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ON THE DEGREE OF AN INDECOMPOSABLE REPRESENTATION OF A FINITE GROUP

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Abstract

Let k be an algebraically closed field of characteristic p, and G a finite group. Let M be an indecomposable kG-module with vertex V and source X, and let P be a Sylow p-subgroup of G containing V. Theorem: If $\dim_k X$ is prime to p and if $N_G(V)$ is p-solvable, then the p-part of $\dim_k M$ equals $[P : V]$; $\dim_k X$ is prime to p if V is cyclic.

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Let k be an algebraically closed field of characteristic p. Suppose that M is an indecomposable kG-module, where G is a finite group. If V is a vertex of M and V is contained in a Sylow p-subgroup P of G, then Green (1959) has shown that $[P : V]$ divides the dimension of M over k. In this note, we show that $[P : V]$ is equal to the p-part of $\dim_k M$ if a source of M has dimension prime to p and if $N_G(V)$ is p-solvable. We may therefore determine the p-part of $\dim_k M$ if V is cyclic and $N_G(V)$ is p-solvable.

NOTATION. All modules are finitely generated left modules. We understand $L \mid M$ to mean that L is isomorphic to a direct summand of M. If M is a kH-module, where H is a subgroup of G, then M^G denotes the induced module. For a positive integer n, we denote the p-part of n by n_p.

THEOREM. *Let G be a finite group and let M be an indecomposable kG-module with vertex V and source X. If $p \nmid \dim_k X$, and $N_G(V)$ is p-solvable, then* $(\dim_k M)_p = [G : V]_p$.

Proof. We first assume that V is normal in G, so $G = N_G(V)$ is p-solvable. Let T be the inertia group of X in G:

$$T = \{g \in G : g \otimes X \cong X\}.$$

According to the results of Conlon (1964) and Tucker (1965), there exists a twisted group algebra A on T/V over k with the following property: if $A = \sum_{i=1}^{\cdot n} U_i$ is a decomposition of A into a direct sum of indecomposable left ideals, then there is a decomposition $X^G = \sum_{i=1}^{\cdot n} M_i$ into indecomposable kG-submodules of X^G such that

$$\dim_k M_i = (\dim_k X)(\dim_k U_i)[G:T], \quad 1 \leqslant i \leqslant n. \tag{1}$$

(The algebra A is a homomorphic image of $\mathrm{End}_{kT}(X^T)$.) Since M has source X, then $M \mid X^G$, so M is isomorphic to one of the U_i. We show that $(\dim_k U_i)_p = [T:V]_p$.

We prove that if U is an indecomposable summand of a twisted group algebra on a finite p-solvable group H over k, then $(\dim_k U)_p = |H|_p$. We use induction on $|H|$. Denote the twisted group algebra by $(kH)_\alpha$, where α is the factor set on H of the algebra. Let R be a normal subgroup of H such that H/R is a p-group or a p'-group; let $(kR)_\alpha$ be the twisted group algebra on R whose factor set is the restriction of α to R. Then there is an indecomposable $(kR)_\alpha$-module W such that $U \mid W^H = W \otimes_{(kR)_\alpha} (kH)_\alpha$. We apply the results of Conlon (1964) to W^H. Let S be the inertial group of W in H; then there is a twisted group algebra A' on S/R over k, with an indecomposable summand U', such that

$$\dim_k U = (\dim_k W)(\dim_k U')[H:S]. \tag{2}$$

By induction on $|H|$, we have $(\dim_k W)_p = |R|_p$. If H/R is a p-group, then so is S/R, hence A' is isomorphic to the (untwisted) group algebra $k(S/R)$ and is therefore indecomposable. Thus $U' = k(S/R)$, so $\dim_k U' = [S:R]$. We have, from (2),

$$(\dim_k U)_p = |R|_p [S:R]_p [H:S]_p = |H|_p.$$

If H/R is a p'-group, then A' is a twisted group algebra on a group whose order is prime to the characteristic of the field and is therefore semi-simple. Then U is irreducible over A', and by Curtis and Reiner (1962), Theorem 53.16, we have $\dim_k U' \mid [S:R]$, hence $\dim_k U'$ is prime to p. (Theorem 53.16 is proved in Curtis and Reiner (1962) when k is the field of complex numbers, but is valid for any algebraically closed field of characteristic prime to the group order.) Now by (2),

$$(\dim_k U)_p = |R|_p [H:S]_p = |H|_p,$$

since S/R being a p'-group implies that $|R|_p = |S|_p$.

Returning to the calculation of $(\dim_k M_p)$, we have from (1) that

$$(\dim_k M)_p = [T:V]_p\,[G:T]_p = |V|_p$$

since $(\dim_k X)_p = 1$ by hypothesis.

We now drop the assumption that V is normal in G. Set $N = N_G(V)$. By the Green correspondence (Green (1964), Theorem 2), there exists an indecomposable kN-module L, with vertex V and source X, such that

$$L^G = M \oplus \sum_{i=1}^{m}{}^{\cdot} L_i, \tag{3}$$

where each L_i is an indecomposable kG-module with vertex V_i conjugate to a subgroup of $V \cap V^{g_i}$ for some $g_i \in G-N$. Thus $|V_i| < |V|$, so the fact that $[G:V_i]_p$ divides $\dim_k L_i$ implies that

$$[G:V]_p < (\dim_k L_i)_p, \quad 1 \leqslant i \leqslant m. \tag{4}$$

Now the kN-module L has vertex V which is normal in N, and the source of L has dimension prime to p, so we have proved above that $(\dim_k L)_p = [N:V]_p$. Since $(\dim_k L^G)_p = [G:N]_p(\dim_k L)_p$, we have

$$(\dim_k L^G)_p = [G:V]_p. \tag{5}$$

Taking the dimensions of both sides of (3) and applying (4) and (5), we have $(\dim_k M)_p = [G:V]_p$, proving the theorem.

COROLLARY. *Let M be an indecomposable kG-module with cyclic vertex V, such that $N_G(V)$ is p-solvable. Then* $(\dim_k M)_p = [G:V]_p$.

PROOF. Let X be a source of M. Since V is cyclic, the indecomposable kV-modules are well known: there is (up to isomorphism) precisely one indecomposable kV-module of dimension n, for $1 \leqslant n \leqslant |V|$. Suppose that $\dim_k X = pm$; let V_1 be the subgroup of V of index p, and let X_1 be the indecomposable kV_1-module of dimension m. Then X and X_1^V are both indecomposable kV-modules of dimension pm, hence are isomorphic. However, X has vertex V, so X cannot be an induced module. We conclude that $p \nmid \dim_k X$, and the corollary follows from the theorem.

REMARK. These results need not hold if $N_G(V)$ is not p-solvable. Let G be the symmetric group on 5 letters, and let k have characteristic 3. There is an indecomposable summand M of kG with $\dim_k M = 9$ but M has vertex 1 and trivial source, and $[G:1]_3 = 3$.

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