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EVEN AND ODD ENTIRE FUNCTIONS

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Abstract

We examine what can be said about a polynomial p and an entire function f given that $p \circ f$ is an even, or an odd, function.

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1. Introduction

An entire function f is an analytic map of the complex plane $\mathbb{C}$ into itself, and any entire function f can be expressed uniquely as the sum $f = f_E + f_O$ of an even entire function f_E and an odd entire function f_O in the usual way. The composition of two maps f and g is denoted by $f \circ g$. We shall use these notations throughout the paper.

In [1, page 228] the author asks for a characterization of polynomials p and entire functions f such that $p \circ f$ is even, and remarks that the existence of an algebraic relation between f_E and f_O is a necessary, but not a sufficient, condition for $p \circ f$ to be even. We provide a simple characterization in Section 2, and this shows quite clearly why the existence of an algebraic relation is necessary but not sufficient. In [1] and [2], the authors discuss criteria that imply that if $f \circ g$ is even, where f and g are entire, then f or g is even. We suggest that this may not be the right question to ask, and we develop this idea in Section 3. Finally, in Section 4 we show that [1, Example 3.1], given to illustrate [1, Theorem 3.1], is essentially the only possible example that could have been given, and we place this example in a more general context.

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2. The characterization

Given a polynomial p in one complex variable, we define polynomials Φ_p and Ψ_p in two variables by

$$\Phi_p(u, v) = p(u+v) - p(u-v), \qquad \Psi_p(u, v) = p(u+v) + p(u-v).$$

THEOREM 2.1. *Let p be a nonconstant polynomial, and let f be a nonconstant entire function. Then*

(a) *$p \circ f$ is even if and only if for all z, $\Phi_p\big(f_E(z), f_O(z)\big) = 0$;*
(b) *$p \circ f$ is odd if and only if for all z, $\Psi_p\big(f_E(z), f_O(z)\big) = 0$.*

PROOF. This is trivial, for obviously

$$\Phi_p\big(f_E(z), f_O(z)\big) = p\big(f(z)\big) - p\big(f(-z)\big),$$
$$\Psi_p\big(f_E(z), f_O(z)\big) = p\big(f(z)\big) + p\big(f(-z)\big).$$

This shows that if $p \circ f$ is even then f_E and f_O are algebraically related but, of course, only the algebraic relation $\Phi_p = 0$ (or a relation which has Φ_p as a factor) will guarantee that $p \circ f$ is even. The corresponding statement holds for odd functions and the polynomial Ψ_p. □

Notice that Theorem 2.1 enables one to characterize, for a given polynomial p, all entire functions f for which $p \circ f$ is even, and an example will suffice to illustrate this. Let $p(z) = z^3 + z$. Then $\Phi_p(u, v) = 2v(3u^2 + v^2 + 1)$ so that $p \circ f$ is even if and only if either $f_O = 0$ (so that f is even), or $3f_E^2 + f_O^2 + 1 = 0$. Of course, for algebraic reasons $\Phi_p(u, v)$ will always have a factor v, and this corresponds to the analytic fact that if f is even, then so is $p \circ f$ for every p.

The example $f(z) = \sinh z + 1$ and $p(z) = z^2 - 2z$ shows that $p \circ f$ may be even while p and f are not. Here, $p \circ f$ is even because $\Phi_p(u, v) = 4v(u-1)$ and $f_E(z) = 1$ for all z. The example $f(z) = \sin z + 1$ and $p(z) = z - 1$ shows that $p \circ f$ may be odd while p and f are not. In this case, $p \circ f$ is odd because $\Psi_p(u, v) = 2(u-1)$ and $f_E(z) = 1$ for all z.

3. Some remarks

In [2] the authors ask whether $f \circ g$ being even (where f and g are entire) implies that either f or g is even, and they then give the example $f(z) = (z-1)^2$ and $g(z) = z + 1$ to show this is not so. We suggest, however, that this may not be the correct question to ask as the given data, namely the *single* function $f \circ g$, does *not* determine f and g uniquely. This suggests the following question: *if a composition F of two entire functions is even, can we express F as a nontrivial composition of two functions at least one of which is even*? Now this question has a trivial answer as any even entire function is an entire function of z^2; so, once again, we have to modify the question. Perhaps the 'right' question is: *if f and g are entire functions, and if $f \circ g$ is even, is there a linear polynomial t such that either $f \circ t$ or $t^{-1} \circ g$ is even*? The choice of the class of linear polynomials here is natural as these are the automorphisms of $\mathbb{C}$. Note that the counterexample in [1] (and given above) is not a counterexample to the modified question for (with the same f and g as above) $f \circ g = (f \circ t) \circ (t^{-1} \circ g)$, and $f \circ t$ is even when $t(z) = z + 1$.

The authors of [1] and [2] also make frequent use of an assumption $f(0) = 0$ or $g(0) = 0$. For example, in [2] they remark that 'the problem becomes more interesting if one assumes that $g(0) = 0$' and they then prove that *if p and q are polynomials with $q(0) = 0$ and $p \circ q$ even, then p or q is even* [2, Theorem 1]. Here, the assumption $q(0) = 0$ seems arbitrary, but it appears naturally in this modified setting for now their Theorem 1 reads as follows: *if $p \circ q$ is even, then there is a linear polynomial t such that $p \circ t$ or $t^{-1} \circ q$ is even*. Thus their Theorem 1 answers the question posed above when f and g are polynomials.

4. An example

Theorem 3.1 in [1] states that: *if f is entire, and if $f_E^2 + f_O^2 = 1$, then $p \circ f$ is even, where $p(z) = z^4 - 2z^2$*, and this is then illustrated by the example $f(z) = \cos z + \sin z$. We shall now show that this is essentially the only example that could have been given here. First, if $p(z) = z^4 - 2z^2$ then $\Phi_p(u, v)$ has a factor $u^2 + v^2 - 1$, and this is why $p \circ f$ is even when $f_E^2 + f_O^2 = 1$.

Suppose now that the two entire functions s and t satisfy $s^2 + t^2 = 1$ throughout $\mathbb{C}$. Then $(s + it)(s - it) = 1$, so that neither factor vanishes, and this means that there is an entire function h such that $s + it = e^{ih}$ and $s - it = e^{-ih}$. Thus $s(z) = \cos h(z)$ and $t(z) = \sin h(z)$. Now suppose that, in addition, s is even and t is odd. Then as

$$\sin h_E(z) \sin h_O(z) = -s_O(z) = 0,$$
$$\sin h_E(z) \cos h_O(z) = t_E(z) = 0,$$

we see that $\sin h_E(z) = 0$ for all z; thus $h_E(z) = k\pi$ for some integer k. It follows from this discussion that if f is entire, and if $f_E^2 + f_O^2 = 1$ then, by taking $f = s + t$, we see that

$$f_E(z) = s(z) = (-1)^k \cos h_O(z), \quad f_O(z) = t(z) = (-1)^k \sin h_O(z).$$

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There are many algebraic relations for which a similar result holds, and the explanation lies in the theory of uniformization of algebraic curves, and in the fact that an algebraic relation or, equivalently, an algebraic curve, is essentially a compact Riemann surface. In the example above, the algebraic curve is given by $u^2 + v^2 - 1 = 0$, the corresponding Riemann surface is the Riemann sphere, and the algebraic curve is uniformized by the pair of functions $\sin z$ and $\cos z$. Not every algebraic curve arises in this way, for Picard proved that if an algebraic curve can be uniformized by a pair of entire functions (in our case, by f_E and f_O), then the corresponding Riemann surface is topologically a sphere. We end with an example to illustrate this idea, and this example will also serve as another application of Theorem 2.1.

EXAMPLE 4.1. Let $P(u, v) = (8u + 1)u^2 - 9v^2$, and suppose that s and t are nonconstant entire functions such that $P(s, t) = 0$, where s is even and t is odd. We shall show that there is an odd entire function g such that

$$s(z) = \tfrac{1}{8}\big(4g(z)^2 - 1\big), \quad t(z) = \tfrac{1}{12}\big(g(z)[4g(z)^2 - 1]\big). \tag{4.1}$$

PROOF. We begin with the graph of P, namely

$$\mathbf{P} = \{(u, v) \in \mathbb{C} \times \mathbb{C} : P(u, v) = 0\},$$

and this is embedded in projective space $\mathbb{P}_2$ in the usual way by using homogeneous co-ordinates. The projective model of this graph is the set $\mathbf{P}^{\#}$ of projective points (u, v, w) for which $(8u + w)u^2 + 9v^2w = 0$, and this meets the line at infinity $(w = 0)$ at the single projective point $(0, 1, 0)$. Thus $\mathbf{P}^{\#}$ is obtained from $\mathbf{P}$ (or, strictly, a projective copy of $\mathbf{P}$) by adding the single projective point $(0, 1, 0)$, and $\mathbf{P}^{\#}$ is conformally equivalent to the extended complex plane $\mathbb{C}_\infty$ (for, as we shall see below, $\mathbf{P}$ is uniformized by two polynomials p and q). We shall not need these facts, but they underpin much of our argument.

Now let

$$p(z) = z(z + 1)/2, \quad q(z) = z(z + 1)(2z + 1)/6,$$

so that $3q(z) = (2z + 1)p(z)$. Then

$$P\big(p(z), q(z)\big) = p(z)^2\Big[8p(z) + 1 - (2z + 1)^2\Big] = 0,$$

so that the map $\theta : z \mapsto \big(p(z), q(z)\big)$ maps $\mathbb{C}$ into $\mathbf{P}$. In fact, θ is a bijection of $\mathbb{C}\backslash\{0, -1\}$ onto $\mathbf{P}\backslash\{(0, 0\}$ because the point (u_0, v_0) on $\mathbf{P}$, where $u_0 \neq 0$, is the θ-image of exactly one point in $\mathbb{C}$, namely $(3v_0 - u_0)/(2u_0)$.

Now suppose that s and t are entire functions with $P(s, t) = 0$, s even and t odd, and let $\mu : \mathbb{C} \to \mathbf{P}$ be given by $\mu(z) = \big(s(z), t(z)\big)$. Then the map $h = \theta^{-1} \circ \mu$

is defined and analytic on the set $\mathbb{C}\backslash Z$, where Z is the set of points z in $\mathbb{C}$ where $s(z) = t(z) = 0$. Clearly, the points of Z are isolated, and if z is near a point of Z, then $h(z)$ is in some neighbourhood of 0 or -1; thus each point of Z is a removable singularity of h, and so h is entire.

Next, $\theta \circ h = \mu$ so that $p \circ h = s$. By assumption, s is even, so, from Theorem 2.1, $\Phi_p(h_E, h_O) = 0$. Now as $p(z) = z(z+1)/2$, we see that $\Phi_p(u, v) = v(2u+1)$, so that h is even or $2h_E + 1 = 0$. As $q \circ h = t$, h is not even (else t is even and odd and hence identically zero); thus $2h_E + 1 = 0$ and this means that $h(z) + h(-z) = -1$. We now let $g = h + 1/2$; then g is odd, and $s = p \circ h = (4g^2 - 1)/8$ and similarly for t. The proof is complete. □

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