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Invariant Theory of Abelian Transvection Groups

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Abstract. Let G be a finite group acting linearly on the vector space V over a field of arbitrary characteristic. The action is called *coregular* if the invariant ring is generated by algebraically independent homogeneous invariants, and the *direct summand property* holds if there is a surjective $k[V]^G$-linear map $\pi\colon k[V] \to k[V]^G$.

The following Chevalley–Shephard–Todd type theorem is proved. Suppose G is abelian. Then the action is coregular if and only if G is generated by pseudo-reflections and the direct summand property holds.

1 Introduction

Let V be a vector space of dimension n over a field k. A linear transformation $\tau\colon V \to V$ is called a *pseudo-reflection* if its fixed-points space $V^\tau = \{v \in V \,;\, \tau(v) = v\}$ is a linear subspace of codimension one. Let $G < \mathrm{GL}(V)$ be a finite group acting linearly on V. Then G acts by algebra automorphisms on the coordinate ring $k[V]$, which is by definition the symmetric algebra on the dual vector space V^*. We shall say that G is a *pseudo-reflection group* if G is generated by pseudo-reflections; it is called a *non-modular* group if the order of G is not divisible by the characteristic of the field. The action is called *coregular* if the invariant ring is generated by n algebraically independent homogeneous invariants. Finally we say that the *direct summand property* holds if there is a surjective $k[V]^G$-linear map $\pi\colon k[V] \to k[V]^G$ respecting the gradings.

For a non-modular group the direct summand property always holds, because in that case we can take the *transfer* Tr^G as projection, defined by

$$\mathrm{Tr}^G\colon k[V] \to k[V]^G : \mathrm{Tr}^G(f) = \sum_{\sigma\in G} \sigma(f),$$

since for any invariant f we have $\mathrm{Tr}^G(|G|^{-1} f) = f$. A theorem of Serre [1, Theorem 6.2.2] implies that if the action is coregular then G is a pseudo-reflection group and the direct summand property holds. We conjectured that the converse also holds [2]. The theorem of Chevalley–Shephard–Todd [1, Chapter 6] says that the converse holds if the group is non-modular. In this note we prove that the converse holds if G is abelian. Elsewhere we show that the converse is also true if V is an irreducible kG-module [3].

Theorem 1.1 *Suppose $G < \mathrm{GL}(V)$ is an abelian group acting on the finite dimensional vector space V. Then the action is coregular if and only if G is a pseudo-reflection group and the direct summand property holds.*

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As corollary we get a special case of a conjecture made by Shank–Wehlau [8]. Suppose the characteristic of the field is $p > 0$.

Corollary 1.2 *Let $G < \mathrm{GL}(V)$ be an abelian p-group acting linearly on the vector space V. The image of the transfer map Tr^G is a principal ideal in $k[V]^G$ if and only if the action is coregular.*

2 Hilbert Ideal and the Direct Summand Property

For elementary facts on the invariant theory of finite groups we refer to [1], for a discussion of the direct summand property and the different see [2]. We recall that the different θ_G of the action can be defined as the largest degree homogeneous form in $k[V]$ such that $\mathrm{Tr}^G(f/\theta) \in k[V]^G$ for all $f \in k[V]^G$; it is unique up to a multiplicative scalar. The direct summand property holds if and only if there exists a $\tilde{\theta}_G$ such that $\mathrm{Tr}^G(\tilde{\theta}_G/\theta_G) = 1$ and then we can take as $k[V]^G$-linear projection

$$\pi\colon k[V] \to k[V]^G : \ \pi(f) := \mathrm{Tr}^G\Big(\frac{\tilde{\theta}_G f}{\theta_G}\Big).$$

If $J \subseteq k[V]^G$ is an ideal, we define $J^e := J \cdot k[V]$, the ideal in $k[V]$ generated by J. If $I \subseteq k[V]$, we define $I^c := I \cap k[V]^G$, the ideal in $k[V]^G$ generated by the invariants contained in I. An important consequence of the direct summand property is that it implies $J = J^{ec}$ [2, Proposition 6].

The *Hilbert ideal* $\mathfrak{H} \subset k[V]$ is the ideal generated by all positive degree homogeneous invariants. Hilbert already noticed that if the direct summand property holds, then any collection of homogeneous G-invariants generating the Hilbert ideal also generates the algebra of invariants. We say that the Hilbert ideal is a *complete intersection ideal*, if it can be generated by n homogeneous invariants where $n = \dim V$. Those invariants necessarily form a (very special) homogeneous system of parameters. We shall use the following criterion for coregularity.

Proposition 2.1 *The action is coregular if and only if the Hilbert ideal $\mathfrak{H}$ is a complete intersection ideal and the direct summand property holds.*

Proof If the action is coregular, then $k[V]^G = k[f_1, \dots, f_n]$ and so $\mathfrak{H} = (f_1, \dots, f_n)$ is a complete intersection ideal. Coregularity also implies the direct summand property [2, Proposition 5(ii)].

Conversely, suppose the direct summand property holds and $\mathfrak{H} = (f_1, \dots, f_n)$, where $f_1, \dots, f_n$ are homogeneous invariants of positive degree. Now we recall Hilbert's argument showing that $R := k[f_1, \dots, f_n]$ is equal to $k[V]^G$. Suppose R is not equal to $k[V]^G$. Then let $f \in k[V]^G$ be of minimal degree such that f is not in R. But $f \in \mathfrak{H}$, so there are $h_1, \dots, h_n \in k[V]$ of degree strictly smaller than the degree of f, such that $f = h_1 f_1 + \cdots + h_n f_n$. By hypothesis there is a $k[V]^G$-linear projection operator $\pi\colon k[V] \to k[V]^G$ respecting grading. We can assume $\pi(1) = 1$. We use it to get $f = \pi(f) = \pi(h_1) f_1 + \cdots + \pi(h_n) f_n$. Each $\pi(h_i)$ is now invariant and of strictly lower degree than f, hence is in R. But then $f \in R$, which is a contradiction. It follows that $k[V]^G$ is generated by $f_1, \dots, f_n$, and so the action is coregular. ■

Let $U \subseteq V^G$ be a linear subspace, and $U^\perp \subset V^* = k[V]_1$ the space of linear forms vanishing on U. Let $I(U)$ be the ideal in $k[V]^G$ generated by $U^\perp$. We shall define $\mathfrak{H}_U$, the *Hilbert ideal relative to* U, to be $I(U)^{ce}$, *i.e.,* $\mathfrak{H}_U$ is the ideal of $k[V]$ generated by all the invariants contained in $I(U)$. In particular, for $U = \{0\}$ we get the original Hilbert ideal $\mathfrak{H}$. Let s be the codimension of U in V. Then we say that $\mathfrak{H}_U$ is a *complete intersection ideal* if it can be generated by s homogeneous invariants.

Lemma 2.2 *Let $\mathfrak{H}_U$ be the Hilbert ideal relative to $U \subset V^G$. If $\mathfrak{H}_U$ is a complete intersection ideal then the Hilbert ideal $\mathfrak{H}$ is also a complete intersection ideal.*

Proof We shall use that the quotient algebra $k[V]^G/I(U)^c$ is a polynomial ring, a result due to Nakajima [7, Proof of Lemma 2.11]. We recall the quick proof.

To prove this result we can suppose that k is algebraically closed so that we can use the language of algebraic geometry. Let $\pi_G\colon V \to V/G$ be the quotient map. The linear algebraic group U acts on V by translations:

$$U \times V \to V\colon (u, v) \mapsto u + v.$$

Since $U \subseteq V^G$, the translations commute with the G-action on V, hence the U-action on V descends to an action on the quotient variety

$$U \times V/G \to V/G\colon (u, \pi_G(v)) \mapsto \pi_G(u + v).$$

It acts simply transitively on itself and on its image $\pi_G(U)$ in V/G. So $\pi_G(U)$ is isomorphic to $U \simeq k^{n-s}$, hence the coordinate ring of $\pi_G(U)$ is isomorphic to a polynomial ring with $n - s$ variables. The coordinate ring of V/G can be identified with $k[V]^G$ and then $\pi_G(U)$ is defined by $I(U)^c$. It follows that $k[V]^G/I(U)^c$ is a polynomial ring in $n - s$ variables. This finishes the proof of Nakajima's result.

So we can find $n - s$ homogeneous invariants $f_{s+1}, f_{s+2}, \dots, f_n$ such that

$$I(U)^c + (f_{s+1}, f_{s+2}, \dots, f_n)k[V]^G = k[V]^G_+,$$

the maximal homogeneous ideal of $k[V]^G$. So

$$\mathfrak{H} = (k[V]^G_+)^e = I(U)^{ce} + (f_{s+1}, f_{s+2}, \dots, f_n)k[V] = \mathfrak{H}_U + (f_{s+1}, f_{s+2}, \dots, f_n)k[V].$$

Now if $\mathfrak{H}_U$ is a complete intersection ideal, hence generated by s elements, it follows that $\mathfrak{H}$ is generated by n elements and is also a complete intersection ideal. ∎

3 Abelian Transvection Groups

For any pseudo-reflection ρ on V there is a vector $e_\rho \in V$ such that $(\rho - 1)(V) = ke_\rho$ and a functional $x_\rho \in V^*$ such that $\rho(v) - v = x_\rho(v)e_\rho$. Then $v \in V^\rho$ if and only if $x_\rho(v) = 0$, or x_ρ is a linear form defining the fixed-points set V^ρ. There also is a unique linear map $\Delta_\rho\colon k[V] \to k[V]$ such that for $f \in k[V]$

$$\rho(f) - f = \Delta_\rho(f)x_\rho.$$

The pseudo-reflection is called a *transvection* if $\rho(e_\rho) = e_\rho$, *i.e.*, $e_\rho \in V^\rho$, or equivalently if $\Delta_\rho(x_\rho) = 0$. The fixed-points set V^ρ is then called a transvection hyperplane. Otherwise the pseudo-reflection is diagonalisable over k, and called *homology*, *i.e.*, there is a basis of V consisting of eigenvectors. A *transvection group* is a group generated by transvections.

Proposition 3.1 *Let G be a finite abelian transvection group acting on V.*

(i) *$\mathfrak{H}_{V^G}$ is a complete intersection ideal, where $\mathfrak{H}_{V^G}$ is the Hilbert ideal relative to V^G.*
(ii) *G is an abelian p-group, where p is the characteristic of the field.*

Proof (i) Let r_1 and r_2 be two transvections in G, whose fixed-point sets are defined by the two linear forms x_1 and x_2. Then for any $f \in k[V]$ there is a unique $\Delta_1(f)$ and $\Delta_2(f)$ such that $r_i(f) = f + \Delta_i(f)x_i$, for $i = 1, 2$. Since the r_i are transvections, we have $\Delta_i(x_i) = 0$. For any linear form y we have that $\Delta_i(y)$ is a scalar and

$$r_1(r_2(y)) = r_1(y + \Delta_2(y)x_2) = y + \Delta_1(y)x_1 + \Delta_2(y)x_2 + \Delta_2(y)\Delta_1(x_2)x_1,$$
$$r_2(r_1(y)) = r_2(y + \Delta_1(y)x_1) = y + \Delta_2(y)x_2 + \Delta_1(y)x_1 + \Delta_1(y)\Delta_2(x_1)x_2.$$

Since G is abelian we get for all $y \in V^*$ that $\Delta_2(y)\Delta_1(x_2)x_1 = \Delta_1(y)\Delta_2(x_1)x_2$.

If x_1 and x_2 are dependent then $\Delta_i(x_j) = 0$. Supposing they are independent, we get $\Delta_2(y)\Delta_1(x_2) = 0$ for all linear forms y, hence $\Delta_1(x_2) = 0$. Similarly $\Delta_2(x_1) = 0$. Therefore we get $r_i(x_j) = x_j$. Since our group is an abelian transvection group, it follows that any linear form defining a transvection hyperplane is a G-invariant linear form.

Let $\mathfrak{T} \subset G$ be the collection of transvections in G. For any $\tau \in \mathfrak{T}$ fix x_τ as above. Since the transvections generate G we get

$$(V^G)^\perp = \Big(\bigcap_{\tau\in\mathfrak{T}} V^\tau\Big)^\perp = \sum_{\tau\in\mathfrak{T}} (V^\tau)^\perp = \sum_{\tau\in\mathfrak{T}} \langle x_\tau\rangle = \langle x_\tau\,;\, \tau \in \mathfrak{T}\rangle.$$

Since we just proved that each $x_\tau \in (V^*)^G \subseteq k[V]^G$, it follows that $(V^G)^\perp$ is generated by linear invariants, say $x_1, \dots, x_{n-s}$, and so $\mathfrak{H}_{V^G}$ is a complete intersection ideal, since

$$I(V^G) = (x_1, \dots, x_{n-s}) = I(V^G)^{ce} = \mathfrak{H}_{V^G}.$$

(ii) Suppose G is not a p-group. Then (by extending the field if necessary) there exists a $\sigma \in G$ and a linear form $y \in V^*$ such that $\sigma(y) = cy$, where $c \neq 1$. Since G is generated by transvections, there must be a transvection $\tau \in G$, with corresponding x_τ and Δ_τ, such that $\tau(y) \neq y$, or $\Delta_\tau(y) \neq 0$. Then

$$\sigma\tau(y) = \sigma(y + \Delta_\tau(y)x_\tau) = cy + \Delta_\tau(y)\sigma(x_\tau)$$
$$\tau\sigma(y) = \tau(cy) = cy + \Delta_\tau(y)cx_\tau.$$

Comparing, we get $\sigma(x_\tau) = cx_\tau$ and so $x_\tau \notin (V^*)^G$, which contradicts (i). So G is a p-group. ∎

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4 Reduction to Abelian Transvection Groups and Diagonalisable Pseudo-Reflection Groups

The following proposition allows us to treat separately abelian transvection groups and diagonalisable pseudo-reflection groups. The first two parts were known to Nakajima [6, Proof of Proposition 2.1].

Proposition 4.1 *Let $G < \mathrm{GL}(V)$ be an abelian pseudo-reflection group G acting on V. Denote T for the subgroup of G generated by the transvections and D for the subgroup generated by the homologies in G.*

(i) *Then D is a non-modular group, T is a p-group and $G = T \times D$.*

(ii) *There is a direct sum decomposition of kG-modules $V = V^D \oplus V_D$, where D acts trivially on V^D and T acts trivially on V_D. For the invariant rings we get*

$$k[V]^G \simeq k[V^D]^T \otimes k[V_D]^D.$$

Consequently, the G-action on V is coregular if and only if the T-action on V^D (or on V) and the D-action on V_D (or on V) are coregular.

(iii) *The direct summand property holds for the G-action on V if and only if the direct summand property holds for the T-action on V^D (or V).*

Proof (i) Since every generator of D is diagonalisable over k and D is abelian, the group D is simultaneously diagonalisable; in particular it is non-modular. Since T is an abelian transvection group, it is a p-group by Lemma 3.1. So $T \cap D = \{1\}$ and $G = T \times D$.

(ii) Let V^D be the space of invariants and V_D the direct sum of the remaining eigenspaces of D, so at least $V = V^D \oplus V_D$ as kG-modules.

If $\tau \in T$, then by commutativity also $\tau(v) \in V^D$, so V^D is a kG-submodule.

Let τ be transvection with corresponding $e_\tau \in V$ and $x_\tau \in V^*$ such that $\tau(v) - v = \delta(v)e_\tau$, for any $v \in V$. Let σ be a homology and $\sigma v = cv$, where v is the eigenvector for σ with eigenvalue $c \neq 1$. Then $\tau\sigma v = \tau c v = cv + x_\tau(v)ce_\tau$ and $\sigma\tau v = \sigma(v + x_\tau(v)e_\tau) = cv + x_\tau(v)\sigma(e_\tau)$. Commutativity implies $x_\tau(v)(\sigma(e_\tau) - ce_\tau) = 0$. If $x_\tau(v) \neq 0$, it follows that e_τ is an eigenvector for σ with eigenvalue c. So v is a scalar multiple of e_τ (since σ is a homology, the eigenspace with eigenvalue $c \neq 1$ is one-dimensional). But since $e_\tau \in V^\tau$ (since τ is a transvection) it follows that $\tau(v) = v$ and so $x_v(v) = 0$, which is a contradiction. So necessarily $x_\tau(v) = 0$ and $\tau(v) = v$.

Since the eigenvectors of homologies with non-identity eigenvalue span V_D (since those homologies generate D), it follows that T acts trivially on V_D. In particular V_D is also a kG-submodule and $V = V^D \oplus V_D$ is a decomposition as kG-modules.

Let $y_1, \dots, y_m$ be a basis of linear forms vanishing on V_D, and $z_1, \dots, z_{n-m}$ a basis of linear forms vanishing on V^D. So $y_1, \dots, y_m$ are coordinate functions on V^D, $z_1, \dots, z_{n-m}$ are coordinate functions on V_D, and

$$\begin{aligned} k[V] &= k[y_1 \dots, y_m, z_1, \dots, z_{n-m}] = k[y_1, \dots, y_n] \otimes k[z_1, \dots, z_{n-m}] \\ &= k[V_D] \otimes k[V^D]. \end{aligned}$$

For the invariants we get $k[V]^G \simeq k[V^D]^T \otimes k[V_D]^D$.

(iii) The different of the G-action θ_G is a product of linear forms x_α, where the zero-set of x_α, say $V_\alpha := \{v \in V; x_\alpha(v) = 0\}$, is the fixed-point set of a pseudo-reflection [2, Proposition 9]. The same holds for θ_T and θ_D. If τ is a transvection, then $V^\tau \supset V_D$; if τ is diagonalisable, then $V^\tau \supset V^D$. It follows that $\theta_T \in k[y_1, \ldots, y_m] = k[V^D]$ and $\theta_D \in k[z_1, \ldots, z_{n-m}] \in k[V_D]$ and $\theta_G = \theta_T \cdot \theta_D$. In particular T acts trivially on θ_D and D acts trivially on θ_T.

Suppose the direct summand property holds for the G-action, *i.e,* there exists a $\tilde{\theta}_G \in k[V]$ such that $\mathrm{Tr}^G(\frac{\tilde{\theta}_G}{\theta_G}) = 1$. Put $\hat{\theta}_T := \mathrm{Tr}^D(\frac{\tilde{\theta}_G}{\theta_D})$, then

$$\mathrm{Tr}^T\Big(\frac{\hat{\theta}_T}{\theta_T}\Big) = \mathrm{Tr}^T\Big(\frac{1}{\theta_T}\mathrm{Tr}^D\Big(\frac{\tilde{\theta}_G}{\theta_D}\Big)\Big) = \mathrm{Tr}^T\Big(\mathrm{Tr}^D\Big(\frac{\tilde{\theta}_G}{\theta_D}\Big)\Big) = 1,$$

since $\mathrm{Tr}^G = \mathrm{Tr}^T \circ \mathrm{Tr}^D$ and θ_T is D-invariant. So the direct summand property holds for the G-action V.

Suppose that $\hat{\theta}_T$ is not in $k[V^D] = k[y_1, \ldots, y_n]$. So we can write

$$\hat{\theta}_T = \tilde{\theta}_T + \sum_{i=1}^{n-m} z_i f_i,$$

where $\tilde{\theta}_T \in k[V^D]$ and $f_i \in k[V]$. Then

$$1 = \mathrm{Tr}^T\Big(\frac{\hat{\theta}_T}{\theta_T}\Big) = \mathrm{Tr}^T\Big(\frac{\tilde{\theta}_T}{\theta_T}\Big) + \sum_{i=1}^{n-m} z_i\,\mathrm{Tr}^T\Big(\frac{f_i}{\theta_T}\Big) = \mathrm{Tr}^T\Big(\frac{\tilde{\theta}_T}{\theta_T}\Big),$$

since $\mathrm{Tr}^T(f_i/\theta_T)$ is of negative degree, hence 0. It follows that the direct summand property also holds for the T-action on V^D.

Conversely, suppose the direct summand property holds for the T-action on V. Then by the foregoing argument the direct summand property also holds for the T-action on V^D. Hence there is a $\tilde{\theta}_T \in k[y_1, \ldots, y_m]$ such that $\mathrm{Tr}^T(\tilde{\theta}_T/\theta_T) = 1$. Put $\tilde{\theta}_G := |D|^{-1} \cdot \theta_D \cdot \tilde{\theta}_T$. This makes sense since D is non-modular. Then

$$\mathrm{Tr}^G\Big(\frac{\tilde{\theta}_G}{\theta_G}\Big) = \mathrm{Tr}^T \circ \mathrm{Tr}^D\Big(\frac{|D|^{-1} \cdot \theta_D \cdot \tilde{\theta}_T}{\theta_T \cdot \theta_D}\Big) = \mathrm{Tr}^T\Big(\frac{\tilde{\theta}_T}{\theta_T}\mathrm{Tr}^D(|D|^{-1})\Big) = 1$$

and so the direct summand property also holds for the G-action on V. ■

5 Proofs of Main Results

We now prove our main theorem and its corollary.

***Theorem* 1.1** *Suppose $G < \mathrm{GL}(V)$ is an abelian group acting on the finite-dimensional vector space V. Then the action is coregular if and only if G is a pseudo-reflection group and the direct summand property holds.*

Proof Even when G is not abelian, by a theorem of Serre it is generally true that if the action is coregular, then G acts as a pseudo-reflection group and the direct summand property holds [2].

Suppose that G is an abelian pseudo-reflection group and the direct summand property holds. Then $G = T \times D$, where T is the subgroup generated by transvections and D the subgroup generated by diagonalisable reflections, as in Proposition 4.1. We use the notation of that proposition. Since D is a non-modular pseudo-reflection group acting on V_D, it follows from the classical Chevalley–Shephard–Todd theorem that $k[V_D]^D$ is a polynomial ring. From Proposition 4.1 it also follows that T is an abelian transvection group acting on V^D and that this action has the direct summand property. From Proposition 3.1 and Lemma 2.2 it follows that the Hilbert ideal $\mathfrak{H}$ of this action is a complete intersection ideal. So by the criterion in Proposition 2.1 it follows that the T-action on V^D is coregular, and so $k[V^D]^T$ is a polynomial ring. So $k[V]^G = k[V^D]^T \otimes k[V_D]^D$ (see Proposition 4.1 again) is a polynomial ring. Hence the G-action is coregular. ■

We get a special case of Shank–Wehlau's conjecture [8].

***Corollary* 1.2** *Let $G < \mathrm{GL}(V)$ be an abelian p-group acting linearly on the vector space V. The image of the transfer map Tr^G is a principal ideal in $k[V]^G$ if and only if the action is coregular.*

Proof In [2] it was already shown for p-groups that the direct summand property holds if and only if the image of the transfer map Tr^G is a principal ideal in $k[V]^G$ and that this condition implies that G is a transvection group, and if G is abelian, then Theorem 1.1 implies that the action is even coregular. Conversely, if the action is coregular, then the direct summand property holds and the image of the transfer is a principal ideal. ■

Example 1 The simplest example of an abelian transvection group that satisfies neither the direct summand property nor the coregularity property is the following. Take $p = 2$, $k = \mathbb{F}_2$, $G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^3$, $V = \mathbb{F}_2^3$ and the action is defined by the three matrices

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \ \sigma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}; \ \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

In fact σ_1, σ_2, and σ_3 are the only transvections in the group, with transvection hyperplanes defined by x_1, x_2, and $x_1 + x_2$, respectively. So the ideal I defining V^G is $I = (x_1, x_2)$ and the Dedekind different is $\theta = x_1 x_2 (x_1 + x_2)$. A minimal generating set of invariants is (see [5]) x_1, x_2 and

$$\begin{aligned} f_3 &:= x_1 x_3 (x_1 + x_3) + x_2 x_4 (x_2 + x_4); \\ N(x_3) &= x_3 (x_3 + x_1)(x_3 + x_2)(x_3 + x_1 + x_2); \\ N(x_4) &= x_4 (x_4 + x_1)(x_4 + x_2)(x_4 + x_1 + x_2). \end{aligned}$$

There is one generating relation among the generators.

The Hilbert ideals are complete intersection ideal $\mathfrak{H} = (x_1, x_2, N(x_3), N(x_4))$, and $\mathfrak{H}_{V^G} = (x_1, x_2)$. But the direct summand property does not hold, since if it would hold we would have for $J = (x_1, x_2)k[V]^G$ that $J = J^{ec}$, but $J^{ec} = (x_1, x_2, f_3)k[V]^G$. Or more directly, a calculation shows that if $f \in k[V]$ is of degree 3, then $\mathrm{Tr}^G(f/\theta_G) = 0$.

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