J. Austral. Math. Soc. (*Series A*) **56** (1994), 384–390

L^p-FUNCTIONS SATISFYING THE MEAN VALUE PROPERTY ON HOMOGENEOUS SPACES

A. SITARAM and G. A. WILLIS

(Received 5 February 1992)

Communicated by I. Raeburn

Abstract

It is proved that on certain kinds of homogeneous spaces, the only L^p function, $1 \leq p < \infty$, satisfying the mean value property is the zero function.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 22 E 99, 43 A 85.

1. Introduction

Let M be an analytic Riemannian manifold with $\dim M > 2$. The spherical mean value operator M with radius $r > 0$ is defined by

$$(M_r f)(x) = \int_{\{X \in T_x(M): \|X\|=1\}} f(\mathrm{Exp}_x rX)\, d\sigma(X)$$

(Here $T_x(M)$ is the tangent space at $x \in M$ equipped with inner product arising from the Riemannian structure, Exp_x the exponential map from $T_x(M)$ into M and $d\sigma$ the normalized measure on the surface of the unit sphere in $T_x(M)$.) Roughly speaking $(M_r f)(x)$ is the mean value of f at a geodesic distance r from x.

Equip M with the canonical Riemannian measure and let Δ be the Laplace-Beltrami operator of M. We will say that a locally integrable function satisfies MVP ('mean value property') if there exists $\epsilon > 0$ such that $M_r f = f$ for all $r < \epsilon$. For a wide class of Riemannian manifolds (see [9]) it is known that f satisfies MVP if and only if $\Delta f \equiv 0$. However in general Sunada [11] has proved that for a complete analytic Riemannian manifold M, there exists a family of self-adjoint elliptic operators $\{P_k\}_{k=1,2,\ldots}$ with $P_1 = \Delta$ such that f satisfies MVP if and only if $P_k f \equiv 0$ for all k. (For instance on $\mathbb{R}^n$, $P_k = \Delta^k$ and the above just reduces to the usual equivalence of MVP and harmonicity.) Note that because of this and the elliptic regularity theorem on such manifolds any f satisfying MVP is real analytic.

There seems to be a lot of deep work (see [3] for example) connecting the differential-geometric properties of M and the existence or non-existence of L^p-functions f satisfying $\Delta f \equiv 0$. Therefore, in view of the previous paragraph, it seems natural to consider the existence of L^p-functions satisfying MVP. In this note we deal with manifolds which are homogeneous spaces of the kind G/H where G is a connected Lie group, H a compact subgroup and G acts on G/H as isometries. Note that a wide class of manifolds including symmetric spaces and Euclidean spaces are of this kind. There is a result of Yau (see [5]) which says that, if M is a complete, non-compact Riemannian manifold and $p > 1$, then there are no non-trivial L^p-functions f satisfying $\Delta f \equiv 0$. However, there is apparently a construction due to Chung (see [5]) of a non-trivial L^1-harmonic function on a complete Riemannian manifold on infinite volume. In view of this, the contents of this paper seem particularly interesting because we prove that on certain kinds of manifolds there are no L^p-functions satisfying MVP, including the case $p = 1$.

In what follows we assume M is an analytic Riemannian manifold of the form G/H where G is a non-compact unimodular connected Lie group, H a compact subgroup and G acts on M by isometries. We will show that non-zero L^p-functions, $1 \leq p < \infty$, satisfying MVP do not exist on manifolds of this type by reducing to an argument on the group G. The contents of the next section are probably well known in the folklore, especially to those people who have looked at the Choquet-Deny equation (that is, the equation $f * \mu = f$) in detail. However, for the sake of completeness we give all the details.

2. A group theoretic proposition.

Since the result of this section is valid for any locally compact unimodular group, let S be such a group.

Let $L^p(S)$ denote the space of functions which are L^p-integrable with respect to the Haar measure on S and $C_0(S)$ the space of continuous functions on S vanishing at infinity.

For each probability measure μ on S, the smallest closed subset of S supporting μ will be denoted by supp μ and $\overline{gp\{\text{supp } \mu\}}$ will denote the smallest closed supgroup supporting μ. If f belongs to $C_0(S)$ (respectively $L^p(S)$ for some $p \geq 1$) and μ is a probability measure, put $(f * \mu)(x) = \int_S f(xs^{-1})d\mu(s)$. Then $(f * \mu)(x)$ is defined for almost every x (with respect to Haar measure) in S and the function $f * \mu$ also belongs to $C_0(S)$ (respectively $L^p(S)$).

LEMMA 1. *Let μ be a probability measure on S such that $S = \overline{gp\{\text{supp}\mu\}}$ and let φ in $C_0(S)$ be such that $\varphi * \mu = \varphi$. Then either*

(i) *φ is identically zero; or*
(ii) *S is compact and φ is a constant function.*

PROOF. Since μ is a real measure it may be supposed that φ is a real valued function. Let $M = \sup\{\varphi(x) \mid x \in S\}$ and $K = \{x \in S \mid \varphi(x) = M\}$. If (i) does not hold it may be supposed that $M > O$. Then K is a non-empty, compact subset of S because φ belongs to $C_0(S)$. By translating φ if necessary, we may suppose that e belongs to K, where e is the identity in S.

Now for every x in K,

$$M = \varphi(x) = (\varphi * \mu)(x) = \int_S \varphi(xs^{-1})d\mu(s)$$

Putting x equal to e shows that s^{-1} belongs to K for almost every s (with respect to μ) in supp μ, whence, since K is closed, $(\text{supp } \mu)^{-1}$ is contained in K. It follows, by repeating this argument, that the closed semigroup generated by $(\text{supp } \mu)^{-1}$ is contained in K and so this semigroup is compact.

Let x be in this compact semigroup. Then $\{x^n \mid n = 1, 2, 3, \ldots\}$ has an accumulation point, y say, in the semigroup. If n and m are such that $n - 2 \geq m \geq 1$ and x^n and x^m are close to y, then x^{n-m-1} will be in the semigroup and close to x^{-1}. It follows that the topologically closed semigroup generated by $(\text{supp } \mu)^{-1}$ is also closed under taking inverses and so is in fact a group. Hence $S = \overline{gp\{\text{supp } \mu\}}$ is compact and $\varphi(x) = M$ for every x in S.

The result we require is the extension of the lemma to L^p-spaces.

PROPOSITION 1. *Let S be a noncompact, unimodular group and μ be a probability measure on S such that $\overline{gp\{\text{supp } \mu\}} = S$. If φ belongs to $L^p(S)$ for some $1 \leq p < \infty$ and $\varphi * \mu = \varphi$, then $\varphi = 0$ a.e.*

PROOF. If φ were nonzero, there would be ϕ in $L^1(S) \cap L^\infty(S)$ such that $\phi * \varphi \neq 0$, where the convolution is with respect to Haar measure. Then $\phi * \varphi$ belongs to $C_0(S)$ and $(\phi * \varphi) * \mu = \phi * (\varphi * \mu) = \phi * \varphi$, which contradicts the lemma. Therefore φ is zero a.e.

(The proposition also holds for groups which are not unimodular but for simplicity we have assumed unimodularity.)

3. The main result

Let M be an analytic Riemannian manifold of the type described at the end of Section 1. Equip M with the canonical Riemannian measure dm. This measure is invariant under isometries of M and hence G-invariant. Functions on M can be thought of as right H-invariant functions on G and with this identification fix the Haar measure dg on G such that $\int_G f(g)dg = \int_M f dm$, for $f \in C_C(M)$. Since H is compact, $L^p(M) = L^p(G/H)$ can be thought of as a subspace of $L^p(G)$, consisting of right H-invariant functions. Similarly a complex measure/probability measure on M can be thought of as a complex measure/probability measure on G which is right H-invariant. So from now on we make this identification.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ the Lie algebra of H. We use 'exp' for the exponential map defined on the Lie algebra $\mathfrak{g}$ and Exp_x for the exponential map defined on the tangent space $T_x M$ at $x \in M$ (see [4] for details).

Define a probability measure μ_r (for $r > 0$) on M as follows:

$$\mu_r(f) = \int_{X \in T_{m_0}(M); \|X\|=1} f(\mathrm{Exp}_{m_0} rX)\, d\sigma(X), \qquad f \in C_C(M).$$

Here $m_0 = eH$. Then μ_r can be viewed as a right H-invariant (in fact an H-bi-invariant) probability measure on G and for f a function on $M = G/H$, it is easy to show that $M_r f = f * \check{\mu}_r$ (where for any measure μ on G, $\check{\mu}$ denotes the measure defined by $\check{\mu}(E) = \mu(E^{-1})$ and the convolution is with respect to the Haar measure dg on G). The above follows easily from the fact that G acts (transitively) by left action on $M = G/H$ as isometries. So, in this set up, f satisfies MVP means that there exists $\epsilon > 0$ such that

$$f * \check{\mu}_r = f \qquad \text{for } r < \epsilon.$$

Let $\mathfrak{h}^\perp$ be any complementary linear subspace of $\mathfrak{h}$ in $\mathfrak{g}$ (that is, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$). One knows that the map ψ from $\mathfrak{g} = \mathfrak{h}^\perp \oplus \mathfrak{h}$ into G defined by $\psi : (X, Y) \to$

(exp X exp Y) is a diffeomorphism from a neighbourhood of 0 in $\mathfrak{g}$ onto a neighbourhood of e in G. Also the map φ from $\mathfrak{h}^\perp$ into $G/H = M$ defined by $\varphi : Y \to (\exp Y)H$ is a diffeomorphism from a neighbourhood (in the subspace topology) of 0 in $\mathfrak{h}^\perp$ onto a neighbourhood of m_0 $(= eH)$ in M. Fix any inner product on $\mathfrak{g}$. Without loss of generality we may take ϵ sufficiently small such that: (a) on the ball of radius $r < \epsilon$ in $\mathfrak{g}$, ψ is a difffeomorphism onto its image; (b) on the ball of radius $r < \epsilon$ in $\mathfrak{h}^\perp$, φ is a diffeomorphism onto its image and (c) on the ball of radius $r < \epsilon$ in $T_{m_0}(M)$, Exp_{m_0} is a diffeomorphism onto its image. (The inner product on $T_{m_0}(M)$ is the one induced by the Riemannian structure on M.) Now fix any $r < \epsilon$. Then the support of μ_r, thought of as a measure on G is $\psi(\phi^{-1}(\mathrm{Exp}_{m_0} S_r))H$. Note: $\varphi^{-1}(\mathrm{Exp}_{m_0} S_r) \subseteq \mathfrak{h}^\perp$. Now ψ is defined on $\mathfrak{g}$. Thus, viewing $\mathfrak{h}^\perp$ as a subset of $\mathfrak{g}$, $\psi(\varphi^{-1}(\mathrm{Exp}_{m_0} S_r)$ makes sense. (If L_1 and L_2 are two subsets of G, $L_1 L_2 = \{l_1 l_2 : l_1 \in L_1, \quad l_2 \in L_2\}$.) Here $S_r = \{X \in T_{m_0}(M) : \|X\| = r\}$. By our choice of ϵ, $\varphi^{-1}(\mathrm{Exp}\, S_r)$ as a subset of $\mathfrak{h}^\perp$ will be diffeomorphic to S_r and $0 \in \mathfrak{h}^\perp$ will be in the interior of the bounded component of $(\varphi^{-1}(\mathrm{Exp}\, S_r))^C$. (The topology in $\mathfrak{h}^\perp$ is of course the subspace topology.) From this it will follow easily that the closure of the group generated by supp $\mu_r = (\varphi^{-1}(\mathrm{Exp}_{m_0} S_r))H$ will contain (exp U)H where U is a neighbourhood of 0 in $\mathfrak{h}^\perp$. This implies, since G is connected, that the closure of the group generated by supp μ_r is all of G. We summarize the preceding discussion in the form of the following proposition:

PROPOSITION 2.

(i) *For any $r > 0$, $M_r f = f * \check{\mu}_r$, for f a function on $M = G/H$.*
(ii) *For sufficiently small r, $\overline{gp\{\mathrm{supp}\, \mu_r\}} = \overline{gp\{\mathrm{supp}\, \check{\mu}_r\}} = G$.*

All this combined with the proposition proved in Section 2 about groups leads to:

THEOREM 1. *If $f \in L^p(M)$, $1 \le p < \infty$, and f satisfies* MVP, *then $f = 0$ a.e.*

4. Further remarks

(i) In the above discussion M is noncompact. The case of compact M is dealt with in [10], [11], [6] et cetera. For a compact M, $f \in L^p(M)$ satisfies MVP if and only if it is a constant.

(ii) Note that we have actually proved something stronger than the theorem in Section 3. In fact we have proved the following: let ϵ be chosen as in Section 3. Then if $f \in L^p(M)$, $1 \leq p < \infty$ and f satisfies the mean value property with respect to a *single radius* $r < \epsilon$ then $f = 0$ a.e. (that is, if r_0 is any positive number less than ϵ and $M_{r_0} f = f$, then $f = 0$ a.e.).

(iii) What about $p = \infty$? In general one can have L^∞-functions which satisfy MVP. For example, there are plenty of such functions on the upper half plane equipped with the Poincaré metric. Let us specialize further and take M to be a symmetric space of the noncompact type. For simplicity let us assume that M has rank 1, that is, $M = G/K$ where G is a semi-simple non-compact connected group of real-rank 1 and K a fixed maximal compact subgroup. Now if μ is an absolutely continuous K-bi-invariant probability measure, $f \in L^\infty(M)$ and $f * \mu = f$, then by a theorem of Furstenberg [2], it follows that f is harmonic, that is, $\Delta f \equiv 0$, where Δ is the Laplace-Beltrami operator of M. By a theorem of Ragozin [7] which is true for irreducible symmetric pairs and hence for symmetric spaces of rank 1, for any $r > 0$, a sufficiently high convolution power of μ_r is absolutely continuous and hence combined with the theorem of Furstenberg quoted above it follows that if $f \in L^\infty(M)$ and $M_r f = f$ for a *single radius* r, then f is harmonic. This gives a partial answer to a question of Rudin [8] whether a 'one radius theorem' along the lines of Theorem 4.3.4 in [8] is true for bounded functions in the unit ball of $\mathbb{C}^\infty$. (Of course Rudin allows r to vary from point to point, while for us r will be the same at all points of the manifold.) For related results without any boundedness assumptions see Theorem 3 in [1].

(iv) If M is a symmetric space of the non-compact type, one can use some Fourier analysis on M to conclude that any L^1-harmonic function is zero. This fact is probably well known to experts.

(v) Thangavelu [12] has considered in detail the spherical mean value operator on the reduced Heisenberg group and also examined connections with Fourier Analysis.

Acknowledgements

Part of the work for this paper was done while one of us (A. S.) was visiting The Centre for Mathematical Analysis, The Australian National University. A. Sitaram thanks Professor N. Trudinger for his invitation and hospitality. He thanks Chris Meaney for several useful conversations and in particular for telling him about the results in [7]. A. S. also thanks V. Pati for several useful conversations.

References

[1] C. A. Berenstein and L. Zalcman, 'Pompeiu's problem on symmetric spaces', *Comment. Math. Helv.* **55** (1980), 593–621.
[2] H. Furstenberg, 'A Poisson formula for semisimple Lie groups', *Annals of Math.* **77** (1963), 335–386.
[3] A. A. Grigoryan, 'Stochastically complete manifolds and summable harmonic functions', *Mathematics of USSR—Izvestiya* **33** (1989), 425–432.
[4] S. Helgason, *Differential geometry, Lie groups and symmetric spaces* (Academic Press, San Diego, 1978).
[5] L. Karp, 'Subharmonic functions, harmonic mappings and isometric immersions', in: *Seminar on differential geometry* (ed. S. T. Yau) (Princeton University Press, Princeton, 1982).
[6] V. Pati, M. Shahshahani and A. Sitaram, 'The spherical mean value operator for compact symmetric spaces', Technical report, (Indian Statistical Institute—Bangalore Technical Report, 1989).
[7] D. Ragozin, 'Zonal measure algebras on isotropy irreducible homogeneous spaces', *J. of Funct. Anal.* **17** (1974), 355–376.
[8] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$ (Springer, Berlin, 1980).
[9] H. S. Ruse, A. G. Walker and T. J. Wilmore, *Harmonic spaces* (Edizione Cremonese, Rome, 1961).
[10] T. Sunada, 'Spherical means and geodesic chains on a Riemannian manifold', *Trans. Amer. Math. Soc.* **267** (1981), 483–501.
[11] ——, 'Mean value theorem and ergodicity of certain random walks', *Compositio Math.* **48** (1983), 129–137.
[12] S. Thangavelu, 'Spherical means on the Heisenberg group and a restriction theorem for the symplectic Fourier transform', preprint.

Indian Statistical Institute
Bangalore 560 059
India

The Australian National University
Canberra
ACT 2601
Australia