Notes on Spherical Harmonics.

By JOHN DOUGALL, M.A., D.Sc.

(Read 12th December 1913. Received 30th March 1914).

This paper contains

(*a*) an explicit solution of the problem of finding a function which is harmonic within a given sphere and takes at the surface the same value as a given rational integral homogeneous function of the rectangular coordinates of a point referred to the centre of the sphere as origin;

(*b*) a concise symbolical expression for the integral, over the surface of the sphere, of the product of any three rational integral spherical harmonics.

I believe that the results are new. The expression, for *two* spherical harmonics, corresponding to that of (*b*) has been given by Cayley.*

1. Write as usual

$$r^2 \equiv x^2 + y^2 + z^2, \quad \ldots\ldots(1)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \quad \ldots\ldots(2)$$

If u_n is a homogeneous function of x, y, z of degree n, it is easy to prove that

$$\nabla^2(r^p u_n) = p(p + 2n + 1) r^{p-2} u_n + r^p \nabla^2 u_n, \quad \ldots\ldots(3)$$

and thence that, if n is a positive integer, the function defined by the finite series

$$u_n - \frac{1}{2(2n-1)} r^2 \nabla^2 u_n + \frac{1}{2 \,.\, 4(2n-1)(2n-3)} r^4 \nabla^4 u_n - \ldots\ldots(4)$$

is a rational integral spherical harmonic of degree n, which we may denote by $H(u_n)$. From (4) it is obvious that, if $H(u_n)$ vanishes, u_n contains r^2 as a factor.

---

* *Cf.* a paper "On a certain expression for a spherical harmonic, with some extensions," *Proc. Edin. Math. Soc.*, Vol. VIII.

Interesting examples of (4) are obtained by taking

$$u_n = z^n \qquad \ldots\ldots (5)$$

$$\left.\begin{aligned} u_n &= (x^2+y^2)^{\frac{n}{2}}, \quad n \text{ even},\\ u_n &= (x^2+y^2)^{\frac{n-1}{2}} z, \quad n \text{ odd}, \end{aligned}\right\} \qquad \ldots\ldots (6)$$

$$u_n = (x+iy)^m z^{n-m}. \qquad \ldots\ldots (7)$$

With regard to (7) it is worth noting that the series (4) will terminate for any value of n, provided $n-m$ is a positive integer.

2. It is well known, and will be proved immediately, that u_n can be expanded in a finite series of the form

$$u_n = Y_n + r^2 Y_{n-2} + r^4 Y_{n-4} + \ldots\ldots, \qquad (8)$$

where Y_n, Y_{n-2}, ... are harmonic functions of the degrees indicated by the suffixes. The usual way of determining Y_n, Y_{n-2}, ... is as follows. By successive applications of ∇^2, we get from (8) with the help of (3)

$$\nabla^2 u_n = 2(2n-1)Y_{n-2} + 4(2n-3)r^2 Y_{n-4} + \ldots\ldots, \qquad (9)$$

$$\nabla^4 u_n = \qquad 2 \cdot 4(2n-3)(2n-5)Y_{n-4} + \ldots\ldots, \qquad (10)$$

and so on.

The last of such equations will give the lowest harmonic Y explicitly; then the last equation but one will give the lowest harmonic Y but one; and so on. But it does not seem to have been noticed that a general formula for any of the harmonics Y can easily be found. For it is obvious that an identity exists of the form

$$u_n = H(u_n) + C_2 r^2 H(\nabla^2 u_n) + C_4 r^4 H(\nabla^4 u_n) + \ldots\ldots, \qquad (11)$$

where C_2, C_4, ... are constants.

This is easily seen if we write down a few terms in full, say

$$\left.\begin{aligned} u_n = u_n &- \frac{1}{2(2n-1)} r^2 \nabla^2 u_n + \frac{1}{2.4(2n-1)(2n-3)} r^4 \nabla^4 u_n - \ldots \\ &+ \frac{r^2}{2(2n-1)}\left\{\nabla^2 u_n - \frac{1}{2(2n-5)} r^2 \nabla^4 u_n + \ldots\right\} \\ &\qquad + \frac{r^4}{2 \cdot 4(2n-3)(2n-5)}\left\{\nabla^4 u_n - \ldots\right\} \end{aligned}\right\} \qquad (12)$$

Here the coefficient of r^2 in the second line is taken so that the term in $r^2\nabla^2 u_n$ may disappear; then the coefficient of r^4 in the third line so that the term in $r^4\nabla^4 u_n$ may disappear; and so on.

But, once the expansion (11) is seen to be possible, the coefficients C_2, C_4, ... are easily obtained thus. Operate on both sides of (11) by ∇^2, ∇^4, ... Then, as at (9), (10)

$$\nabla^2 u_n = C_2 . 2(2n-1) H(\nabla^2 u_n) + C_4 . 4(2n-3) r^2 H(\nabla^4 u_n) + \ldots\ldots, \quad (13)$$

$$\nabla^4 u_n = \qquad C_4 . 2 . 4(2n-3)(2n-5) H(\nabla^4 u_n) + \ldots, \quad (14)$$

and so on.

Now the process described at (9), (10) shows that only one expansion of the form (8) is possible. And (11) then shows that the highest harmonic in the expansion of $\nabla^2 u_u$ in the form (8) is $H(\nabla^2 u_n)$. Thus from (13)

$$C_2 . 2(2n-1) = 1 \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

Similarly, $\quad C_4 . 2 . 4(2n-3)(2n-5) = 1, \ldots\ldots\ldots\ldots \quad (16)$

and, generally,

$$C_{2p} . (2 . 4 \ldots 2p)(2n-2p+1)(2n-2p-1) \ldots (2n-4p+3) = 1 \ldots \quad (17)$$

The values of u_n in (5), (6), (7) give useful examples of (11). The solution of the problem stated in (*a*) follows easily, for it is seen at once from (11) that the function which is harmonic within the sphere $r = a$, and takes the same value as u_n at the surface, is

$$H(u_n) + C_2 a^2 H(\nabla^2 u_n) + C_4 a^4 H(\nabla^4 u_n) + \ldots\ldots\ldots \quad (18)$$

3. Since the surface integral of any rational integral harmonic of degree greater than zero vanishes when taken over the surface of a sphere with centre at the origin, it follows that the surface integral of the function u_n depends only on the constant term, or harmonic of degree zero, in the expansion (11).

If n is odd, this harmonic does not occur, and the surface integral vanishes.

If n is even, the last term of (11) is

$$\frac{1}{(n+1)!} r^n \nabla^n u_n,$$

and in this case, if $n = 2p$, and the radius of the sphere is a,

$$\iint u_{2p} dS_a = \frac{4\pi}{(2p+1)!} a^{2p+2} \nabla^{2p} u_{2p} \ldots\ldots\ldots\ldots \quad (19)$$

4. For an application of (19) take

$$u_{2p} = (a_1 x + b_1 y + c_1 z)^l (a_2 x + b_2 y + c_2 z)^m,$$

where

$$\left. \begin{array}{l} a_1^2 + b_1^2 + c_1^2 = 0, \\ a_2^2 + b_2^2 + c_2^2 = 0. \end{array} \right\} \ldots\ldots\ldots\ldots \quad (20)$$

https://doi.org/10.1017/S0013091500035008 Published online by Cambridge University Press

As a consequence of (18) the functions $(a_1x+b_1y+c_1z)^l$ and $(a_2x+b_2y+c_2z)^m$ are both harmonic.

Hence

$$\nabla^2 u_{2p} = 2lm(a_1a_2+b_1b_2+c_1c_2)(a_1x+b_1y+c_1z)^{l-1}(a_2x+b_2y+c_2z)^{m-1},$$

and, continuing the process of taking ∇^2, we see that $\nabla^{2p}u_{2p}$ will vanish unless $l=m$, but that

$$\nabla^{2p}\{(a_1x+b_1y+c_1z)^p(a_2x+b_2y+c_2z)^p\} \\ = 2^p(p\,!)^2(a_1a_2+b_1b_2+c_1c_2)^p \quad \text{............ (21)}$$

Hence, if l, m are different,

$$\iint(a_1x+b_1y+c_1z)^l(a_2x+b_2y+c_2z)^m dS_a = 0\,; \quad \text{......... (22)}$$

but

$$\iint(a_1x+b_1y+c_1z)^p(a_2x+b_2y+c_2z)^p dS_a \\ = 4\pi a^{2p+2}\frac{2^p(p\,!)^2}{(2p+1)!}(a_1a_2+b_1b_2+c_1c_2)^p. \quad \text{......... (23)}$$

5. These results may be generalised so as to give expressions for the surface integral of any two spherical harmonics. For if u_l is a harmonic of degree l, we may write symbolically

$$u_l(x,\,y,\,z) = \frac{1}{l\,!}\left(x\frac{\partial}{\partial a_1}+y\frac{\partial}{\partial b_1}+z\frac{\partial}{\partial c_1}\right)^l . \, u_l(a_1,\,b_1,\,c_1) \quad \text{...... (24)}$$

where $$\left\{\left(\frac{\partial}{\partial a_1}\right)^2+\left(\frac{\partial}{\partial b_1}\right)^2+\left(\frac{\partial}{\partial c_1}\right)^2\right\} . \, u_l(a_1,\,b_1,\,c_1)=0. \quad \text{............ (25)}$$

Applying the processes of Art. 4 to any harmonics u_l, u_m, u_p, in their symbolical forms, we find

$$\iint u_l u_m dS_a = 0, \text{ if } l \neq m, \quad \text{.................................. (26)}$$

but

$$\iint u_p v_p \, dS_a \\ = 4\pi a^{2p+2}\frac{2^p}{(2p+1)\,!}\left(\frac{\partial}{\partial a_1}\frac{\partial}{\partial a_2}+\frac{\partial}{\partial b_1}\frac{\partial}{\partial b_2}+\frac{\partial}{\partial c_1}\frac{\partial}{\partial c_2}\right)^p . \\ u_p(a_1,\,b_1,\,c_1)\,v_p(a_2,\,b_2,\,c_2) \quad \text{......... (27)}$$

6. Results analogous to those of the last two articles may be obtained for the integral of the product of three harmonics.

https://doi.org/10.1017/S0013091500035008 Published online by Cambridge University Press

Let
$$\left.\begin{aligned} u_l &= (a_1x + b_1y + c_1z)^l, \\ u_m &= (a_2x + b_2y + c_2z)^m, \\ u_n &= (a_3x + b_3y + c_3z)^n, \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

where
$$\left.\begin{aligned} a_1^2 + b_1^2 + c_1^2 &= 0, \\ a_2^2 + b_2^2 + c_2^2 &= 0, \\ a_3^2 + b_3^2 + c_3^2 &= 0. \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

If $l+m+n$ is odd, the integral of the product of the three harmonics obviously vanishes.

Suppose then that $l+m+n$ is even and equal to $2p$.

In order to find $\nabla^{2p}(u_l u_m u_n)$ consider

$$\begin{aligned} u &= \{\lambda(a_1x + b_1y + c_1z) + \mu(a_2x + b_2y + c_2z) + \nu(a_3x + b_3y + c_3z)\}^{2p} \\ &= \{(\lambda a_1 + \mu a_2 + \nu a_3)x + (\lambda b_1 + \mu b_2 + \nu b_3)y + (\lambda c_1 + \mu c_2 + \nu c_3)z\}^{2p}. \end{aligned}$$

We have

$$\begin{aligned} \nabla^{2p}u &= (2p)!\{(\lambda a_1 + \mu a_2 + \nu a_3)^2 + (\lambda b_1 + \mu b_2 + \nu b_3)^2 + (\lambda c_1 + \mu c_2 + \nu c_3)^2\}^p \\ &= (2p)!\,2^p\{\mu\nu(a_2a_3 + b_2b_3 + c_2c_3) + \nu\lambda(a_3a_1 + b_3b_1 + c_3c_1) \\ &\qquad + \lambda\mu(a_1a_2 + b_1b_2 + c_1c_2)\}^p, \ldots (30) \end{aligned}$$

on account of (29).

Now equate the coefficients of $\lambda^l\mu^m\nu^n$ on the two sides of (30). On the right side the coefficient will come from a single term involving, say, $(\mu\nu)^\alpha(\nu\lambda)^\beta(\lambda\mu)^\gamma$,

where $$\beta + \gamma = l, \quad \gamma + \alpha = m, \quad \alpha + \beta = n,$$

so that

$$\alpha = \frac{m+n-l}{2}, \quad \beta = \frac{n+l-m}{2}, \quad \lambda = \frac{l+m-n}{2}.$$

Hence if any one of l, m, n is greater than the sum of the other two, then

$$\nabla^{2p}(u_l u_m u_n) = 0, \quad\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

but if any two of l, m, n are together not less than the third, then

$$\frac{(2p)!}{l!\,m!\,n!}\nabla^{2p}(u_l u_m u_n) = (2p)!\,2^p\frac{p!}{\alpha!\,\beta!\,\gamma!}\,.$$
$$(a_2a_3 + b_2b_3 + c_2c_3)^\alpha(a_3a_1 + b_3b_1 + c_3c_1)^\beta(a_1a_2 + b_1b_2 + c_1c_2)^\gamma \ldots\ldots (32)$$

Thus, by (19)

$$\iint u_l u_m u_n \, dS_a = 0, \quad \text{.........................} \quad (33)$$

if any one of l, m, n is greater than the sum of the other two; otherwise

$$\iint u_l u_m u_n dS_a$$

$$= 4\pi a^{l+m+n+2} \frac{2^{\frac{l+m+n}{2}} \left(\frac{l+m+n}{2}\right)! \, l! \, m! \, n!}{(l+m+n+1)! \left(\frac{m+n-l}{2}\right)! \left(\frac{n+l-m}{2}\right)! \left(\frac{l+m-n}{2}\right)!} \cdot$$

$$(a_2 a_3 + b_2 b_3 + c_2 c_3)^{\frac{m+n-l}{2}} (a_3 a_1 + b_3 b_1 + c_3 c_1)^{\frac{n+l-m}{2}} (a_1 a_2 + b_1 b_2 + c_1 c_2)^{\frac{l+m-n}{2}}. \quad (34)$$

Verify that this reduces to (23) when $l = p$, $m = p$, $n = 0$.

The generalised results for any three spherical harmonics may now be written down at once as at (26), (27).

https://doi.org/10.1017/S0013091500035008 Published online by Cambridge University Press