski declare none. The views expressed in this paper are those of the authors and do not necessarily represent those of STU ERGO Hestia SA.

Acknowledgments. The authors would like to thank the reviewers for all their comments, which were used to improve and extend the first version of this paper.

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Appendix A Proofs

For reader's convenience, we first recall some known results on the distribution of a multivariate Gaussian vector which are used in this paper, and also used by Merz et al. (2012) and Wüthrich (2015) in their works.

Theorem A.1. *Let* $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)^T \sim N\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right)$ *with*

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}.$$

We have the conditional distribution

$$\mathbf{X}_2 | \mathbf{X}_1 = \mathbf{x}_1 \sim N\left(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}\right),$$

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{x}_1 - \boldsymbol{\mu}_1), \quad \tilde{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12},$$

and the marginal distributions

$$\mathbf{X}_1 \sim N\left(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1\right), \quad \mathbf{X}_2 \sim N\left(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2\right).$$

Theorem A.2. *Let* $\mathbf{X} | \boldsymbol{\Theta} = \mathbf{v} \sim N\left(\mathbf{v}, \boldsymbol{\Sigma}\right)$ *and* $\boldsymbol{\Theta} \sim N\left(\boldsymbol{\mu}, \mathbf{T}\right)$. *We have the joint distribution*

$$\left(\mathbf{X}, \boldsymbol{\Theta}\right)^T \sim N\left(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}\right),$$

$$\tilde{\boldsymbol{\mu}} = \begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\mu} \end{bmatrix}, \quad \tilde{\boldsymbol{\Sigma}} = \begin{bmatrix} \boldsymbol{\Sigma} + \mathbf{T} & \mathbf{T} \\ \mathbf{T} & \mathbf{T} \end{bmatrix},$$

and the marginal distribution

$$\mathbf{X} \sim N\left(\boldsymbol{\mu}, \boldsymbol{\Sigma} + \mathbf{T}\right).$$

Theorem A.3. *Let* $\mathbf{X} \sim N\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right)$. *Let* **a** *and* **b** *denote vectors of the same dimension as* **X**. *We have the distribution*

$$\mathbf{a}^T\mathbf{X} \sim N\left(\mathbf{a}^T\boldsymbol{\mu}, \mathbf{a}^T\boldsymbol{\Sigma}\mathbf{a}\right).$$

We also have the formulas for the exponential moments

$$\mathbb{E}\left[e^{\mathbf{a}^T\mathbf{X}}\right] = e^{\mathbf{a}^T\boldsymbol{\mu} + \frac{1}{2}\mathbf{a}^T\boldsymbol{\Sigma}\mathbf{a}}, \quad cov\left[e^{\mathbf{a}^T\mathbf{X}}, e^{\mathbf{b}^T\mathbf{X}}\right] = \mathbb{E}\left[e^{\mathbf{a}^T\mathbf{X}}\right] \cdot \mathbb{E}\left[e^{\mathbf{b}^T\mathbf{X}}\right] \cdot \left(e^{\mathbf{a}^T\boldsymbol{\Sigma}\mathbf{b}} - 1\right).$$

Below, we present the proofs of the results from the paper.

The proof of Corollary 2.1: Let us consider the joint multivariate normal distribution of $\left(\boldsymbol{\xi}^{D_t}, \boldsymbol{\xi}^{D_t^c}\right)^T$ with the covariance matrix

$$cov\left[\left(\boldsymbol{\xi}^{D_t}, \boldsymbol{\xi}^{D_t^c}\right)^T\right] = \begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{Q} & \mathbf{P} \end{bmatrix},$$

where $\mathbf{Q} = cov\left[\boldsymbol{\xi}^{D_t}, \boldsymbol{\xi}^{D_t^c}\right]$. The elements of **Q** are calculated with the formula

$$cov\left[\xi_{i,j,n}, \xi_{l,z,m}\right] = \mathbb{E}\left[cov\left[\xi_{i,j,n}, \xi_{l,z,m} \middle| \boldsymbol{\Theta}\right]\right] + cov\left[\theta_{j,n}, \theta_{z,m}\right].$$

Under dependence A and B and without parameters' uncertainty, we have $cov\left[\xi_{i,j,n}, \xi_{l,z,m}\right] = 0$ for $i + j \neq l + z$, that is, for $(i, j, n) \in \mathcal{D}_t$ and $(l, z, m) \in \mathcal{D}_t^c$. □

The proof of Theorem 3.1: We define $u = x/y$ and consider the function

$$R(u) = \frac{u^2 + 1 + 2up}{u^2 + 1 + 2u\rho}.$$

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We calculate the derivative $R'(u)$ and we conclude that $R(u)$ is maximal if $u = 1$. If $x = y$, then $R(1) = \frac{1+p}{1+\rho}$. □

The proof of Theorems 4.2 and 4.4: To prove the formulas for the correlations, we use Corollary 2.1 and directly substitute the assumed covariance structures into Eqs. (4.3) and (4.10). The upper bound for the ultimate and the one-year correlations in dependence A can be immediately proved by the Cauchy–Schwarz inequality. To prove the lower bound for the one-year correlations in dependence B, we set $x_i = \hat{C}^t_{i,J,n}\sigma_{t+k-i+1,n}$ and $y_l = \hat{C}^t_{l,J,m}\sigma_{t+k-l+1,m}$. Next, we deduce that

$$\frac{\sum_{i,l} x_i y_l \rho}{\sqrt{\sum_i x_i^2(1-\rho) + \sum_{i,l} x_i x_l \rho} \cdot \sqrt{\sum_i y_i^2(1-\rho) + \sum_{i,l} y_i y_l \rho}}$$
$$= \frac{\big(\sum_i x_i\big)\big(\sum_i y_i\big)\rho}{\sqrt{\sum_i x_i^2(1-\rho) + \big(\sum_i x_i\big)^2\rho} \cdot \sqrt{\sum_i y_i^2(1-\rho) + \big(\sum_i y_i\big)^2\rho}} \geq \rho,$$

since $\sum_i x_i^2 \leq \big(\sum_i x_i\big)^2$. Under the assumptions of Theorem 4.6, the ultimate correlation is lower than the one-year correlation and the ratio of the ultimate correlation to the one-year correlation can be sufficiently small (if α_1 is different from α_2). Hence, there exists a claims development process for which the ultimate correlation is lower than ρ. □

The proof of Proposition 4.1: We derive

$$\begin{aligned}
\hat{C}^{t+k}_{i,J,n} &= C_{i,t+k-i,n} e^{\left(\mathbf{e}^T_{t+k|i,j\leq J,n}\boldsymbol{\mu}^{post}_{\mathcal{D}^c_{t+k}} + \frac{1}{2}\mathbf{e}^T_{t+k|i,j\leq J,n}\mathcal{S}^{post}_{\mathcal{D}^c_{t+k}}\mathbf{e}_{t+k|i,j\leq J,n}\right)\mathbf{1}\{i+J>t+k\}} \\
&= C_{i,t-i,n} e^{\sum_{j=t-i+1}^{t-i+k}\xi_{i,j,n} + \left(\mathbf{e}^T_{t+k|i,j\leq J,n}\boldsymbol{\mu}^{post}_{\mathcal{D}^c_{t+k}} + \frac{1}{2}\mathbf{e}^T_{t+k|i,j\leq J,n}\mathcal{S}^{post}_{\mathcal{D}^c_{t+k}}\mathbf{e}_{t+k|i,j\leq J,n}\right)\mathbf{1}\{i+J>t+k\}} \\
&= C_{i,t-i,n} e^{\mathbf{e}^T_{t|i,j\leq t-i+k,n}\boldsymbol{\xi}^{\mathcal{D}^c_t} + \mathbf{e}^T_{t+k|i,j\leq J,n}\mathcal{Q}_{\mathcal{D}_{t+k},\mathcal{D}^c_{t+k}}\boldsymbol{\xi}^{\mathcal{D}_{t+k}}\mathbf{1}\{i+J>t+k\} + r_{t|i,k,n}} \\
&= C_{i,t-i,n} e^{\mathbf{e}^T_{t|i,j\leq t-i+k,n}\boldsymbol{\xi}^{\mathcal{D}^c_t} + \mathbf{e}^T_{t+k|i,j\leq J,n}\mathcal{Q}_{\mathcal{D}_{t+k},\mathcal{D}^c_{t+k}}\mathcal{P}_{\mathcal{D}_{t+k}}\mathcal{P}^T_{\mathcal{D}^c_t}\mathbf{1}\{i+J>t+k\}\boldsymbol{\xi}^{\mathcal{D}^c_t} + r_{t|i,k,n}} \\
&= C_{i,t-i,n} e^{\mathbf{p}^T_{t|i,k,n}\boldsymbol{\xi}^{\mathcal{D}^c_t} + r_{t|i,k,n}}.
\end{aligned}$$

In the derivation above, first, we use the estimate (4.2) after time $t+k$, the claims development process (2.1), and the definition of the conditional mean from Theorem 2.1. Next, we collect all $\mathcal{F}_{t+k}$-measurable terms and the residual term $r_{t|i,k,n}$ collects all $\mathcal{F}_t$-measurable terms. We notice that $\mathcal{P}^T_{\mathcal{D}^c_t}\boldsymbol{\xi}^{\mathcal{D}^c_t}$ creates a vector of dimension $\mathbb{R}^d$, which contains $\xi_{i,j,n}$ for $(i,j,n) \in \mathcal{D}^c_t$ and sets $\xi_{i,j,n} = 0$ for $(i,j,n) \in \mathcal{D}_t$, which allows us to represent the $\mathcal{F}_{t+k}$-elements from $\boldsymbol{\xi}^{\mathcal{D}_{t+k}}$, which are not $\mathcal{F}_t$-measurable, with a linear transformation of $\boldsymbol{\xi}^{\mathcal{D}^c_t}$. Finally, $r_{t|i,k,n}$ is derived by the property that

$$\mathbb{E}\big[\hat{C}^{t+k}_{i,J,n}\big|\mathcal{F}_t\big] = \hat{C}^t_{i,J,n},$$

which holds for any $k = 0, 1, \ldots$. □

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The proof of Theorem 4.3: We use the definition of the loss, Eq. (4.9), Proposition 4.1, Theorem A.3 and classical formulas for covariance. Moreover, we prove

$$cov\Big[\hat{C}^{t+k}_{i,J,n}\hat{C}^{t+k+1}_{l,J,m}\big|\mathcal{F}_t\Big] = \mathbb{E}\Big[cov\Big[\hat{C}^{t+k}_{i,J,n}\hat{C}^{t+k+1}_{l,J,m}\big|\mathcal{F}_{t+k}\Big]\big|\mathcal{F}_t\Big]$$
$$+cov\Big[\mathbb{E}\big[\hat{C}^{t+k}_{i,J,n}\big|\mathcal{F}_{t+k}\big], \mathbb{E}\big[\hat{C}^{t+k+1}_{l,J,m}\big|\mathcal{F}_{t+k}\big]\big|\mathcal{F}_t\Big] = cov\Big[\hat{C}^{t+k}_{i,J,n}, \hat{C}^{t+k}_{l,J,m}\mathcal{F}_t\Big].$$

□

The proof of Theorem 4.5:. The result follows from (4.15) and Cassels's inequality, see Eq. (3.2) in Watson (1955). Let us assume that $R^{t+k+1}_1 \neq \alpha R^{t+k+1}_2$, for some $k = 0, \ldots, J-1$ and all $\alpha > 0$. If

$$corr\Big[L^{Ult,t}_1, L^{Ult,t}_2\big|\mathcal{F}_t\Big] = corr\Big[L^{1YR,t+k+1}_1, L^{1YR,t+k+1}_2\big|\mathcal{F}_t\Big], \quad k = 0, \ldots, J-1,$$

then we get the contradiction

$$1 = \sum_{k=0}^{J-1} R^{t+k+1}_1 R^{t+k+1}_2 < \sqrt{\sum_{k=0}^{J-1} |R^{t+k+1}_1|^2}\sqrt{\sum_{k=0}^{J-1} |R^{t+k+1}_2|^2} = 1.$$

□

The proof of Theorem 4.6: We substitute the exponential functions assumed for the volatility parameters into the formulas for the ultimate correlations from Theorem 4.2 and match them with the one-year correlations from Theorem 4.4.
Dependence A. We derive

$$\sum_{j=t-i+1}^{J} \sigma_{j,1}\sigma_{j,2}\rho = \sum_{j=t-i+1}^{J} \sigma_{0,1}\sigma_{0,2}e^{-\alpha_1\cdot j}e^{-\alpha_2\cdot j}\rho$$
$$= \sigma_{0,1}\sigma_{0,2}\frac{e^{-(\alpha_1+\alpha_2)(t-i+1)} - e^{-(\alpha_1+\alpha_2)(J+1)}}{1-e^{-(\alpha_1+\alpha_2)}}\rho$$
$$\approx \sigma_{0,1}\sigma_{0,2}\frac{e^{-(\alpha_1+\alpha_2)(t-i+1)}}{1-e^{-(\alpha_1+\alpha_2)}}\rho = \frac{\sigma_{t-i+1,1}\sigma_{t-i+1,2}\rho}{1-e^{-(\alpha_1+\alpha_2)}}.$$

In the same way, we handle $\sum_{j=t-i+1}^{J}\sigma^2_{j,n}$. In order to have a good approximation, the following conditions should be satisfied

$$\sigma_{J+1,1}\sigma_{J+1,2} \approx 0, \quad \sigma^2_{J+1,1} \approx 0, \quad \sigma^2_{J+1,2} \approx 0. \tag{A.1}$$

Dependence B. First, we derive

$$\sum_{j=t-i+1}^{J\wedge(J+l-i)} \sigma_{j,1}\sigma_{i+j-l,2}\rho = \sum_{j=t-i+1}^{J\wedge(J+l-i)} \sigma_{0,1}\sigma_{0,2}e^{-\alpha_1\cdot j}e^{-\alpha_2\cdot(i+j-l)}\rho$$
$$= \sigma_{0,1}\sigma_{0,2}\frac{e^{-\alpha_1\cdot(t-i+1)}e^{-\alpha_2\cdot(t+1-l)} - e^{-(\alpha_1+\alpha_2)\cdot(J\wedge(J+l-i)+1)-\alpha_2\cdot(i-l)}}{1-e^{-(\alpha_1+\alpha_2)}}\rho.$$

Next, we show that

$$(\alpha_1+\alpha_2)\cdot(J\wedge(J+l-i)+1) + \alpha_2\cdot(i-l) \geq \alpha_1(J+1) + 2\alpha_2,$$

if $i \leq l$, and

$$(\alpha_1+\alpha_2)\cdot(J\wedge(J+l-i)+1) + \alpha_2\cdot(i-l) \geq \alpha_2(J+1) + 2\alpha_1,$$

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if $i \geq l$, since we consider i and l from $I-J+1$ up to I and, consequently, $|i-l| \leq J-1$. We conclude that

$$\sum_{j=t-i+1}^{J \wedge (J+l-i)} \sigma_{j,1}\sigma_{i+j-l,2}\rho \approx \sigma_{0,1}\sigma_{0,2}\frac{e^{-\alpha_1\cdot(t-i+1)}e^{-\alpha_2\cdot(t-l+1)}}{1-e^{-(\alpha_1+\alpha_2)}}\rho = \sigma_{t-i+1,1}\sigma_{t-l+1,2}\rho.$$

We handle $\sum_{j=t-i+1}^{J}\sigma_{j,n}^2$ as for dependence A. In order to have a good approximation, this time the following conditions should be satisfied

$$\sigma_{J+1,1}\sigma_{2,1} \approx 0, \quad \sigma_{J+1,1}\sigma_{2,2} \approx 0, \quad \sigma_{J+1,2}\sigma_{2,1} \approx 0, \quad \sigma_{J+1,2}\sigma_{2,2} \approx 0,$$
$$\sigma_{J+1,1}^2 \approx 0, \quad \sigma_{J+1,2}^2 \approx 0. \tag{A.2}$$

We observe that (A.2) is stronger than (A.1). □

Cite this article: Delong Ł and Szatkowski M (2025). One-year and ultimate correlations in dependent claims run-off triangles, *Annals of Actuarial Science*, **19**, 159–192. https://doi.org/10.1017/S1748499524000204