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ABELIAN VARIETIES ATTACHED TO CYCLES OF INTERMEDIATE DIMENSION

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0. Introduction

The group of cycles of codimension one algebraically equivalent to zero of a nonsingular projective variety modulo rational equivalence forms an abelian variety, i.e., the Picard variety. To the group of cycles of dimension zero and of degree zero, there corresponds an abelian variety, the Albanese variety. Similarly, Weil, Lieberman and Griffiths have attached complex tori to the cycles of intermediate dimension in the classical case. The aim of this article is to give a purely algebraic construction of such "intermediate Jacobian varieties."

We denote the group of cycles of codimension p of a nonsingular projective variety X modulo rational equivalence by $CH^p(X)$, the subgroup of $CH^p(X)$ consisting of cycles algebraically equivalent to zero by $A^p(X)$. Then $CH(X) = \oplus_{p>0} CH^p(X)$ has a ring structure, the Chow ring of X [19]. Lieberman has introduced an "axiom of intermediate Jacobian" [13]: For each nonsingular projective variety X and for each integer p ($1 \leq p \leq \dim X$), there exist

(i) a subgroup $K^p(X)$ of $A^p(X)$ and
(ii) an abelian variety $J^p(X)$.

These should satisfy the following conditions:

(iii) $J^p(X)$ is the Albanese variety of X if $p = \dim X$.
(iv) There is an isomorphism $A^p(X)/K^p(X) \rightleftarrows J^p(X)$ of groups.
(v) Functoriality: for arbitrary varieties X and Y, and an element z of $CH^{p+q}(X \times Y)$, there exists an abelian variety homomorphism

$$[z]: J^{m-q}(X) \longrightarrow J^p(Y)$$

($m = \dim X$) such that the diagram

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$$\begin{array}{ccc} A^{m-q}(X) & \xrightarrow{z(?)} & A^{p}(Y) \\ \downarrow & & \downarrow \\ J^{m-q}(X) & \xrightarrow{[z]} & J^{p}(Y) \end{array}$$

is commutative, where the vertical maps are defined via the isomorphism (iv) and the upper horizontal map is given by $x \mapsto \pi_Y(x \times Y \cdot z)$, $\pi_Y : X \times Y \to Y$ being the projection. Then $J^p(X)$ is said to be a theory of intermediate Jacobian. It follows from the axiom that $J^1(X)$ is the Picard variety of X. Using the Griffiths (or Weil) intermediate Jacobian, he has shown that over the field of complex numbers, there exists a theory of intermediate Jabocian $J_a^p(X)$. The Picard variety and the Albanese variety are dual; it is therefore natural to ask whether $J^p(X)$ and $J^{m-p+1}(X)$ ($m = \dim X$) are dual or not. He has shown using $J_a^p(X)$, by an argument based on an idea of Grothendieck, that there exists a theory of intermediate Jacobian for which the duality holds up to isogeny; he has also shown that, over an arbitrary fields, one can construct an isogeny class of abelian varieties which in the classical case, contains the latter intermediate Jacobian [10, 13].

In **1**, we give the definition of regular homomorphisms and that of a class of regular homomorphisms in which we work. In **2**, we consider some equivalent conditions one of which says the existence of an abelian variety and a regular homomorphism with some universal property. In **3**, we give the definition of Picard homomorphisms essentially due to Kleiman. For the class of Picard homomorphisms, a condition in **2** is satisfied and there exists a theory of intermediate Jacobian $\mathrm{Pic}^p X$. As a consequence, the group of cycles algebraically equivalent to zero modulo incidence equivalence has a structure of abelian variety $\mathrm{Pic}^p X$. In **4**, we show that any abelian variety homomorphism $\mathrm{Pic}^q X \to \mathrm{Pic}^p Y$ is essentially induced by a cycle via (v) above. In **5**, we give a remark about the relation between $J_a^p(X)$ and $\mathrm{Pic}^p X$. Our remark is incomplete because it relies upon an announcement of Griffiths whose proof does not seem to appear. In **6**, we give an example that for an abelian variety A over an uncountable field of characteristic zero, the kernel of the map $A^p(A) \to \mathrm{Pic}^p(A)$ is not finite if $1 < p \leq \dim A$, partially generalizing (and using) a result of Bloch [4].

We suppose that the ground field k is algebraically closed and of arbitrary characteristic unless the contrary is explicitly stated. We understand by a variety a nonsingular projective variety defined over k. By abuse of language, we will often refer to elements of $CH(X)$ as cycles. For cycles $u \in CH(X \times Y)$ and $v \in CH(Y \times Z)$, we define

$$v \circ u = \pi_{X \times Y}(\pi^*_{X \times Y}(u) \cdot \pi^*_{Y \times Z}(v)) ,$$

where $\pi_{Y \times Z}$, $\pi_{X \times Y}$, and $\pi_{X \times Z}$ are projections from $X \times Y \times Z$. If

$$u \in CH^p(X \times Y) , \quad v \in CH^q(Y \times Z) \quad \text{and} \quad \dim Y = n ,$$

then $v \circ u \in CH^{p+q-n}(X \times Z)$. For $u \in CH(X \times Y)$, $x \in CH(X)$, we put $u(x) = \pi_Y(x \times Y \cdot u)$. If $x \in CH^{m-q}(X)$ and $u \in CH^{p+q}(X \times Y)$ then $u(x) \in CH^p(Y)$ ($m = \dim X$). For $u \in CH(X \times Y)$, $v \in CH(Y \times Z)$, $x \in CH(X)$, we have $(v \circ u)(x) = v(u(x))$. For morphisms $f: X' \to X$ and $g: Y' \to Y$ and for a cycle $z \in CH(X \times Y)$,

$$(f \times g)^*(z) = {}^t\Gamma_g \circ z \circ \Gamma_f \qquad \text{in } CH(X' \times Y')$$

where $\Gamma_f \in CH(X' \times X)$ and $\Gamma_g \in CH(Y' \times Y)$ are graphs of f and g respectively. For $x \in CH(X)$, ${}^t\Gamma_f(x) = f^*(x)$, and for $x' \in CH(X')$, $\Gamma_f(x') = f_*(x')$ [12]. We will often write $CH_q(X) = CH^{m-q}(X)$ and $A_q(X) = A^{m-q}(X)$ and so on if $m = \dim X$. When we talk about points of a variety, we consider only closed points. If p is a point of a variety, we denote the point regarded a cycle by (p).

1.

DEFINITION (1.1). A group homomorphism

$$h: A^p(X) \to A$$

from the subgroup $A^p(X)$ of the Chow group $CH^p(X)$ into an abelian variety A is said to be *regular* if the following condition is satisfied: for any variety T, for any $z \in CH^p(T \times X)$ and for any $t_0 \in T$, the map

$$F(h, z; t_0): T \to A$$

defined by $t \mapsto h(z((t) - (t_0)))$ is a morphism of varieties.

Remarks (1.1.1). In the above definition, since

$$F(h, z; t_0) = F(h, z; t_1) - F(h, z; t_1)(t_0)$$

for $t_1 \in T$, the condition that $F(h, z, t_0)$ be a morphism is independent of $t_0 \in T$. We say a map of the form

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$$T \ni t \mapsto z((t)) \in CH^p(X)\,, \qquad \text{where } z \in CH^p(T \times X)\,,$$

an algebraic family of cycles. Since $z((t)-(t_0)) = (z - T \times z((t_0)))(t)$, we can say that $h\colon A^p(X) \to A$ is regular if for any algebraic family $f\colon T \to A^p(X)$, the composite

$$T \xrightarrow{f} A^p(X) \xrightarrow{h} A$$

is a morphism.

(1.1.2) Suppose that h is regular, that T is an abelian variety and that t_0 is the origin of T. Then $F(h, z, t_0)$ is an abelian variety homomorphism since it is a morphism and sends the origin of T to that of A.

(1.1.3) Let $p = 1$, let P be the Picard variety of a variety X, and $\mathfrak{P} \in CH^1(P \times X)$ be a Poincaré divisor. The map $\varphi\colon P \to A^1(X)$ such that $t \mapsto \mathfrak{P}((t)-(0))$ is a group isomorphism. Let $\theta^{(1)}\colon A^1(X) \to P$ be its inverse. Then $\theta^{(1)}$ is a regular homomorphism because of the universality of φ for algebraic family [20], and hence $\theta^{(1)}$ has the universal property: for any abelian variety A, the map

$$\begin{array}{ccc} \mathrm{Hom}\,(P, A) & \longrightarrow & \{\text{The regular homomorphisms } A^1(X) \longrightarrow A\} \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ f & \longmapsto & f \circ \theta^{(1)} \end{array}$$

is bijective, the inverse being given by $h \mapsto h \circ \varphi$.

(1.1.4) Let $m = \dim X = p$, let $\mathrm{Alb}\,(X)$ be the Albanese variety of X and $c\colon X \to \mathrm{Alb}\,(X)$ the canonical map. Then c induces a regular homomorphism $\theta^{(m)}\colon A^m(X) \to \mathrm{Alb}\,(x)$ which has the universal property: for any abelian variety A, the map

$$\begin{array}{ccc} \mathrm{Hom}\,(\mathrm{Alb}\,(X), A) & \longrightarrow & \{\text{the regular homomorphisms } A^m(X) \longrightarrow A\} \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ f & \longmapsto & f \circ \theta^{(m)} \end{array}$$

is bijective (cf. [20]). Note that $\theta^{(m)}$ is not necessarily bijective while $\theta^{(1)}$ is so. For $m = 2$, $k = C$, it is known that if $p_g > 0$, then $\theta^{(2)}$ is not isomorphic and the kernel is "big" [16], and that if $p_g = 0$ and X is not of general type, then $\theta^{(2)}$ is an isomorphism [5].

PROPOSITION (1.2). *Let $h\colon A^p(X) \to A$ be a regular homomorphism. Then*:

(i) *the image of h is an abelian subvariety of A; and*

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(ii) *there exist an abelian variety* B *and a cycle* $z \in CH^p(B \times X)$ *such that the homomorphism* $F(h, z, 0): B \to A$ *has a finite kernel and that the image of* $F(h, z, 0)$ *is that of* h.

Proof. Consider the pair of an abelian variety B and a cycle $z \in CH^p(B \times X)$ such that the kernel of $F(h, z, 0)$ is finite; and take a pair (B, z) with the dimension of the image of $F(h, z, 0)$ maximum. Since the kernel of $F(h, z, 0)$ is zero and hence finite if $B = 0$ and $z = 0$, and since the dimension of the image does not exceed that of A, such a pair exists. As $\operatorname{Im}(F(h, z, 0)) \subset \operatorname{Im}(h)$, it is sufficient to show that they are equal; then $\operatorname{Im} h$ is an abelian subvariety of A as the image of a morphism of abelian varieties sending zero to zero, and (B, z) has the required property for (ii). Assume $\operatorname{Im} F(h, z, 0) \subsetneqq \operatorname{Im} h$. Then there exists $x \in A^p(X)$ with $h(x) \notin \operatorname{Im} F(h, z, 0)$. Since $x \in A^p(X)$, there is an abelian variety J, $u \in CH^p(J \times X)$ and $p \in J$ with $u((p) - (0)) = x$[21]. Let $\pi_{B,X}: B \times J \times X \to B \times X$, $\pi_{J,X}: B \times J \times X \to J \times X$ be the projections, and put

$$z' = \pi_{B,X}^*(z) + \pi_{J,X}^*(u) \in CH^p(B \times J \times X)$$

Denote the inclusions $B = B \times 0 \hookrightarrow B \times J$ and $J = 0 \times J \hookrightarrow B \times J$ by i_1 and i_2 respectively, then we have

$$F(h, z, 0) = F(h, z', 0) \circ i_1 ,$$
$$F(h, u, 0) = F(h, z', 0) \circ i_2 ;$$

in particular

$$F(h, z', 0) \circ i_2(p) = h(x) .$$

Let N be the connected component of the origin of the kernel of $F(h, z, 0)$. By Poincaré's theorem of complete reducibility, there exists an abelian subvariety C of $B \times J$ with $B \times J = C + N$ and $C \cap N$ finite. Put

$$\varphi = F(h, (j \times id_X)^*(z'), 0) = F(h, z', 0) \circ j: C \to A$$

where $j: C \hookrightarrow B \times J$ is the inclusion map. The image of $F(h, z', 0)$ and that of φ coincide since $B \times J = C + N$ and $F(h, z', 0)(N) = 0$; hence

$$\operatorname{Im} \varphi = \operatorname{Im} F(h, z; 0) \supset \operatorname{Im} F(h, z, 0) ,$$

and

$$\dim \operatorname{Im} \varphi \geq \dim \operatorname{Im} F(h, z, 0) .$$

On the other hand, $\operatorname{Ker}\varphi = \operatorname{Ker} F(h, z', 0) \cap C$ is a finite set. The couple $(C, (j \times \mathrm{id}_X)_* z')$, therefore, satisfies the same condition as the pair (B, z). We have

$$h(x) = F(h, z'; 0) \circ i_2(p) \in \operatorname{Im} F(h, z'; 0) = \operatorname{Im}\varphi ,$$

while $\operatorname{Im} F(h, z, 0) \not\ni h(x)$. Since $\operatorname{Im} F(h, z, 0)$ and $\operatorname{Im}\varphi$ are irreducible as images of abelian varieties, $\operatorname{Im}\varphi \supsetneqq \operatorname{Im} F(h, z, 0)$ implies $\dim \operatorname{Im}\varphi > \dim \operatorname{Im} F(h, z, 0)$ and this contradicts the choice of (B, z).

(1.3) In the following, we denote by $\underline{\underline{A}}$ a class of regular homomorphisms satisfying the following conditions (I)–(V):

(I) For every variety X, each abelian variety A and each $p \geq 0$, the zero map $A^p(X) \to A$ belongs to $\underline{\underline{A}}$.

Fixing a variety X and a number $p \geq 0$, we put

$$K^p(X) = \bigcap \operatorname{Ker}(A^p(X) \to A)$$

where h runs over all regular homomorphisms belonging to $\underline{\underline{A}}$ from $A^p(X)$ to any abelian variety A.

(II) $K(X) = \oplus_p K^p(X)$ defines an adequate equivalence relation [18] over the set of cycles on X, i.e.,

a) The equivalence relation defined by $K(X)$ is compatible with the addition.

b) For a cycle z on X and a finite number of subvarieties Y_j (not necessarily nonsingular), there exists a cycle z' equivalent to z such that $z' \cdot Y_j$ are defined for all j.

c) Let X and Y be varieties, u a cycle on X equivalent to zero, z a cycle on $X \times Y$ such that $z(u) = \pi_Y(u \times X \cdot z)$ is defined. Then the cycle $z(u)$ on Y is equivalent to zero. We put

$$G^p(X) = A^p(X)/K^p(X), \quad \text{and} \quad G_q(X) = G^{m-q}(X) , \quad m = \dim X .$$

(III) $G_0(X)$ and $\operatorname{Alb}(X)$ are naturally isomorphic, i.e., the kernel of the canonical map $A(X) \to \operatorname{Alb}(X)$ (1.1.4) is $K_0(X)$.

(IV) If $h\colon A^p(X) \to A$ belongs to $\underline{\underline{A}}$ and if h is factorized as

$$h\colon A^p(X) \xrightarrow{k} B \hookrightarrow A ,$$

where B is an abelian subvariety of A, then k belongs to $\underline{\underline{A}}$.

(V) If $h\colon A^p(X) \to A$ and $k\colon A^p(X) \to B$ belong to $\underline{\underline{A}}$, then

$$(h, k): A^p(X) \to A \times B$$

also belongs to $\underline{\underline{A}}$.

EXAMPLE (1.3.1). Let $\underline{\underline{A}}$ be the class of all regular homomorphisms. Then $\underline{\underline{A}}$ satisfies the conditions (I)–(V).

(1.4) Fix a class $\underline{\underline{A}}$ as above. For $u \in CH^{p+q}(X \times Y)$ and $x \in A_q(X)$, we have $u(x) \in A^p(Y)$, hence group homomorphism

$$CH^{p+q}(X \times Y) \to \operatorname{Hom}(A_q(X), A^p(Y)) .$$

An element of $CH^{p+q}(X \times Y)$ is called (q, p)-trivial if, by the above map, it is sent to a map whose image is in $K^p(Y)$. The (q, p)-trivial elements form a subgroup of $CH^{p+q}(X \times Y)$ and we set

$$\operatorname{Cor}^{q,p}(X, Y) = CH^{p+q}(X \times Y)/\{\text{the } (q, p)\text{-trivial elements}\} .$$

The above homomorphism induces an injection

$$\operatorname{Cor}^{q,p}(X, Y) \to \operatorname{Hom}(G_q(X), G^p(Y)) . \tag{1.4.1}$$

(1.5) Let $f: X' \to X$ be a morphism, $z \in CH^{p+q}(X \times Y)$ be a (q, p)-trivial element. Since $(f \times id_Y)^*(z)(x') = z(f_*(x'))$ for $x' \in A_q(X')$, $(f \times id_Y)^*z$ is also (q, p)-trivial. The homomorphism $(f \times id_Y)^*: CH^{p+q}(X \times Y) \to CH^{p+q}(X' \times Y)$, therefore, induces a group homomorphism

$$f^*: \operatorname{Cor}^{q,p}(X, Y) \to \operatorname{Cor}^{q,p}(X', Y) ,$$

and

$$X \mapsto \operatorname{Cor}^{q,p}(X, Y)$$

defines a contravariant functor. Regarding $\operatorname{Hom}(G_q(X), G^p(Y))$ a functor in X, the map (1.4.1) is a natural transformation of functors.

(1.6) Let $h: A^p(Y) \to A$ belong to $\underline{\underline{A}}$. By definition of $G^p(Y)$, $h: A^p(Y) \to A$ is decomposed into $A^p(Y) \to G^p(Y) \to A$, where $A^p(Y) \to G^p(Y)$ is the projection. The uniquely determined map $G^p(Y) \to A$ is denoted by ${}^a h$. We put

$$\operatorname{Cor}^p(X, Y) = \operatorname{Cor}^{0,p}(X, Y) .$$

Using the map $\operatorname{Cor}^p(X, Y) \to \operatorname{Hom}(G_0(X), G^p(Y))$ of (1.4.1), the map ${}^a h: G^p(Y) \to A$ and (1.3, III) and noting that h is regular, we get a group homomorphism

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(1.6.1) $$\beta\colon \mathrm{Cor}^p(X, Y) \to \mathrm{Hom}(\mathrm{Alb}\, X, A)$$

where the second member is the set of abelian variety homomorphisms. The group homomorphism is a natural transformation of functors if we regard both of the members of (1.6.1) functors in X. We say that β is induced by h.

(1.7) Suppose X and T varieties, A an abelian variety and $z \in CH^p(A \times X)$. Let $f\colon \mathrm{Alb}\, T \to A$ be an abelian variety homomorphism and $i\colon T \to \mathrm{Alb}\, T$ be "a" canonical map. We will show that the map from $G_0(T)$ into $G^p(X)$ induced by $(f \circ i \times \mathrm{id}_X)^* z \in CH^p(T \times X)$ is independent of choice of the map i. Let $i'\colon T \to \mathrm{Alb}\, T$ be another canonical map. Then there exists $a \in \mathrm{Alb}\, T$ with $i' = T_a \circ i$, where T_a is the translation by a. For $t \in T$, we have

$$\{(f \circ i' \times \mathrm{id}_X)^* z\}(t) = z(f \circ i'(t)) = z(f \circ i(t) + f(a))$$

Hence, in order to verify that $(f \circ i \times \mathrm{id}_X)^* z$ and $(f \circ i' \times \mathrm{id}_X)^* z$ induce the same map $G_0(T) \to G^p(X)$, it suffices to show

$$z(x + f(a)) - z(f(a)) = z((x) - (0)) \qquad \text{in } G^p(X)$$

for any x in A, a fortiori to show $z((x + y) - (x) - (y) + (0)) = 0$ in $G^p(X)$ for any x and y in A. Note that $A \times A \to CH^p(X)$, $(x, y) \mapsto z((x + y) - (x) - (y) + (0))$ is an algebraic family of cycles. Therefore for any regular homomorphism $h\colon A^p(X) \to B$, the map $h'\colon A \times A \to B$, $(x, y) \mapsto h(z((x + y) - (x) - (y) + (0)))$ is a morphism with $h'(x, 0) = h'(0, y) = 0$. A morphism of abelian varieties with this property is a zero map, i.e. $h(z((x + y) - (x) - (y) + (0))) = 0$. This shows $z((x + y) - (x) - (y) + (0)) = 0$ in $G^p(X)$ for any x and y in A. Thus z induces a well-defined map $f \mapsto (f \circ i \times \mathrm{id}_X)^* z$,

$$\gamma\colon \mathrm{Hom}(\mathrm{Alb}\, T, A) \to \mathrm{Cor}^p(T, X)\,.$$

It is easy to see that γ is a group homomorphism and that when both members are viewed as functors in T, it is a natural transformation.

2.

(2.1) Fix a class $\underline{\underline{A}}$ as in (1.3). Fixing a variety X and a natural number $p > 0$ we consider the following conditions.

(A) There exist an abelian variety P and a cycle $\mathfrak{P} \in CH^p(P \times X)$ such that

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$$\varphi: P \to G^p(X)\,, \qquad \varphi(x) = \pi(\mathfrak{P}(x) - (0)))$$

is a group isomorphism, where $\pi: A^p(X) \to G^p(X)$ is the surjection.

(A′) There exist a variety Y and a cycle $\mathfrak{P} \in CH^p(Y \times X)$ such that the map $G_0(Y) \to G^p(X)$ induced by $\mathfrak{P}$ is surjective.

(B) There exist an abelian variety P, a cycle $\mathfrak{P} \in CH^p(P \times X)$ and a homomorphism $\theta: A^p(X) \to P$ in the class, $\underline{\underline{A}}$, such that $\beta \circ \gamma = \mathrm{id}$ and $\gamma \circ \beta = \mathrm{id}$, where

$$\beta: \mathrm{Cor}^p\,(T, X) \to \mathrm{Hom}\,(\mathrm{Alb}\,T, P) \text{ is induced by } \theta \text{ as in (1.6)}$$

and

$$\gamma: \mathrm{Hom}\,(\mathrm{Alb}\,T, P) \to \mathrm{Cor}^p\,(T, X) \text{ is induced by } \mathfrak{P} \text{ as in (1.7)}\,,$$

and these are regarded a natural transformation of functors (in T).

(B′) The same condition as (B) except that the conditions $\beta \circ \gamma = \mathrm{id}$ and $\gamma \circ \beta = \mathrm{id}$ are replaced by the conditions $\beta \circ \gamma = r\cdot\mathrm{id}$, $\gamma \circ \beta = r\cdot\mathrm{id}$ for some positive integer r.

(C) There exists a number $M \geq 0$ such that, for any homomorphism $h: A^p(X) \to A$ of the class $\underline{\underline{A}}$, it holds that $\dim \mathrm{Im}\,(h) \leq M$.

(D) There exist an abelian variety P and a homomorphism $\theta: A^p(X) \to P$ in the class $\underline{\underline{A}}$ with the following property: for any homomorphism $h: A^p(X) \to A$ in the class $\underline{\underline{A}}$, there exists one and only one abelian variety homomorphism $f: P \to A$ such that $h = f \circ \theta$.

THEOREM (2.2). *The condition* (B) *implies the condition* (*A*). *The converse is true if the characteristic of k is zero.*

(ii) *The conditions* (A′), (B′), (C) *and* (D) *are equivalent.*

(iii) *The condition* (A) *implies the conditions of* (ii).

Proof. We show the following implications

$$\begin{array}{ccccc} (A) & \Longrightarrow & (A') & \Longrightarrow & (C) \\ \Uparrow & & \Uparrow & & \Downarrow \\ (B) & & (D) & \Longleftarrow & (B') \end{array}$$

(A) $\Rightarrow$ (A′): For any abelian variety P, $P \ni x \mapsto (x) - (0) \in G_0(X)$ is an isomorphism.

(A′) $\Rightarrow$ (C): The map $G_0(Y) \xrightarrow{\mathfrak{P}(?)} G^p(X)$ is surjective. For any homomorphism $h: A^p(X) \to A$ in the class $\underline{\underline{A}}$, the map ${}^a h \circ \mathfrak{P}(?): G_0(Y) \to G^p(X) \xrightarrow{{}^a h} A$ is, if we identify $G_0(Y)$ with Alb Y naturally, an abelian variety

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homomorphism. Since $\operatorname{Im} h = \operatorname{Im} {}^a h = \operatorname{Im}({}^a h \circ \mathfrak{P}(?))$, we have $\dim \operatorname{Im} h = \dim \operatorname{Im} {}^a h \circ \mathfrak{P}(?) \leq \dim Y = : M$.

(C) $\Rightarrow$ (B′): Take a pair (P, θ) of an abelian variety P and a surjective homomorphism $\theta: A^p(X) \to P$ in the class $\underline{\underline{A}}$ such that for any surjective homomorphism $h: A^p(X) \to A$ in the class, $\dim A \leq \dim P$: the existence of such a pair (P, θ) is guaranteed by the condition (C). Consider the homomorphism

$$\beta: \operatorname{Cor}^p(T, X) \to \operatorname{Hom}(\operatorname{Alb} T, P)$$

induced by θ. We will show that this homomorphism is injective for any variety T. Suppose that for some variety T, there exists $u \in \operatorname{Cor}^p(T, X)$ with $u \neq 0$ and $\beta(u) = 0$. Since $u \neq 0$, the map $u(?): G_0(T) \to G^p(X)$ is not zero, hence there is $t \in G_0(T)$ with $u(t) \neq 0$ in $G^p(X)$. This implies the existence of a homomorphism $k: A^p(X) \to A$ in the class $\underline{\underline{A}}$ such that ${}^a k(u(t)) \neq 0$. The map $(\theta, k): A^p(X) \to P \times A$ belongs to $\underline{\underline{A}}$; and denoting $B = \operatorname{Im}(\theta, k)$, we see that (θ, k) is factorized into $(\theta, k): A^p(X) \to B \subset P \times A$, where the map $h: A^p(X) \to B$ is in the class $\underline{\underline{A}}$. Let $\pi: B \subset P \times A \to A$ be the projection and we get $\pi \circ h = k$, therefore ${}^a h(u(t)) \neq 0$. On the other hand if we denote the projection $B \subset P \times A \to P$ by p, we have $\theta = p \circ h$. Since θ is surjective, so is $p: B \to P$. Consider $f: \operatorname{Alb} T \to B$ induced by ${}^a h \circ u(?): G_0(T) \to B$ and $g: \operatorname{Alb} T \to P$ induced by ${}^a \theta \circ u(?): G_0(T) \to P$. We have $p \circ f = g$. The assumption $\beta(u) = 0$ implies $g = 0$ and we obtain $\operatorname{Im}(f) \subset \operatorname{Ker}(p)$. Since h is surjective, by the choice of P and by the fact that P is surjective, p is an isogeny; in particular $\operatorname{Ker}(p)$ is finite and so is $\operatorname{Im}(f)$. On the other hand, $\operatorname{Im}(f)$ is irreducible as the image of an abelian variety, hence $\operatorname{Im}(f) = 0$. If $t' \in \operatorname{Alb} T$ is a point corresponding to $t \in G_0(T)$ we have $f(t') = {}^a h(u(t)) \neq 0$, the required contradiction; hence β is injective.

LEMMA (2.3). *For any surjective regular homomorphism $h: A^p(X) \to A$ consider the map $\beta: \operatorname{Cor}^p(T, X) \to \operatorname{Hom}(\operatorname{Alb} T, A)$ induced by h. If $T = A$ the image of β contains an element of the form $r \cdot \mathrm{id}_A$, where r is a positive integer.*

By (1.2), for the surjective regular map $h: A^p(X) \to A$, there exist an abelian variety A' and a cycle $z \in CH^p(A', X)$ such that $F(h, z, 0) = \varphi: A' \to A$ is an isogeny. Then, there is an isogeny $\psi: A \to A'$ such that $\varphi \circ \psi = r \cdot \mathrm{id}_A$, r being a positive integer. Putting $u = (\psi \times \mathrm{id}_X)^* z \in CH^p(A \times X)$, we have

$$F(h, u, 0) = F(h, z, 0) \circ \psi = r \cdot \mathrm{id}\ ,$$

for $u((x)) = z(\psi_*((x)))$ for $x \in A$ (Cf. **0**). Therefore if the map

$$\gamma \colon \mathrm{Hom}\,(\mathrm{Alb}\, T, A) \to \mathrm{Cor}^p\,(T, X)$$

is induced by u (1.7), $F(h, u, 0) = r \cdot \mathrm{id}_A$ implies $\beta \circ \gamma(\mathrm{id}_A) = \beta(\gamma(\mathrm{id}_A)) = r \cdot \mathrm{id}_A$.

By the proof of the lemma, $\beta \circ \gamma(\mathrm{id}_P) = r \cdot \mathrm{id}_P$ for some integer r, where γ is induced by some $\mathfrak{P} \in CH^p(P \times X)$. Since $\beta \circ \gamma \colon \mathrm{Hom}\,(\mathrm{Alb}\, T, P) \to \mathrm{Hom}\,(\mathrm{Alb}\, T, P)$ is a natural transformation of functors in $T, \beta \circ \gamma(\mathrm{id}_P) = r \cdot \mathrm{id}_P$ implies $\beta \circ \gamma = r \cdot \mathrm{id}$. Then $\beta \circ \gamma \circ \beta = r \cdot \mathrm{id} \circ \beta = \beta(r \cdot \mathrm{id})$, and since β is injective, we get $\gamma \circ \beta = r \cdot \mathrm{id}$.

(B′) $\Rightarrow$ (D): Let $h \colon A^p(X) \to A$ be a surjective homomorphism in the class $\underline{\underline{A}}$ and $\pi \colon A \to P$ be an abelian variety homomorphism with $\pi \circ h = \theta$. We shall show that π is an isogeny and that $\deg \pi \leq \deg\,(r \cdot \mathrm{id}_P)$. If

$$\beta' \colon \mathrm{Cor}^p\,(T, X) \to \mathrm{Hom}\,(\mathrm{Alb}\, T, A)$$

is induced by h(1.6), we obtain $\pi_* \circ \beta' = \beta$ where

$$\pi_* = \mathrm{Hom}\,(\mathrm{id}, \pi) \colon \mathrm{Hom}\,(\mathrm{Alb}\, T, A) \to \mathrm{Hom}\,(\mathrm{Alb}\, T, P)\ .$$

For $T = A$, by (2.3), there exists $u \in \mathrm{Cor}^p\,(A, X)$ such that $\beta'(u) = s \cdot \mathrm{id}_A$, s an integer. Then $\beta' \circ \gamma \circ \pi_*(s \cdot \mathrm{id}_A) = \beta' \circ \gamma \circ \pi_* \circ \beta'(u) = \beta' \circ \gamma \circ \beta(u) = rs \cdot \mathrm{id}_A$. The natural transformation $\beta' \circ \gamma \colon \mathrm{Hom}\,(\mathrm{Alb}\, T, P) \to \mathrm{Hom}\,(\mathrm{Alb}\, T, A)$ is determined by $\rho = \beta' \circ \gamma(\mathrm{id}_P) \colon P \to A$, and the equality above means $\rho_* \circ \pi_*(s \cdot \mathrm{id}_A) = r(s \cdot \mathrm{id}_A)$, hence $r \circ \rho = r \cdot \mathrm{id}_A$. The equality $\beta \circ \gamma = r \cdot \mathrm{id}$ implies $F(\theta, \mathfrak{P}, 0) = r \cdot \mathrm{id}_P$, hence θ is surjective. π is also surjective since $\pi \circ h = \theta$. By $\rho \circ \pi = r \cdot \mathrm{id}_A$ we see that $\mathrm{Ker}\,(\pi)$ is finite and hence π is an isogeny; therefore

$$\deg\,(r \cdot \mathrm{id}_P) = \deg\,(\rho \circ \pi) = \deg \rho \cdot \deg \pi \geq \deg \pi\ .$$

Take a pair (h, π) of a surjective homomorphism $h \colon A^p(X) \to A$ and $\pi \colon A \to P$ with $\pi \circ h = \theta$ such that $\deg \pi (\leq \deg r \cdot \mathrm{id}_P)$ is maximum, and call it (θ^*, π), $\pi \colon P^* \to P$. Then we show that (P^*, θ^*) satisfies the condition (D). Suppose given a homomorphism $h \colon A^p(X) \to A$ of the class $\underline{\underline{A}}$. By (1.3, V), $(\theta^*, h) \colon A^p(X) \to P^* \times A$ belongs to $\underline{\underline{A}}$ and putting $B = \mathrm{Im}\,(\theta^*, h)$, we obtain a factorization

$$(\theta^*, h) \colon A^p(X) \xrightarrow{\ k\ } B \hookrightarrow P^* \times A\ ,$$

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where k belongs to $\underline{\underline{A}}$ by (1.3. IV). We set $\pi'\colon B \hookrightarrow P^* \times A \longrightarrow P^*$; then $\pi' \circ k = \theta^*$ and $\pi \circ \pi' \circ k = \pi \circ \theta^* = \theta$. Since k is surjective, $\pi \circ \pi'$ is an isogeny and

$$\deg(\pi \circ \pi') = \deg \pi \cdot \deg \pi' \geq \deg \pi .$$

The maximality of $\deg \pi$ implies $\deg \pi' = 1$ and π' is an isomorphism. Put

$$f\colon P^* \xrightarrow{\pi'^{-1}} B \subset P^* \times A \longrightarrow A ,$$

then $f \cdot \theta^* = h^* = h$. Since θ^* is surjective, such an f is unique.

(D) $\Rightarrow$ (A′): The implication will be clear if we note that $G_0(A) \simeq A$ for an abelian variety A and the Lemma (2.3) and if we show the following lemma.

LEMMA (2.4). *Let* (P, θ) *be as in the condition* (D). *The homomorphism* ${}^a\theta\colon G^p(X) \to P$ *induced by* θ *is an isomorphism.*

We will show that θ is surjective. The image Q of θ is an abelian subvariety of P and if we define θ' by $\theta\colon A^p(X) \xrightarrow{\theta'} Q \overset{i}{\hookrightarrow} P$, θ' belongs to $\underline{\underline{A}}$. Therefore there exist uniquely a homomorphism $\varphi\colon P \to Q$ with $\theta' = \varphi \circ \theta = \varphi \circ \theta = \varphi \circ i \circ \theta'$. Since θ' is surjective, $\varphi \circ i = \mathrm{id}_Q$ and hence $P = Q \times N$, where $N = \ker \varphi$; therefore N is irreducible (and reduced). Then θ can be identified with $(\theta', 0)\colon A^p(X) \to Q \times N$, and $\theta = (\mathrm{id}_Q \times \mathrm{id}_N) \circ \theta = (\mathrm{id}_Q \times 0_N) \circ \theta$. By the uniqueness, $\mathrm{id}_Q \times \mathrm{id}_N = \mathrm{id}_A \times 0_N$ and $\mathrm{id}_N = 0_N$, i.e., $N = 0$; θ is surjective, hence so is ${}^a\theta$. To see ${}^a\theta$ injective it is sufficient to show $K^p(X) = \mathrm{Ker}\,\theta$. By definition $\mathrm{Ker}\,\theta \supset K^p(X)$. We have only to show that $h(x) = 0$ for any $x \in \mathrm{Ker}\,\theta$ and any homomorphism $h\colon A^p(X) \to A$ in the class $\underline{\underline{A}}$. By the hypothesis, there exists a homomorphism $\varphi\colon P \to A$ such that $h = \varphi \circ \theta$; whence $h(x) = \varphi(\theta(x)) = \varphi(0) = 0$.

(B) $\Rightarrow$ (A): By $\beta \circ \gamma = \mathrm{id}$, $\beta(\gamma(\mathrm{id}_P)) = \mathrm{id}_P$. Clearly, (B) $\Rightarrow$ (B′) and (D) holds; and by (2.4), ${}^a\theta\colon G^p(X) \simeq P$. Therefore $\mathrm{id}_P = F(\theta, \mathfrak{P}, 0)\colon P \to G^p(X) \simeq P$, where $\mathfrak{P} \in CH^p(P \times X)$ is a representative of $\gamma(\mathrm{id}_P) \in \mathrm{Cor}^p(P, X)$ implies (A).

(A) $\Rightarrow$ (B) (char. $k = 0$): It is already verified that (A) implies (D). Hence by (2.4) there exists a homomorphism $\theta\colon A^p(X) \to P^*$ in the class $\underline{\underline{A}}$ with bijective ${}^a\theta\colon G^p(X) \to P^*$. Then $P \to G^p(X) \xrightarrow{{}^a\theta} P^*$ is a bijection, hence we may suppose $P = P^*$ because char. $k = 0$. The subfunctor $\mathrm{Cor}^p(T, X)$ of $\mathrm{Hom}(\mathrm{Alb}\,T, P)$ induced by via $\mathfrak{P}$ (1.6) is, since $\mathfrak{P}$ is mapped to id_P for $T = P$, $\mathrm{Hom}(\mathrm{Alb}\,T, P)$ itself.

Remark (2.5). Let $J^p(X)$ be a theory of intermediate Jacobian. We define a class $\underline{\underline{A}}$ of homomorphisms $h\colon A^p(X) \to A$ for various X and p as follows: a homomorphism $h\colon A^p(X) \to A$ belongs to $\underline{\underline{A}}$ if (and only if) the map h is factorized as $h\colon A^p(X) \xrightarrow{\pi} J^p(X) \xrightarrow{h'} A$, where π is the canonical map (**0**) and $h'\colon J^p(X) \to A$ is an abelian variety homomorphism. By **0**, (V), we see that such a homomorphism $h\colon A^P(X) \to A$ is regular. We verify easily that the class satisfies $\underline{\underline{A}}$ the condition of (1.3). By definition, for each variety X and each natural number p $(1 \leq p \leq \dim X)$, the condition (D) of (2.1) is satisfied, hence the equivalent conditions of the Theorem (2.2, (ii)) hold. Further has $\underline{\underline{A}}$ the following properties:

(2.5.1) For any homomorphism $h\colon A^p(Y) \to A$ in the class $\underline{\underline{A}}$ and for any cycle $z \in CH^{p+q}(X \times Y)$ the homomorphism $h(z(?))\colon A^{m-q}(X) \to A$ defined by $x \mapsto h(z(x))$ belongs to the class $\underline{\underline{A}}$.

(2.5.2) For any homomorphism $h\colon A^p(X) \to A$ in the class $\underline{\underline{A}}$ and for any abelian variety homomorphism $f\colon A \to B$, the composite $f \circ h\colon A^p(X) \to B$ is also in the class $\underline{\underline{A}}$.

Conversely given a class $\underline{\underline{A}}$ satisfying the conditions (I)–(V) of (1.3), suppose that the equivalent conditions of the Theorem (2.2, (ii)) and the condition (2.5.1) hold. We will denote the abelian variety P of (2.1, D) by $J^p(X)$, $\theta\colon A^p(X) \to P$ by $\theta^{(p)}$. The kernel $\theta^{(p)}$ of is equal to $K^p(X)$ by (2.4). Then $J^p(X)$ satisfies the conditions **0**, (I)–(V), and hence is a theory of intermediate Jacobian. It is easily verified that there is a one-to-one correspondence between the theories of intermediate Jacobians and the classes of regular homomorphisms satisfying the conditions (2.5.1 and 2). But the writer does not know whether a theory of intermediate Jacobian is unique or not.

Remark (2.6). Suppose the condition (2.1, A) holds. If we have a regular surjective homomorphism $h\colon A^p(X) \to A$ in the class $\underline{\underline{A}}$ and a homomorphism $\psi\colon A \to P$ with $\psi \circ h \circ \varphi = \mathrm{id}_P$, then necessarily ψ is an isomorphism.

PROPOSITION (2.7). *We denote a theory of intermediate Jacobian by* $J^p(X)$.

(i) *Let E be a vector bundle of rank $(r+1)$ over a variety X and $P = \boldsymbol{P}(E)$. Then*

$$J^p(P) = \bigoplus_{0 \leq i \leq p} J^{p-i}(X)\,.$$

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(ii) *Let $X \supset Y$ be a variety and its subvariety, X' be the blowing up of X with center Y. If Y is of codimension r in X, then*

$$J^p(X') = J^p(X) \oplus \bigoplus_{1 \le i \le r-1} J^{p-i}(Y) .$$

(i) let $\xi \in CH^1(P)$ be the class of $O_P(-1)$. Denote the projection of P onto X by g and put

$$u(x) = (g_*(x \cdot \xi^{r-i}))_{0 \le i \le r} \in \bigoplus_{0 \le i \le r} A^{p-i}(X) \qquad \text{for } x \in A^p(P) ,$$

$$v((a_i)) = \sum_{0 \le i \le r} g^*(a_i)\xi^i \in A^p(P) , \qquad \text{for } (a_i) \in \bigoplus_{0 \le i \le r} A^{p-i}(X) .$$

Then u and v are inverse each other [17] and they define isomorphisms between $J^p(P)$ and $\bigoplus_{0 \le i \le r} J^{p-i}(X)$ by **0**, (V).

(ii) Similar method as (i). Let $f: X' \to X$ be the canonical map, N be the normal bundle of Y in X, $Y' = f^{-1}(Y), f'$ the restriction of f to Y' onto Y, and let

$$0 \to F \to f'^*N \to O_{Y'}(+1) \to 0$$

be exact. Put $\Phi_i = c_i(F) \in CH^i(Y^i), \xi$ the class of $O_{Y'}(+1)$ in $CH^1(Y')$. The maps

$$u: A^p(X') \to A^p(X) \oplus \bigoplus_{1 \le i \le r-1} A^{p-i}(Y)$$

defined by $u(x') = f_*(x') + \sum_{1 \le i \le r-1} f'_*(\Phi_{r-1-i} \cdot i^*x')$, and

$$v: A^p(X) \oplus \bigoplus_{1 \le i \le r-1} A^{p-i}(Y) \to A^p(X')$$

defined by $v(x, (y_i)) = f^*(x) - i_*(\sum_{1 \le i \le r-1} \xi^{i-1} \cdot f'^*y_i)$, where $i: Y' \hookrightarrow X'$ is the closed immersion, are inverse to each other and define the required isomorphisms by **0**, (V) [2].

3.

DEFINITION (3.1). Let X be a variety of dimension m and A be an abelian variety. A group homomorphism $h: A^p(X) \to A$ is said to be a Picard homomorphism if there exist a variety Y and a cycle $z \in CH^{m-p+1}(X \times Y)$ such that A is an abelian subvariety of the Picard variety $J^1(Y)$ of Y and that the diagram

$$\begin{array}{ccc} A^p(X) & \xrightarrow{z(?)} & A^1(Y) \\ & \searrow^{h} & \downarrow \wr\, \theta^{(1)} \\ & A & \hookrightarrow J^1(Y) \end{array}$$

is commutative.

(3.2) A Picard homomorphism is regular. It is enough to show that, for any variety T and a cycle $u \in CH^p(T \times X)$, $t_0 \in T$, $\varphi = F(h, u, t_0): T \to A^p(X) \to A$ is a morphism. Since T is reduced, to say φ a morphism is equivalent to the assertion that, in the notation of (3.1), the map $T \to A \to J^1(Y)$ is a morphism; we are reduced to the case $A = A^1(Y) = J^1(Y)$. Since $\varphi(t) = z \circ u((t) - (t_0))$ and $z \circ u \in CH^1(T \times Y)$, the assertion follows from a property of Picard varieties [20, 8–24].

(3.3) We denote the class of Picard homomorphisms by $\underline{\underline{P}}$. We shall show that the class $\underline{\underline{P}}$ satisfies the condition (1.3, (I)–(V)) and the condition (2.5.1). By definition it is trivial that the condition (2.5.1) and (1.3, I) hold. To show (II) of (1.3), we must verify the three conditions a)–c) of (1.3, II):

Now a) and b) are easily verified. The equivalence relation defined by $K(X)$ is compatible with graduation by codimension and we may, therefore, suppose z, u homogeneous for codimension: $u \in CH^p(X)$, $z \in CH^q(X \times Y)$. Then $z(u) \in CH^{p+q-m}(Y)$ where $\dim X = m$; since $u \in A^p(X)$, $z(u) \in A^{p+r-m}(Y)$. It is sufficient to verify $h(z(u)) = 0$ for any Picard homomorphism $h: A^{p+q-m}(Y) \to A$. For this purpose, we can restrict ourselves to the case where h is of the form

$$A^{p+q-m}(Y) \xrightarrow{v(?)} A^1(Z) = J^1(Z) = A ,$$

with $v \in CH^{m+n-p-q+1}(Y \times Z)$ and $\dim Y = n$. Since $h(z(u)) = v(z(u))$ and $v \circ z \in CH^{m-p+1}(X \times Z)$, and since $u \in K(X)$, by definition, $v \circ z(u) = 0$ i.e., $h(z(u)) = 0$.

Let $m = \dim X$. Because the canonical homomorphism $\theta^{(m)}: A^m(X) \to \operatorname{Alb} X$ is universal for regular homomorphisms, in order to verify (III) of (1.3), we have only to show $\theta^{(m)}$ a Picard homomorphism. Let $P = J^1(X)$ and $\mathfrak{P} \in CH^1(P \times X)$ be a Poincaré divisor. If we denote $\mathfrak{P}^t$ by the image of $\mathfrak{P}^t$ by the isomorphism $P \times X \rightrightarrows X \times P$, the transposition, then the map $A^m(X) \to J^1(P)$, $x \mapsto \mathfrak{P}^t(x)$ is nothing but $\theta^{(m)}$ if we identify $J^1(P)$ with $\operatorname{Alb}(X)$ [11]. The condition (IV) clearly holds. The condition (V) is verified using the canonical isomorphism [6]

$$J^1(Y) \times J^1(Z) = J^1(Y \times Z) .$$

PROPOSITION 3.4 (Grothendieck, [10, 13]). The class $\underline{\underline{P}}$ defined in (3.3) satisfies the condition (C) of (2.1)

We denote the l-adic cohomology by $H\cdot(X)$ ($l \neq$ char. k is prime, or the singular cohomology if $k = C$). We note the following: if an algebraic family $T \to CH^1(X)$ defined by $z \in CH^1(T \times X)$ is zero, then there are cycles $x \in CH^1(T)$ and $y \in CH^1(X)$ such that $z = x \times X + T \times y$, hence the induced map ${}^t z \colon H^{\cdot}(X) \to H^{\cdot}(T)$ is zero. It is sufficient to show that $\dim A \leq \frac{1}{2} \dim H^{2m-2p+1}(X)$ if A is the image of a Picard homomorphism $h \colon A^p(X) \to J^1(Y)$ defined by a cycle $z \in CH^{m-p+1}(X \times Y)$ and $m = \dim X$. Denote $B = J^1(Y)$ and a Poincaré divisor $\mathfrak{P} \in CH^1(B \times Y)$. By (2.3), there exists $u \in CH^1(A \times X)$ such that $F(u, h, 0)$ is an isogeny $r_A = r \cdot \mathrm{id}_A (r > 0)$ of A. If we denote the inclusion $A \subset B$ by i, then we have a commutative diagram:

$$\begin{array}{ccccc} A & & \xrightarrow{u((?)-(0))} & & A^p(X) \\ {\scriptstyle r_A}\downarrow & & & & \downarrow{\scriptstyle z(?)} \\ A & \xrightarrow{i} & B & \xrightarrow[\mathfrak{P}((?)-(0))]{} & A^1(Y) \end{array}$$

Consider

$$w = z \circ (u - A \times u(0)) - (i \circ r_A \times \mathrm{id}_Y)^*(\mathfrak{P} - B \times \mathfrak{P}(0)) \in CH^1(A \times Y) .$$

The commutativity implies that the algebraic family defined by w is zero and by the above remark ${}^t w = 0$. Since $u(0) \times A$ and $\mathfrak{P}(0) \times B$ induce the zero maps on the cohomology groups, ${}^t u \circ {}^t z = r_A^* \circ i^* \circ {}^t \mathfrak{P} \colon H^{2n-1}(Y) \to \to H^1(A)$, where $n = \dim Y$. Since r_A is an isogeny, r_A^* is an isomorphism. ${}^t \mathfrak{P}$ of $H^{2n-1}(Y)$ to $H^1(B)$ is bijective [9]. Since i is an injective abelian variety homomorphism, by Poincaré's theorem of complete reducibility, i^* is surjective, hence ${}^t u \colon H^{2m-2p+1}(X) \to H^1(A)$ is surjective and $2 \cdot \dim A = \dim H^1(A) \leq \dim H^{2m-2p+1}(X)$.

(3.5) By (2.2, ii), there exists an abelian variety P and a Picard homomorphism $\theta \colon A^p(X) \to P$, universal for Picard homomorphisms in the sense of (2.1, D). We write $\mathrm{Pic}^p X$ for P and $\theta^{(p)}$ for θ. For convention we put $\mathrm{Pic}^p X = 0$ if $p \leq 0$ or $p > \dim X$. Also $\mathrm{Pic}_{m-p} X = \mathrm{Pic}^p X$ if $m = \dim X$.

THEOREM (3.6). Pic X *satisfies the axiom of intermediate Jacobian. Explicitly, we have the following*:

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(i) *Put* $K^p(X) = \left\{\begin{array}{l} u \in CH^p(X);\ u \in A^p(X) \textit{ and for any variety } Y \textit{ and} \\ \quad a \textit{ cycle } z \in CH^{m-p+1}(X \times Y) \\ \qquad\qquad (m = \dim X),\ z(u) = 0 \textit{ in } CH^1(Y)\ . \end{array}\right\}$

Then $K^p(X)$ *is the set of cycles incidence equivalent to zero* [8] *and* $\theta^{(p)}: A^p(X) \to \mathrm{Pic}^p X$ *induces an isomorphism* $A^p(X)/K^p(X) \simeq \mathrm{Pic}^p X$.

(ii) $\mathrm{Pic}^1 X$ *is the Picard variety of* X; $\mathrm{Pic}_0 X$ *is the Albanese variety of* X.

(iii) *For any cycle* $z \in CH^{p+q}(X \times Y)$ $(m = \dim X)$, *there exists one and only one abelian variety homomorphism* $[z] = [z]^p_q: \mathrm{Pic}_q X \to \mathrm{Pic}^p Y$ *such that the diagram*

$$\begin{array}{ccc} A_q(X) & \xrightarrow{z(?)} & A^p(Y) \\ \downarrow & & \downarrow \\ \mathrm{Pic}_q X & \xrightarrow{[z]^p_q} & \mathrm{Pic}^p Y \end{array}$$

is commutative.

Since the class of Picard homomorphisms satisfies (1.3, I–V) (2.1, C) and (2.5.1), the assertion follows from the Remark (2.5).

(3.7) By (3.5, iii), for fixed p and $q \leq 0$, the attachments

$$X \mapsto \mathrm{Pic}^p X \quad \text{and} \quad X \mapsto \mathrm{Pic}_q X$$

define contravariant and covariant functors from the category of varieties into that of abelian varieties.

Remark (3.8). We define a class $\underline{\underline{P}}'$ of regular homomorphisms as follows: A regular homomorphism $h: A^p(X) \to A$ belongs to $\underline{\underline{P}}'$ if there exist a Picard homomorphism $k: A^p(X) \to B$ and an abelian variety homomorphism $\pi: A \to B$ such that $k = \pi \circ h$ and that the kernel of π is finite. We can verify the conditions (1.3, I–V), (2.1, C) and (2.5.1) for the class $\underline{\underline{P}}'$, and the class defines a theory of intermediate Jacobian $J^p(X)$. Clearly there is a canonical map $J^p(X) \to \mathrm{Pic}^p X$ with the finite kernel. As remarked above, this is bijective if there exists a cycle $\mathfrak{P} \in CH^p(\mathrm{Pic}^p X \times X)$ such that the induced map $[\mathfrak{P}]: \mathrm{Pic}_0(\mathrm{Pic}^p X) \simeq \mathrm{Pic}^p X \to \mathrm{Pic}^p X$ is an identity. While $J^p(X)$ may appear more general than $\mathrm{Pic}^p X$, the writer knows no example for which the kernel of the above homomorphism is different from zero. We note that the results of 4 will hold also for $J^p(X)$ defined in this way.

4.

(4.1) Consider the natural transformation

$$\beta\colon \mathrm{Cor}^p(T, X) \to \mathrm{Hom}(\mathrm{Alb}\, T, \mathrm{Pic}^p X)$$

defined by $\theta^{(p)}\colon A^{(p)}(X) \to \mathrm{Pic}^p X$. By definition, β is injective (for any T). The image of β when $T = \mathrm{Pic}^p X$ is a subgroup of $\mathrm{End}(\mathrm{Pic}^p X)$ and the intersection with $Z\cdot \mathrm{id}_{\mathrm{Pic}^p X}$ is of the form $Z\cdot k_X^{(p)}\cdot \mathrm{id}_{\mathrm{Pic}^p X}$ ($k_X^{(p)} > 0$). Let $\mathfrak{P}_X^{(p)} \in CH^p(\mathrm{Pic}^p X \times X)$ be a representative of the element whose image by $\mathfrak{P}$ is $k_X^{(p)}\cdot \mathrm{id}_{\mathrm{Pic}^p X}$. Such $\mathfrak{P}_X^{(p)}$ is not necessarily unique. Consider the natural transformation

$$\gamma\colon \mathrm{Hom}(\mathrm{Alb}\, T, \mathrm{Pic}^p X) \to \mathrm{Cor}^p(T, X)$$

defined by the cycle $\mathfrak{P}_X^{(p)}$: we have then

$$\beta \circ \gamma = k_X^{(p)}\cdot \mathrm{id} \quad \text{and} \quad \gamma \circ \beta = k_X^{(p)}\cdot \mathrm{id}\,.$$

The condition $k_X^{(p)} = 1$ and that of (2.1, B) are equivalent. Except the case $p = 1$, in general, there is few case for which the condition $k_X^{(p)}$ is known to hold or not, even for $p = \dim X$ (while $k_X^{(p)} = 1$ if X is an abelian variety and $p = \dim X$).

(4.2) The cycle ${}^t\mathfrak{P}_X^{(m-p+1)} \in CH^{m-p+1}(X \times \mathrm{Pic}^{m-p+1} X)$ ($m = \dim X$) defines an abelian variety homomorphism

$$\lambda_X^{(p)} := [{}^t\mathfrak{P}_X^{(m-p+1)}]^1_{m-p}\colon \mathrm{Pic}^p X \to \mathrm{Pic}^1(\mathrm{Pic}^{m-p+1} X) = (\mathrm{Pic}^{m-p+1} X)^\wedge$$

where $\hat{?}$ indicates the dual abelian variety of ?. While the cycle $\mathfrak{P}_X^{(m-p+1)}$ is, in general, not uniquely determined, and $\lambda_X^{(p)}$ may appear not to be unique, it will be shown that $\lambda_X^{(p)}$ obtained as above is unique (4.5). We choose such a $\mathfrak{P}_X^{(m-p+1)}$ once for all.

Remark (4.3). $\lambda_X^{(p)}$ is an isomorphism if $p = m$.

PROPOSITION (4.4). *Let X and Y be varieties of dimension m and n respectively and $z \in CH^{p+q}(X \times Y)$. Then, for*

$$[z] = [z]^p_q\colon \mathrm{Pic}^{m-q} X \to \mathrm{Pic}^p Y$$
$$[{}^tz] = [{}^tz]^{q+1}_{p-1}\colon \mathrm{Pic}^{n-p+1} Y \to \mathrm{Pic}^{q+1} X\,,$$

we have

(4.4.1) $$k_X^{(p)}[z]^\wedge \circ \hat{\lambda}_X^{(p)} = k_Y^{(n-p+1)}\lambda_X^{(q+1)} \circ [{}^tz]\,.$$

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In particular if $X = Y$, $p + q = \dim X$ *and* $z = \Delta_X \in CH^m(X \times X)$ *the diagonal*

$$(4.4.2) \qquad k_X^{(p)} \hat{\lambda}_X^{(p)} = k_X^{(m-p+1)} \lambda_X^{(m-p+1)} .$$

Using this formula (4.4.1) *is also expressed as*

$$(4.4.3) \qquad k_X^{(m-q)}[z]^\wedge \circ \lambda_Y^{(n-p+1)} = k_Y^{(p)} \lambda_X^{(q+1)} \circ [{}^t z] ,$$

$$(4.4.4) \qquad k_X^{(q+1)}[z]^\wedge \circ \hat{\lambda}_Y^{(p)} = k_Y^{(n-p+1)} \hat{\lambda}_X^{(m-q)} \circ [{}^t z] .$$

Moreover $\lambda_X^{(p)}$ *is an isogeny for any* p *and any* X (*the duality*).

For the proof we refer to [13].

COROLLARY (4.5). *The isomorphism* $CH^{p+q}(X \times Y) \to CH^{p+q}(Y \times X)$ *induced by the isomorphism* $X \times Y \to Y \times X$, *the transposition, induces an isomorphism*

$$\mathrm{Cor}^{q,p}(X, Y) \rightrightarrows \mathrm{Cor}^{p-1,q+1}(Y, X)$$

defined by class of $z \mapsto$ *class of* ${}^t z$. *In particular, letting* $q = 0$, $X = \mathrm{Pic}^{n-p+1} Y$, *and* $p = m - r + 1$, *we see that* $\lambda_Y^{(r)}$ *is independent of choice of a representative* $\mathfrak{P}_Y^{(n-r+1)} \in CH^{n-r+1}(\mathrm{Pic}^{n-r+1} Y \times Y)$ *and uniquely determined.*

It is sufficient to show that the map

$$CH^{p+q}(X \times Y) \to CH^{p+q}(Y \times X) \to \mathrm{Cor}^{p-1,q+1}(Y, X)$$

has, for value, zero on the (q, p)-trivial elements. By (4.4), if

$$z \in CH^{p+q}(X \times Y)$$

is a (q, p)-trivial element, then $k_Y^{(n-p+1)} \lambda_Y^{(q+1)} \circ [{}^t z] = k_x^{(p)}[z]^\wedge \circ \hat{\lambda}_Y^{(p)} = 0$. Since $k_Y^{(n-p+1)} \lambda_X^{(q+1)}$ is an isogeny, we have $[{}^t z] = 0$.

THEOREM (4.6). *For any integer* p *and* $q \geq 0$, *consider the homomorphism*

$$\beta : \mathrm{Cor}^{q,p}(X, Y) \to \mathrm{Hom}(\mathrm{Pic}_q X, \mathrm{Pic}^p Y)$$

defined by (3.6, iii). *Then there exist a natural number* r *and a homomorphism*

$$\Gamma : \mathrm{Hom}(\mathrm{Pic}_q X, \mathrm{Pic}^p Y) \to \mathrm{Cor}^{q,p}(X, Y)$$

such that

$$\beta \circ \Gamma = r \cdot \mathrm{id} , \qquad \Gamma \circ \beta = r \cdot \mathrm{id} .$$

(Γ and a positive integer r are, in general, dependent upon X, Y, p and q). The cokernel of β is, therefore, finite and $\beta\otimes \boldsymbol{Q}$ is an isomorphism.

The map β is defined by $z \mapsto [z]^p_q$. Since β is injective, it is sufficient to show the existence of r and Γ with $\beta\circ\Gamma = r\cdot\mathrm{id}$. We suppose the following, and prove the theorem: if A is an abelian variety, there exist a cycle $\pi \in CH_1(A\times\hat{A})$ and $s>0$ such that $s\cdot\mathrm{id} = [\pi]\colon \mathrm{Pic}^1(A) \to \mathrm{Alb}(\hat{A}) = \hat{A}$. By the assumption, there exist a cycle $\pi\in CH_1(\mathrm{Pic}^{q+1}X \times (\mathrm{Pic}^{q+1}X)^\wedge)$ and $s>0$ such that $s\cdot\mathrm{id} = [\pi]\colon \mathrm{Pic}^1(\mathrm{Pic}^{q+1}X) \to \mathrm{Alb}(\mathrm{Pic}^{q+1}X^\wedge) = (\mathrm{Pic}^{p+1}X)^\wedge$. Let $m = \dim X$. Since $\lambda_X^{(m-q)}\colon \mathrm{Pic}^{(m-q)}(X) \to (\mathrm{Pic}^{q+1}X)^\vee$ is an isogeny, so is $s\cdot\lambda_X^{(m-q)}$; hence there is an isogeny $\varphi\colon (\mathrm{Pic}^{q+1}X)^\wedge \to \mathrm{Pic}^{m-q}X$ such that $\varphi\circ(s\cdot\lambda_X^{(m-q)}) = r'\cdot\mathrm{id}_{\mathrm{Pic}^{m-q}X}$ for some $r'>0$. Then we define the map

$$\Gamma\colon \mathrm{Hom}(\mathrm{Pic}_q X, \mathrm{Pic}^p Y) \to \mathrm{Cor}^{q,p}(X, Y)$$

as follows: for a homomorphism $f\colon \mathrm{Pic}_q X \to \mathrm{Pic}^p Y$,

$$\Gamma(f) = \{(f\circ\varphi\times\mathrm{id}_Y)^*(\mathfrak{P}_Y^{(p)})\}\circ\pi\circ{}^t\mathfrak{P}_X^{(q+1)}\,.$$

It is easily verified that $\Gamma(f)$ defines an element of $\mathrm{Cor}^{q,p}(X, Y)$ and that the element is independent of choice of $\mathfrak{P}$'s. Put $r = r'\cdot k_Y^{(p)}$ and we will show $\beta\circ\Gamma = r\cdot\mathrm{id}$. For a homomorphism $f\colon \mathrm{Pic}_q X \to \mathrm{Pic}^p Y$, it is sufficient to show $\beta\circ\Gamma(f) = r\cdot f$. For $x\in\mathrm{Pic}_q X$,

$$\begin{aligned}(\beta\circ\Gamma)(f)(x) &= [\{(f\circ\varphi\times\mathrm{id}_Y)^*(\mathfrak{P}_Y^{(p)})\}\circ\pi\circ{}^t\mathfrak{P}_X^{(q+1)}]^p_q(x)\\ &= [(f\circ\varphi\times\mathrm{id}_Y)^*(\mathfrak{P}_Y^{(p)})]^p_0\circ[\pi]\circ[{}^t\mathfrak{P}_X^{(q+1)}]^1_q(x)\\ &= [\mathfrak{P}_Y^{(p)}]^p_0\circ f\circ\varphi(s\cdot\mathrm{id}\circ\lambda_X^{(m-q)}(x))\\ &= k_Y^{(p)}f(r'\cdot x)\\ &= (r\cdot f)(x)\,,\end{aligned}$$

i.e., $\beta\circ\Gamma(f) = r\cdot f$. We now show the above assertion. Let A be an abelian variety of dimension a. We identify $\hat{\hat{A}}$ with A. Let D be an ample divisor on $\hat{A}$, and let $C = D^{a-1}\in CH_1(\hat{A})$. Then for $x\in\hat{A}$, in $G_0(\hat{A}) = \hat{A}$, $a\cdot C\cdot(D_x - D) = (D^a)((x)-(0))$ where D_x is the translation of the divisor D by $x\in\hat{A}$, and (D^a) is the intersection number [15]. Hence

$$\hat{A}\longrightarrow A^1(\hat{A})\xrightarrow{\;C\;}A_0(\hat{A})\longrightarrow\hat{A}$$

is an isogeny, where the first map defined by $x\mapsto D_x - D$ is an isogeny, whence

$$\varphi\colon A = A^1(\hat{A})\xrightarrow{\;C\;}A_0(\hat{A})\longrightarrow\hat{A}$$

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is also an isogeny. Let $\varDelta$ denote the diagonal of $\hat{A} \times \hat{A}$ and put $C' = (C \times \hat{A})\cdot \varDelta \in CH_1(\hat{A} \times \hat{A})$. Then $\varphi = [C']: A = \mathrm{Pic}^1(\hat{A}) \to \mathrm{Pic}_0(\hat{A}) = \hat{A}$. Since φ is an isogeny, there is an isogeny $\psi: \hat{A} \to A$ with $\varphi \circ \hat{\psi} = r\cdot \mathrm{id}_A$ for some $r > 0$. Put $\pi\ (\psi \times \mathrm{id}_A)_*(C')$, and we have $r\cdot \mathrm{id} = [\pi]^{a}_{a-1}: \mathrm{Pic}^1(A) \to \mathrm{Pic}_0(\hat{A})$, for $\pi(?) = C'(\psi^*(?))$.

Remark (4.7). Let X, Y and Z be varieties and $n = \dim Y$. If $u \in CH^{p+q}(X \times Y)$ and $v \in CH^{n-p+r}(Y \times Z)$ and if u is (q, p)-trivial or v is $(n-p, r)$-trivial, it is then easy to verify the composite $v \circ u$ is (q, r)-trivial. Hence by composition we have well-defined bilinear forms

$$\delta: \mathrm{Cor}^{q,p}(X, Y) \times \mathrm{Cor}^{n-p,r}(Y, Z) \to \mathrm{Cor}^{q,r}(X, Z)\,,$$

$$o: \mathrm{Hom}(\mathrm{Pic}_q X, \mathrm{Pic}^p Y) \times \mathrm{Hom}(\mathrm{Pic}^p Y, \mathrm{Pic}^r Z) \to \mathrm{Hom}(\mathrm{Pic}_q X, \mathrm{Pic}^r Z)\,.$$

Then the diagram

$$\begin{array}{ccc} \mathrm{Cor}^{q,p}(X, Y) \times \mathrm{Cor}^{n-p,r}(Y, Z) & \xrightarrow{\delta} & \mathrm{Cor}^{q,r}(X, Z) \\ \downarrow{\scriptstyle \beta\times\beta} & & \downarrow{\scriptstyle \beta} \\ \mathrm{Hom}(\mathrm{Pic}_q X, \mathrm{Pic}^p Y) \times \mathrm{Hom}(\mathrm{Pic}^p Y, \mathrm{Pic}^r Z) & \xrightarrow{\circ} & \mathrm{Hom}(\mathrm{Pic}_q X, \mathrm{Pic}^r Z) \end{array}$$

is commutative. In particular, if $m = \dim X$, the set $\mathrm{Cor}^{m-p,p}(X, X)$ has a ring structure whose multiplication is defined by composition, and $\beta: \mathrm{Cor}^{m-p,p}(X, X) \to \mathrm{End}(\mathrm{Pic}^p X)$ is a ring homomorphism, and $\beta \otimes \boldsymbol{Q}$ is an isomorphism of rings by (4.6)

5.

(5.1) In this section, we suppose $k = \boldsymbol{C}$. Let $T^p(X)$ by the Griffiths intermediate Jacobian (or Weil's one). By $\underline{H}^p(X)$, we denote the subgroup of $CH^p(X)$ consisting of cycles homologically equivalent to zero; then homomorphism (called an Abel-Jacobi homomorphism in [8])

$$\Phi: \underline{H}^p(X) \to T^p(X)$$

is defined and is a regular homomorphism [7, 12] (i.e., for any variety T and a cycle $z \in CH^p(T \times X)$ such that $z(t) \in \underline{H}^p(X)$ for every $t \in T$, the composite $T \to \underline{H}^p(X) \to T^p(X)$ is a holomorphic map). The image of $\Phi \mid A^p(X)$ is an abelian subvariety $J^p_a(X)$ of $T^p(X)$ (even if $T^p(X)$ is not an abelian variety but a complex torus). The restriction of $\Phi: A^p(X) \to J^p_a(X)$ is a regular homomorphism and $J^p_a(X)$ defines a theory of intermediate Jacobian [12, 13]. Since for $z \in CH^{p+q}(X \times Y)$ and $m = \dim X$, there is a homomorphism $[z]: J^{m-q}_a(X) \to J^p_a(X)$ such that the diagram

$$\begin{array}{ccc} A^{m-q}(X) & \xrightarrow{z(?)} & A^p(Y) \\ \downarrow & & \downarrow \\ J^{m-q}(X) & \xrightarrow{[z]} & J^p(Y) \end{array}$$

is commutative, we see

(5.1.1) $$\operatorname{Ker}(A^p(X) \to J^p_a(X)) \subset K^p(X)$$

where $K^p(X)$ is defined in (3.6), and obtain a natural surjection

(5.1.2) $$\pi_X^{(p)} : J_a^p(X) \to \operatorname{Pic}^p X \,.$$

PROPOSITION (5.2). *If X is an abelian variety, $\pi_X^{(p)}$ is an isogeny.* *cf.* [12].

PROPOSITION (5.3). *Let X be a variety and Y a hyperplane section of X. If $2p-1 < \dim X$ and if $\pi_Y^{(p)}$ is an isogeny, then $\pi_X^{(p)}$ is also an isogeny.*

By a theorem of Lefschetz [1, for example], for $2p-1 \leq \dim Y$

$$i^* : H^{2p-1}(X, Z) \to H^{2p-1}(Y, Z) \,.$$

induced by the inclusion $i: Y \hookrightarrow X$ is injective. Since

$$T^p(X) = H^{2p-1}(X, R)/\operatorname{Im} H^{2p-1}(X, Z)$$

as real tori, the kernel of $i^* : T^p(X) \to T^p(Y)$ is finite.*) Hence the kernel of $i^* : J_a^p(X) \to J_a^p(Y)$ is also finite. Consider the commutative diagram

$$\begin{array}{ccc} J_a^p(X) & \xrightarrow{i^*} & J_a^p(Y) \\ \downarrow & & \downarrow \\ \operatorname{Pic}^p(X) & \xrightarrow{i^*} & \operatorname{Pic}^p Y \end{array}$$

Since $\operatorname{Ker}(\pi_Y^{(p)})$ is finite, so is $\operatorname{Ker}(i^* \circ \pi_X^{(p)}) = \operatorname{Ker}(\pi_Y^{(p)} \circ i^*)$, a fortiori $\operatorname{Ker}(\pi_X^{(p)})$ is finite.

COROLLARY (5.4). *If X is a variety containing an abelian variety of dimension d as a complete intersection, $\pi_X^{(p)}$ is then an isogeny for $2p-1 \leq d$.*

COROLLARY (5.5). *Fix a positive integer p, if $\pi_X^{(p)}$ is an isogeny for every variety X of dimension $2p-1$, then $\pi_X^{(q)}$ is also an isogeny for every $q \leq p$ and every variety X.*

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*) [added in proof]. Ker (i^*) is compact and is a finitely generated abelian group; hence it is finite.

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By iterated use of Prop. (5.3), we see that if $2p-1<\dim X$, $\pi_X^{(p)}$ is an isogeny. For an integer $r>0$, put $P=X\times P^r$; then we have a commutative diagram

$$\begin{array}{ccc} J_a^p(P) & \xrightarrow{\sim} & \bigoplus_{0\le i\le r} J_a^{p-i}(X) \\ \downarrow & & \downarrow \\ \mathrm{Pic}^p(P) & \xrightarrow{\sim} & \bigoplus_{0\le i\le r} \mathrm{Pic}^{p-i} X \end{array}$$

Taking r large enough, we may suppose $2p-1<\dim P$ and $p<r$, then $\pi_P^{(p)}$ is an isogeny, and so $\pi_X^{(q)}$ $(q\le p)$ are isogenies.

Remark (5.6). Griffiths [8] announced the assertion that $\pi_X^{(p)}$ is an isogeny for $\dim X=2p-1$, and this implies that for every $p, \pi_X^{(p)}$ is an isogeny*). Assuming this, the duality of type (4.4) holds for $J_a^p(X)$.

6.

We know a few examples:

(6.1) Let X be a complete intersection in $\boldsymbol{P}^r$. Then $\mathrm{Pic}^p X=0$ if $2p-1\neq\dim X$. We see that for $2p-1<\dim X$, $H^{2p-1}(X)=0$ and $\mathrm{Pic}^p X=0$. By duality we obtain our assertion.

(6.2) Let X be a cubic threefold. Then $A^2(X)\xrightarrow{\sim}\mathrm{Pic}^2 X$ is a Prym variety [17]. More generally, let X be a conic bundle, i.e., a connected complete smooth scheme over k with a morphism $X\to P^2$ where fibers are conics in a projective space. If $\dim X=2n+1$, $\theta^{(n+1)}: A^{n+1}(X)\xrightarrow{\sim}\mathrm{Pic}^{n+1} X$ is a (generalized) Prym variety [2]. For these example θ are universal for the regular homomorphisms.

(6.3) For a unirational threefold X, $A^2(X)$ is "isogenous" to $\mathrm{Pic}^2 X$, i.e. the kernel of the canonical map $\theta^{(2)}: A^2(X)\to\mathrm{Pic}^2 X$ is a finite set. Moreover, if $k=\boldsymbol{C}$ and if Q is a quartic threefold, $A^2(Q)$ is isogenous to $\mathrm{Pic}^2(Q)$ [3].

(6.4) Let X be a general abelian variety defined over $\boldsymbol{C}$. Then for any p, $1\le p\le\dim X$, $\mathrm{Pic}^p X$ and X are isogenous [12, 14].

(6.5) Let A be an abelian variety of dimension m over an uncountable field of characteristic zero. Then for $1<p\le m$, the canonical map $\theta^{(p)}: A^p(A)\to\mathrm{Pic}^p A$ is not an isogeny. We prove a stronger result. Let $G^p(A)$, $K^p(A)$ be as defined in (1.3) for the class (1.3.1). We will show

*) [added in proof]. A proof of a special case of his announcement can be found in [22, **6**, (c)].

that the orders of elements of $K^p(A)$ are not bounded. This implies, in particular, for $k = C$, the kernel of $A^p(A) \to J_a^p(A)$ is non-zero. Let D be an ample divisor on A. For $a_1, \cdots, a_p$ in A, we show $(D_{a_1} - D) \cdots (D_{a_p} - D) \in K^p(A)$. For this purpose, it is sufficient to show the image of the intersection product of $A^p(A)$ and $A^q(A)$ are in $K^{p+q}(A)$. Suppose $x \in A^p(A)$ and $y \in A^q(A)$. Then there exist abelian varieties $B, C, b \in B$, $c \in C$ and $u \in CH^p(B \times A)$, $v \in CH^q(C \times A)$ such that $x = u((b) - (0))$ and $y = v((c) - (0))$. Since $xy = (p_1^* u \cdot p_2^* v)((b, c) - (b, 0) - (0, c) + (0, 0))$ where p_1 and p_2 are projections of $B \times C \times A$ onto $B \times A$ and $C \times A$ respectively, the argument in (1.7) shows the cycle of this form is mapped to zero by any regular homomorphism, and we obtain our assertion. By [4], $m!(D_{a_1} - D) \cdots (D_{a_m} - D) = (D^m)((a_1) - (0)) * \cdots * ((a_m) - (0))$ in $CH_0(A)$ where $*$ denotes the Pontrjagin product. Suppose the orders of elements of $K^p(A)$ are bounded by N.

$$(D_{a_1} - D) \cdots (D_{a_p} - D) \in K^p(A) \ (a_1, \cdots, a_p \in A) \text{ implies}$$
$$N(D_{a_1} - D) \cdots (D_{a_p} - D) = 0 \text{ in } CH^p(A), \text{ hence}$$
$$N(D_{a_1} - d) \cdots (D_{a_m} - D) = 0 \text{ in } CH_0(A), \text{ i.e.,}$$
$$N(D^m)((a_1) - (0)) * \cdots * ((a_m) - (0)) = 0 \text{ in } CH_0(A) \ (a_1, \cdots, a_m \in A).$$

Since the $((a_1) - (0)) * \cdots * (a_m) - (0))$ $(a_1, \cdots, a_m \in A)$ generate I^{*m} where $I = A_0(A)$ and I^{*m} is m-folded Pontrjagin product, and since I^{*m} is divisible, $N \cdot (D^m) \cdot I^{*m} = 0$ implies $I^{*m} = 0$; this contradicts a proposition in [4]. Note, however, that if k is the algebraic closure of a finite field then for a any abelian variety A over k, $A_0(A) \rightleftarrows A$ by use of $I^{*2} = 0$ [4].

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