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A Bilinear Fractional Integral on Compact Lie Groups

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Abstract. As an analog of a well-known theorem on the bilinear fractional integral on $\mathbb{R}^n$ by Kenig and Stein, we establish the similar boundedness property for a bilinear fractional integral on a compact Lie group. Our result is also a generalization of our recent theorem about the bilinear fractional integral on torus.

1 Introduction

Let G be a connected, simply connected, compact semisimple Lie group of dimension n. Following Stein [6, p. 58], the Riesz potential on G is defined by (see [3])

$$I_\alpha(f)(x) = \int_G f(xy^{-1})K_\alpha(y)dy, \quad 0 < \alpha < n$$

where

$$K_\alpha(y) = -\Gamma(\frac{\alpha}{2})^{-1}\int_0^\infty t^{\frac{\alpha}{2}}\Delta W_t(y)dt,$$

and W_t is the heat kernel on G. Thus, naturally, we define the bilinear Riesz potential

$$R_\alpha(f,g)(x) = \int_G f(xy^{-1})g(xy)K_\alpha(y)dy, \quad 0 < \alpha < n.$$

Later in this paper, we will use the property of the heat kernel to show that

$$K_\alpha(y) \simeq d(y,I)^{-n+\alpha},$$

where d is a bi-invariant metric on G and I is the identity in G. Thus $B_\alpha(f,g)$ is equivalent to the bilinear fractional integral

$$B_\alpha(f,g)(x) = \int_G f(xy^{-1})g(xy)d(y,I)^{-n+\alpha}dy.$$

Clearly, the above formulation of B_α is analogous to the bilinear fractional integral operator on $\mathbb{R}^n$

$$ß_\alpha(f,g)(x) = \int_{\mathbb{R}^n} f(x-y)g(x+y)|y|^{-n+\alpha}dy.$$

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In [6], among other things, Kenig and Stein established the boundedness of $ß_\alpha(f,g)$ from $L^r(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with $1/p = 1/q + 1/r - \alpha/n > 0$. This result was also recently obtained in the n-torus T^n, by using a transference method (see [2]). As T^n is the n-dimensional Abelian compact Lie group, it is more interesting to obtain Kenig–Stein's theorem on a general compact Lie group. This is the main purpose of this paper. We will establish the following boundedness property of R_α.

Theorem 1.1 *Assume that* $0 < \alpha < n$, $1/p = 1/q + 1/r - \alpha/n > 0$, *and* $1 \le q, r \le \infty$. *Then*

(i) *if* $1 < q, r$, *then* $\|R_\alpha(f,g)\|_{L^p(G)} \preceq \|f\|_{L^q(G)} \|g\|_{L^r(G)}$;
(ii) *if* $1 \le q, r$ *and either* q *or* r *is one, then* $\|R_\alpha(f,g)\|_{L^{p,\infty}(G)} \preceq \|f\|_{L^q(G)} \|g\|_{L^r(G)}$.

Notice that this theorem is exactly the same version of the result on $\mathbb{R}^n$ by Kenig and Stein ([6, Theorem 2]), but we want to remark that such extension to a general compact Lie group is not a trivial one. Checking the proof of Kenig and Stein, one finds that the argument involving scaling plays a significant role in their proof. But, the dilation, an important feature on $\mathbb{R}^n$, is not available on a compact Lie group G. Thus, though we will follow the idea used in [6], it becomes technically more difficult to execute. To overcome this obstacle, we will carefully treat G locally as an Euclidean space, then use compactness to achieve the global result. The plan of this paper is as follows: in Section 2, we will recall some necessary notation and definitions on a compact Lie group; we will show some basic lemmas in Section 3 and complete the proof of the theorem in Section 4.

In this paper, we use the notation $A \preceq B$ to mean that there is a positive constant C independent of all essential variables such that $A \le CB$. We use the notation $A \approx B$ to mean that there are two positive constants c_1 and c_2 independent of all essential variables such that $c_1 A \le B \le c_2 A$.

2 Notations and Definitions

Let G be a connected, simply connected, compact, semisimple Lie group of dimension n. Let g be the Lie algebra of G and τ the Lie algebra of a fixed maximal torus T in G of dimension m. Let A be a system of positive roots for (g,τ), so that $Card(A) = \frac{n-m}{2}$ and let $\delta = \sum_{\alpha \in A} \alpha$.

Let $|\cdot|$ be the norm of g induced by the negative of the Killing form B on $g^{\mathbb{C}}$, the complexification of g, then $|\cdot|$ induces a bi-invariant metric d on G. Furthermore, since $B|_{\tau^{\mathbb{C}} \times \tau^{\mathbb{C}}}$ is nondegenerate, given $\lambda \in \hom_{\mathbb{C}}(\tau^{\mathbb{C}}, \mathbb{C})$, there is a unique H_λ in $\tau^{\mathbb{C}}$ such that $\lambda(H) = B(H, H_\lambda)$ for each $H \in \tau^{\mathbb{C}}$. We let $\langle \cdot\,, \cdot \rangle$ and $\|\cdot\|$ denote the inner product and norm transferred from τ to $\hom_{\mathbb{C}}(\tau, i\mathbb{R})$ by means of this canonical isomorphism.

Let $\mathbb{N} = \{H \in \tau, \exp H = I\}$, where I is the identity in G. The weight lattice P is defined by $P = \{\lambda \in \tau : \langle \lambda, n \rangle \in 2\pi\mathbb{Z} \text{ for any } n \in \mathbb{N}\}$ with dominant weights defined by $\Lambda = \{\lambda \in P, \langle \lambda, \alpha \rangle \ge 0 \text{ for any } \alpha \in A\}$. Λ provides a full set of parameters for the equivalent classes of unitary irreducible representation of G: for $\lambda \in \Lambda$, the representation U_λ has dimension

$$d_\lambda = \prod_{\alpha \in A} \frac{\langle \lambda + \delta, \alpha \rangle}{\langle \delta, \alpha \rangle},$$

and its associated character is

$$\chi_\lambda(\xi) = \frac{\sum_{w \in W} \epsilon(w) e^{i\langle w(\lambda+\delta), \xi \rangle}}{\sum_{w \in W} e^{i\langle w\delta, \xi \rangle}},$$

where $\xi \in \tau, W$ is the Weyl group and $\epsilon(w)$ is the signature of $w \in W$. Let $X_1, X_2, \dots, X_n$ be an orthonormal basis of g. Form the Casimer operator

$$\Delta = \sum_{i=1}^{n} X_i^2.$$

This is an elliptic bi-invariant operator on G that is independent of the choice of orthonormal basis of g. The solution of the heat equation on $G \times \mathbb{R}^+$,

$$\Delta\Phi(x,t) = \frac{d\Phi}{dt}(x,t), \quad \Phi(x,0) = f(x),$$

$f \in L^1(G)$ is given by $\Phi(x,t) = W_t * f(x)$, where W_t is the Gauss–Weierstrass kernel (heat kernel). It is well known that W_t is a central function, and one can write it as for $\xi \in \tau$ and $t > 0$,

$$W_t(\xi) = \sum_{\lambda \in \Lambda} e^{-t(\|\lambda+\delta\|^2 - \|\delta\|^2)} d_\lambda \chi_\lambda(\xi)$$

for $\xi \in \tau$ and $t > 0$. It is easy to see that W_t satisfies the semi-group property $W_{t+s} = W_t * W_s$ for any $s, t > 0$. Using the heat kernel, we define the bilinear Riesz potential on G by

$$R_\alpha(f,g)(x) = \int_G f(xy^{-1}) g(xy) K_\alpha(y) dy, \quad 0 < \alpha < n,$$

where

$$K_\alpha(y) = -\Gamma\Big(\frac{\alpha}{2}\Big)^{-1} \int_0^\infty t^{\frac{\alpha}{2}} \Delta W_t(y) dt.$$

It is easy to check that if $g(x) \equiv 1$, then B_α is the Riesz potential I_α studied in [4,7].

3 Some Lemmas

For an s-multi-index $J = (j_1, j_2, \dots, j_s)$, denote $X^J = \Pi_{k=1}^{s} X_{j_k}$. Let $H^{2,s}$ be the Sobolev space of functions f on G for which any $X_{j_1}, X_{j_2}, \dots, X_{j_s} \in g$, $X^J f \in L^2(G)$. A norm on the subspace of central functions in $H^{2,s}$ is

$$\|f\|_{H^{2,s}} = \Big\{ \sum_{\lambda \in \Lambda} |f_\lambda|^2 \|\lambda + \delta\|^{2s} d_\lambda \Big\}^{\frac{1}{2}},$$

where f_λ are the Fourier coefficients of f. Since the heat kernel W_t is a central function, we have the following estimate of W_t.

Lemma 3.1 *Fix a $\sigma > 0$. For any multi-index J, $\|X^J W_t\|_{L^\infty(G)} \preceq t^{-N}$ for any $N > 0$, uniformly for $t > \sigma$.*

Proof Using Hölder's inequality, the semi group property of W_t, and the left invariance of X^J, one has

$$\begin{aligned}\|X^J W_t\|_{L^\infty(G)} &= \|X^J W_{t/2} * W_{t/2}\|_{L^\infty(G)} \preceq \|X^J W_{t/2}\|_{L^2(G)} \|W_{t/2}\|_{L^2(G)} \\ &\approx \|W_{t/2}\|_{H^{2,s}(G)} \|W_{t/2}\|_{L^2(G)}, \quad \text{with} \quad s = |J|.\end{aligned}$$

Thus, the lemma follows easily from the definition of W_t. ∎

By the Poisson summation formula (see [3], or [1]), we know that

$$W_t(\xi) = \frac{e^{t\|\rho\|^2} t^{-m/2}}{D(\xi)} \sum_{\lambda \in \mathbb{N}} \Big(\prod_{\alpha \in A} \langle \xi + \lambda, \alpha \rangle e^{-\frac{\|\xi+\lambda\|^2}{4t}} \Big),$$

where

$$D(\xi) = \sum_{w \in W} e^{i\langle w\delta, \xi \rangle}.$$

Using this expression of the heat kernel, we can obtain the following estimate.

Lemma 3.2 $|K_\alpha(y)| \preceq d(y, I)^{-n+\alpha}$.

Proof Fix a positive $\sigma > 0$. We write

$$|K_\alpha(y)| \preceq \Big| \int_0^\sigma t^{\frac{\alpha}{2}} \Delta W_t(y) dt \Big| + \Big| \int_\sigma^\infty t^{\frac{\alpha}{2}} \Delta W_t(y) dt \Big|.$$

By Lemma 3.1, the second integral above is $O(1)$. Let U be a neighborhood of 0 in τ such that it translates by elements of Λ are all disjoint, and let $\eta(x)$ be a C^∞ function supported on U, radial and identically one on a neighborhood of 0. One defines two modified kernels K_t and V_t by

$$V_t(\xi) = e^{2t\|\rho\|^2} t^{-n/2} \sum_{\lambda \in \mathbb{N}} e^{-\frac{\|\xi+\lambda\|^2}{4t}},$$

$$K_t(\xi) = e^{2t\|\rho\|^2} t^{-n/2} \sum_{\lambda \in \mathbb{N}} \eta(\xi + \lambda) e^{-\frac{\|\xi+\lambda\|^2}{4t}}.$$

By [3, Theorem 4], it is known that for any pair of integers s and N,

$$\|V_t - K_t\|_{H^{2,s}(G)} = O(t^N), \quad t \to 0.$$

Also, by [3, Theorem 2], we know that given any pair of integers s and N, there is an integer L such that

$$\|\Delta_{L,t} V_t - W_t\|_{H^{2,s}(G)} = O(t^N), \quad t \to 0,$$

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where

$$\Delta_{L,t} = \sum_{j=0}^{M} t^j D_{j,L}, \quad M = L(n-m)/2,$$

and $D_{j,L}$, $j = 0, 1, \dots, M$, are differential operators of order j, which are invariant under both left and right translations. Thus, we have

$$\Big| \int_0^\sigma t^{\frac{\alpha}{2}} \Delta W_t(y) dt \Big| \le \int_0^\sigma t^{\frac{\alpha}{2}} \{ \|\Delta(W_t - \Delta_{L,t} V_t)\|_\infty + \|\Delta\{\Delta_{L,t}(V_t - K_t)\}\|_\infty \} dt + \Big| \int_0^\sigma t^{\frac{\alpha}{2}} (\Delta\Delta_{L,t} K_t)(y) dt \Big|.$$

By the Sobolev embedding theorem

$$\int_0^\sigma t^{\frac{\alpha}{2}} \Big\{ \|\Delta(W_t - \Delta_{L,t} V_t)\|_\infty + \|\Delta\Delta_{L,t}(V_t - K_t)\|_\infty \Big\} dt \preceq \int_0^\sigma t^{\frac{\alpha}{2}} \Big\{ \|\Delta(W_t - \Delta_{L,t} V_t)\|_{H^{2,s}(G)} + \|\Delta\Delta_{L,t}(V_t - K_t)\|_{H^{2,s}(G)} \Big\} dt$$

for some $s > n/2 + 3 + M$. By [3, Theorems 2 and 4], we now obtain that

$$\Big| \int_0^\sigma t^{\frac{\alpha}{2}} \Delta W_t(y) dt \Big| \preceq O(1) + \Big| \int_0^\sigma t^{\frac{\alpha}{2}} (\Delta\Delta_{L,t} K_t)(y) dt \Big|.$$

Recalling that the function K_t, considered as a function on G, is supported on a small neighborhood V_I of I, one introduces on this neighborhood the regular coordinates $(\xi_1, \dots, \xi_n)$, where $(\xi_1, \dots, \xi_n) \to \exp(\sum_{j=1}^n \xi_j X_j) = y$. In these coordinates,

$$K_t(y) = e^{2t\|\rho\|^2} t^{-n/2} \eta(\xi) e^{-\frac{\|\xi\|^2}{4t}}, \ \|\xi\| \simeq d(y, I).$$

By the proof of [3, Lemma 5], it is easy to see that

$$|\Delta\Delta_{L,t} K_t(\xi)| \preceq t^{-n/2-2} \|\xi\|^2 e^{-\frac{\|\xi\|^2}{4t}}.$$

Thus,

$$\Big| \int_0^\sigma t^{\frac{\alpha}{2}} (\Delta\Delta_{L,t} K_t)(y) dt \Big| \preceq \Big| \int_0^\sigma t^{\frac{\alpha}{2}} t^{-n/2-2} \|\xi\|^2 e^{-\frac{\|\xi\|^2}{4t}} dt \Big| \preceq \|\xi\|^{-n+\alpha} \int_0^\infty u^{n/2+2-\frac{\alpha}{2}} e^{-u} du \simeq d(y, I)^{-n+\alpha}.$$

Notice $(\Delta\Delta_{L,t} K_t)(y) \equiv 0$ if $y \notin V_I$. We obtain

$$|K_\alpha(y)| \preceq d(y, I)^{-n+\alpha} + O(1).$$

Since we may assume $\mathrm{diam}(G) = 1$, the lemma is proved. ∎

Now we fix a sufficiently larger integer $k_0 > 0$ that is to be determined later. Let $r > 0$ be a fixed small positive number and Φ be a C^∞-diffeomorphism from the neighborhood $V_I(r) = \{ y \in G : d(y, I) < r\}$ to a neighborhood $\widetilde{N}$ of the origin in $\mathbb{R}^n$, which satisfies $d(u, v) \approx |\Phi(u) - \Phi(v)|$, for all $u, v \in V_I(r)$, where $|\Phi(u) - \Phi(v)|$ is the Euclidean distance between $\Phi(u)$ and $\Phi(v)$. Recall that $d(u, v) \approx |\Phi(u) - \Phi(v)|$ means that there are positive constants c_1 and c_2 such that

$$c_1|\Phi(u) - \Phi(v)| \leq d(u, v) \leq c_2|\Phi(u) - \Phi(v)|$$

for all $u, v \in V_I(r)$. Without loss of generality, we may assume $c_1 = 1/2$ and $c_2 = 2$. For each $x \in G$, let

$$V_x(r) = \{xu \in G : u \in V_I\}.$$

Let Φ_x be defined on $V_x(r)$ by $\Phi_x(y)=\Phi(x^{-1}y)$ for $y \in V_x(r)$. Clearly, $V_x(r)$ is a neighborhood of x and Φ_x is a C^∞-diffeomorphism from $V_x(r)$onto $\widetilde{N}$. In addition, for any $\xi, \eta \in V_x(r)$, there are $u, v \in V_I(r)$ such that $xu = \xi$, $xv = \eta$. Thus

$$d(\xi, \eta) = d(u, v) \simeq |\Phi(u) - \Phi(v)| = |\Phi_x(\xi) - \Phi_x(\eta)|.$$

For this r, we fix a large integer k_0 for which $2^{-k_0} < \frac{r}{64}$. From the open cover $\{ V_x(\frac{r}{16}) : x \in G\}$ of G,we pick a finite subcover $\{V_j(\frac{r}{16}) = V_{x_j}(\frac{r}{16}), j = 1, 2, \dots, N\}$.

Lemma 3.3 *For any $x \in V_j(\frac{r}{16})$, and $d(y, I) \leq 2^{-k_0}$ (I is the identity of G), we have $xy^{-1}, xy \in V_j(\frac{r}{8})$.*

Proof $d(xy^{-1}, x_j) \leq d(xy^{-1}, x) + d(x, x_j) \leq 2^{-k_0} + \frac{r}{16} < \frac{r}{8}$. ∎

Lemma 3.4 *Let $k \geq 0$, and*

$$B_k(f, g)(x) = \int_{d(y,I)\simeq 2^{-k}} \frac{f(xy^{-1})g(xy)}{d(y, I)^n} dy.$$

Then one has

$$\|B_k(f, g)\|_{L^{\frac{1}{2}}(G)} \leq C\|f\|_{L^1(G)}\|g\|_{L^1(G)},$$

where the constant C is independent of f, g, and k.

Proof By Fubini's Theorem, clearly we may assume $k > k_0$. Also, it is sufficient to show that for each $j = 1, 2, \dots, N$,

$$\|B_k(f, g)\|_{L^{\frac{1}{2}}(V_j(\frac{r}{16}))} \preceq \|f\|_{L^1(G)}\|g\|_{L^1(G)}.$$

Fix a j, let x_j be the center of $V_j(\frac{r}{16})$, and let f_j and g_j be defined by $f_j(x) = f(x_jx)$. Then

$$\|f\|_{L^1(G)} = \|f_j\|_{L^1(G)}, \quad \|g\|_{L^1(G)} = \|g_j\|_{L^1(G)}.$$

Thus, by a group translation, it suffices to show

$$\|B_k(f, g)\|_{L^{\frac{1}{2}}(V_I(\frac{r}{16}))} \preceq \|f\|_{L^1(G)}\|g\|_{L^1(G)}.$$

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Without loss of generality, we may assume that both f and g are Schwartz functions of nonnegative values. Recall that $\Phi(V_I(\frac{r}{16}))$ is a neighborhood of the origin in $\mathbb{R}^n$. Let $\{Q_i\}$ be the family of disjoint dyadic cubes of $\mathbb{R}^n$ with sidelength 2^{-k} and let $A_i = \Phi^{-1}(Q_i \cap \Phi(V_I(\frac{r}{16})) = \Phi^{-1}(Q_i) \cap V_I(\frac{r}{16})$, $i = 1, 2, \ldots$, where we assume that $\Phi^{-1}(Q_i)$ is the empty set if $Q_i \cap \Phi(V_I(\frac{r}{4}))$ is empty. Clearly, all these $A_i s$ are mutually disjoint, and

$$V_I\Big(\tfrac{r}{16}\Big) = \bigcup_i \Phi^{-1}(Q_i \cap \Phi\Big(V_I\Big(\tfrac{r}{16}\Big)\Big).$$

Thus,

$$\|B_k(f,g)\|^{\frac{1}{2}}_{L^{\frac{1}{2}}(V_I(\frac{r}{16}))} = \sum_i \int_{\Phi^{-1}(Q_i \cap \Phi(V_I(\frac{r}{16}))} |B_k(f,g)(x)|^{\frac{1}{2}} dx.$$

By Hölder's inequality, we have

$$\begin{aligned}&\int_{\Phi^{-1}(Q_i \cap \Phi(V_I(\frac{r}{16}))} |B_k(f,g)(x)|^{\frac{1}{2}} dx \\ &\qquad \le \Big\{ \mathrm{Vol}\Big(\Phi^{-1}\Big(Q_i \cap \Phi\Big(V_I\Big(\tfrac{r}{16}\Big)\Big)\Big)\Big) \int_{\Phi^{-1}(Q_i)\cap V_I(\frac{r}{16})} |B_k(f,g)(x)| dx \Big\}^{\frac{1}{2}}.\end{aligned}$$

Notice that we can view Φ as an isometry. It is easy to check that the volume of $\big(\Phi^{-1}\big(Q_i \cap \Phi(V_I(\frac{r}{16}))\big)\big)$ satisfies

$$\mathrm{Vol}\Big(\Phi^{-1}\Big(Q_i \cap \Phi\big(V_I(\tfrac{r}{16})\big)\Big)\Big) \preceq 2^{-nk}.$$

In addition, by Lemma 3.3,

$$\begin{aligned}&\int_{\Phi^{-1}\left(Q_i \cap \Phi(V_I(\frac{r}{16}))\right)} |B_k(f,g)(x)| dx \\ &= \int_{\Phi^{-1}\left(Q_i \cap \Phi(V_I(\frac{r}{16}))\right)} \int_{d(y,I)\simeq 2^{-k}} \frac{f(xy^{-1})g(xy)}{d(y,I)^n} dy dx \\ &\preceq 2^{nk} \int_{\Phi^{-1}(4Q_i)\cap(V_I(\frac{r}{8})} f(x)dx \int_{\Phi^{-1}(4Q_i)\cap V_I(\frac{r}{8})} g(x)dx.\end{aligned}$$

Thus

$$\begin{aligned}\|B_k(f,g)\|^{\frac{1}{2}}_{L^{\frac{1}{2}}(V_I(\frac{r}{8}))} &\preceq \sum_i \Big(\int_{\Phi^{-1}(4Q_i)\cap V_I(\frac{r}{8})} f(x)dx\Big)^{\frac{1}{2}} \Big(\int_{\Phi^{-1}(4Q_i)\cap V_I(\frac{r}{8})} g(x)dx\Big)^{\frac{1}{2}} \\ &\preceq \Big(\sum_i \int_{\Phi^{-1}(4Q_i)\cap V_I(\frac{r}{8})} f(x)dx\Big)^{\frac{1}{2}} \Big(\sum_i \int_{\Phi^{-1}(4Q_i)\cap V_I(\frac{r}{8})} g(x)dx\Big)^{\frac{1}{2}} \\ &\preceq (\|f\|_{L^1(G)}\|g\|_{L^1(G)})^{\frac{1}{2}}.\end{aligned}$$

The lemma is proved. ■

4 Proof of the Theorem

Now the proof follows the idea of Kenig–Stein in [6]. For completeness, we outline the proof. Without loss of generality, we assume that both f and g are nonnegative valued functions. By Lemma 3.1, it suffices to show the theorem for the operator $B_\alpha(f,g)$. Using a standard method we write

$$B_\alpha(f,g)(x) \simeq \sum_{k=0}^{\infty} 2^{-k\alpha} \int_{d(y,I)\simeq 2^{-k}} \frac{f(xy^{-1})g(xy)}{d(y,I)^n} dy.$$

Thus we further write $B_\alpha(f,g)(x) = D_1 + D_2$, where

$$D_1 = \sum_{k\leq K_0} 2^{-k\alpha} \int_{d(y,I)\simeq 2^{-k}} \frac{f(xy^{-1})g(xy)}{d(y,I)^n} dy,$$

$$D_2 = \sum_{k>K_0} 2^{-k\alpha} \int_{d(y,I)\simeq 2^{-k}} \frac{f(xy^{-1})g(xy)}{d(y,I)^n} dy,$$

and K_0 is to be chosen. Applying Fubini's theorem on D_1 and Lemma 3.4 on D_2, we obtain

$$\|D_1\|_{L^1(G)} \preceq 2^{K_0(n-\alpha)} \|f\|_{L^1(G)} \|g\|_{L^1(G)}.$$

$$\|D_2\|^{\frac{1}{2}}_{L^{\frac{1}{2}}(G)} \preceq 2^{-K_0\alpha/2} \|f\|^{\frac{1}{2}}_{L^1(G)} \|g\|^{\frac{1}{2}}_{L^1(G)}.$$

Fix a sufficiently large $\lambda_0 > 0$. For any $\lambda > \lambda_0$, we let

$$K_0 = \frac{\log_2 \lambda}{2n-\alpha}.$$

Then

$$\begin{aligned}|\{x \in G : B_\alpha(f,g)(x) > \lambda\}| &\leq |\{x \in G : D_1 > \lambda/2\}| + |\{x \in G : D_2 > \lambda/2\}| \\ &\preceq \frac{2^{K_0(n-\alpha)}}{\lambda} \|f\|_{L^1(G)} \|g\|_{L^1(G)} + \frac{2^{-K_0\alpha/2}}{\lambda^{\frac{1}{2}}} \|f\|^{\frac{1}{2}}_{L^1(G)} \|g\|^{\frac{1}{2}}_{L^1(G)}.\end{aligned}$$

We may assume that $\|f\|_{L^1(G)} = \|g\|_{L^1(G)} = 1$. By the choice of λ, one easily sees

$$|\{x \in G : B_\alpha(f,g)(x) > \lambda\}| \preceq \lambda^{-p} \text{ with } \tfrac{1}{p} = 2 - \tfrac{\alpha}{n}.$$

This shows

$$\|B_\alpha(f,g)\|_{L^{p,\infty}} \preceq \|f\|_{L^1(G)} \|g\|_{L^1(G)}, \text{ with } \tfrac{1}{p} = 2 - \tfrac{\alpha}{n}.$$

On the other hand, we have

$$|B_\alpha(f,g)(x)| \preceq \|g\|_{L^\infty(G)} \int_G f(xy^{-1}) d(y,I)^{-n+\alpha} dy = \|g\|_{L^\infty(G)} I_\alpha(f)(x),$$

$$|B_\alpha(f,g)(x)| \preceq \|f\|_{L^\infty(G)} \int_G g(xy) d(y,I)^{-n+\alpha} dy = \|f\|_{L^\infty(G)} J_\alpha(f)(x).$$

The boundedness of these two fractional integrals J_α and I_α are well known on G, and they have exactly the same boundedness as their Euclidean analogs. Actually, one can prove this fact by following exactly the same argument as the proof in the Euclidean case (see also [4, 8]). By the known boundedness of I_α, we have

$$\begin{aligned}\|B_\alpha(f,g)\|_{L^p(G)} &\preceq \|g\|_{L^\infty(G)} \|I_\alpha(f)\|_{L^p(G)} \\ &\preceq \|g\|_{L^\infty(G)} \|f\|_{L^q(G)} \text{ with } \tfrac{1}{p} = \tfrac{1}{q} - \tfrac{\alpha}{n}.\end{aligned}$$

Similarly, one has

$$\|B_\alpha(f,g)\|_{L^p(G)} \preceq \|f\|_{L^\infty(G)} \|g\|_{L^r(G)} \text{ with } \tfrac{1}{p} = \tfrac{1}{r} - \tfrac{\alpha}{n}.$$

Now the theorem follows by a multilinear interpolation theorem by Janson [5].

5 Extension

We can study a more general fractional integral F_α defined by

$$F_\alpha(f,g)(x) = \int_G f(x(y_m)^{-1}) g(xy_\mu) K_\alpha(y) dy, \quad 0 < \alpha < n,$$

where $y_\mu = yy\cdots y$, is the μ-product of y. Noting that $d(y_m, I) \le md(y, I)$ for any $y \in G$, and by checking the proof of Theorem 1.1, it is not difficult see that the results in Theorem 1.1 are also available for the operator $F_\alpha(f,g)$.

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