IDENTITIES FOR MULTIPLICATIVE FUNCTIONS

M.V. Subbarao[1] and A.A. Gioia[2]

(received November 11, 1965)

1. Introduction. Throughout this paper the arithmetic functions $L(n)$ and $w(n)$ denote respectively the number and product of the distinct prime divisors of the integer $n > 1$, with $L(1) = 0$ and $w(1) = 1$. Also let

$$C(m, n) = \begin{cases} (-1)^{L(n)}, & \text{if } w(m) = w(n) \\ 0, & \text{otherwise;} \end{cases}$$

$$E_o(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

We recall that an arithmetic function $f(n)$ is said to be multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$, where (m, n) denotes as usual the greatest common divisor of m and n. It is known (Vaidyanathaswamy [6], [7, section VI]; for another proof, Gioia [3],) that every multiplicative function f satisfies the identity

$$(1.1) \qquad f(mn) = \sum_{\substack{a \mid m \\ b \mid n}} f(m/a)\, f(n/b)\, f^{-1}(ab)\, C(a, b),$$

---

[1] Partially supported by National Science Foundation Grant No. GP 1222.

[2] This author's contribution formed a part of his Ph.D. thesis submitted to the University of Missouri in January, 1964.

Canad. Math. Bull. vol. 10, no. 1, 1967

https://doi.org/10.4153/CMB-1967-007-x Published online by Cambridge University Press

where m and n are arbitrary positive integers and f^{-1} is the Dirichlet inverse of f defined by

$$\sum_{d|n} f(d)f^{-1}(n/d) = E_o(n) .$$

We give here a generalization of this identity which holds in the case of generalized Dirichlet products of arithmetic functions introduced by the authors [5]. We also obtain another identity valid in the case of unitary products.

2. Preliminaries. Let K(n) be a fixed arithmetic function satisfying K(1) = 1 and for arbitrary positive integers a, b, c,

(2.1) $$K((a, b))K((ab, c)) = K((a, bc))K((b, c)) .$$

For any arithmetic functions f and g, their generalized Dirichlet product f.g is the arithmetic function defined by

$$(f.g)(n) = \sum_{d|n} f(d)\, g(n/d)\, K((d, n/d)) .$$

It can be verified (see [4]) that (2.1) assures the associativity of the product, and together with the condition K(1) = 1 it implies that the kernel K(n) is multiplicative. In the sequel we shall refer to the generalized Dirichlet product as the K-product. We note without proof that under the K-product operation the set of multiplicative functions forms an Abelian group G with $E_o(n)$ as the identity element. The group inverse of f in G will be denoted by f^{-1} .

On taking K(n) = 1 for all n, and $K(n) = E_o(n)$, the K-product of f and g becomes, respectively, the ordinary Dirichlet product $\sum_{d|n} f(d)\, g(n/d)$ and the unitary product

$$\sum_{\substack{d|n \\ (d, n/d)=1}} f(d)\, g(n/d) .$$

The latter of these has been studied extensively by Eckford Cohen ([1], [2]).

https://doi.org/10.4153/CMB-1967-007-x Published online by Cambridge University Press

3. A generalized identity for the K-product. We will first note the following

LEMMA. If $(a,b) = 1$, $(a,d) = 1$, and $(b,c) = 1$ then $K((ab, cd)) = K((a,c))\,K((b,d))$.

Proof. The result follows immediately from the multiplicativity of K after observing that under the hypotheses of the lemma we have $((a,c), (b,d)) = 1$ and $(a,c)(b,d) = (ab, cd)$.

COROLLARY.

$$K\Big(\Big(\prod_{i=1}^{t} p_i^{x_i}, \prod_{i=1}^{t} p_i^{y_i}\Big)\Big) = \prod_{i=1}^{t} K((p_i^{x_i}, p_i^{y_i})), \qquad x_i, y_i \geq 0 .$$

From the definition of the function $C(a,b)$, we notice that we also have

$$(3.1) \qquad C\Big(\prod_{i=1}^{t} p_i^{x_i}, \prod_{i=1}^{t} p_i^{y_i}\Big) = \prod_{i=1}^{t} C(p_i^{x_i}, p_i^{y_i}), \qquad x_i, y_i \geq 0 .$$

We can now prove

THEOREM 1. For arbitrary positive integers m and n, every multiplicative function f satisfies the identity

$$f(mn) = \sum_{\substack{a|m \\ b|n}} f(m/a)\, f(n/b)\, f^{-1}(ab)\, K((mn/ab, ab))\, K((m/a, n/b))\, C(a,b) .$$

Proof. Define the function

$$S(m,n) = \sum_{\substack{a|m \\ b|n}} f(m/a) f(n/b) f^{-1}(ab) K((mn/ab, ab)) K((m/a, n/b)) C(a,b) .$$

We shall show that $S(m,n) = f(mn)$ for all $m, n \geq 1$. First, let $m = p^x$ and $n = p^y$, where p is prime and $x, y \geq 1$. Then

$$S(m,n) = f(m)f(n)K((m,n))$$

$$- \sum_{\substack{a|m \\ a>1}} f(\frac{m}{a})K((\frac{m}{a}, na)) \sum_{\substack{b|n \\ b>1}} f(\frac{n}{b})f^{-1}(ab)K((\frac{n}{b}, ab)),$$

where we have used equation (2.1).

Since $a|p^x$,

$$0 = E_o(na) = \sum_{b|a} f(nb)\, f^{-1}(a/b)\, K((nb, a/b))$$

$$+ \sum_{\substack{b|n \\ b>1}} f(n/b)\, f^{-1}(ab)\, K((n/b, ab)),$$

so that

$$S(m,n) = f(m)f(n)K((m,n))$$

$$+ \sum_{\substack{a|m \\ a>1}} f(\frac{m}{a})K((\frac{m}{a}, na)) \sum_{b|a} f(nb)f^{-1}(\frac{a}{b})K((nb, \frac{a}{b}))$$

$$= \sum_{a|m} \sum_{b|a} f(m/a)f(nb)f^{-1}(a/b)K((nb, a/b))K((na, m/a)) .$$

Interchanging the order of summation and using (2.1) again,

$$S(m,n) = \sum_{b|m} f(nb)K((nb, m/b)) \sum_{\substack{a|m \\ b|a}} f(m/a)f^{-1}(a/b)K((a/b, m/a))$$

$$= \sum_{b|m} f(nb)K((nb, m/b))\, E_o(m/b) = f(mn) .$$

https://doi.org/10.4153/CMB-1967-007-x Published online by Cambridge University Press

Furthermore, since $S(1,n) = f(n)$ and $S(m,1) = f(m)$, we see that $S(m,n) = f(mn)$ for $m = p^x$, $n = p^y$ with $x, y \geq 0$. Now from the above corollary and (3.1) we have

$$S\left(\prod_{i=1}^{t} p_i^{x_i}, \prod_{i=1}^{t} p_i^{y_i}\right) = \sum_{\substack{a_1 | p_1^{x_1} \\ \vdots \\ a_t | p_t^{x_t}}} \sum_{\substack{b_1 | p_1^{y_1} \\ \vdots \\ b_t | p_t^{y_t}}} f\left(\Pi \frac{p_i^{x_i}}{a_i}\right) f\left(\Pi \frac{p_i^{y_i}}{b_i}\right) f^{-1}(\Pi a_i b_i)$$

$$\times K\left(\left(\Pi \frac{p_i^{x_i}}{a_i}, \Pi \frac{p_i^{y_i}}{b_i}\right)\right) K\left(\left(\Pi \frac{p_i^{x_i+y_i}}{a_i b_i}, a_i b_i\right)\right)$$

$$\times C(\Pi a_i, \Pi b_i)$$

$$= \prod_{i=1}^{t} S(p_i^{x_i}, p_i^{y_i})$$

$$= \prod_{i=1}^{t} f(p_i^{x_i+y_i}) = f(\Pi p_i^{x_i+y_i}),$$

and the theorem is proved.

In addition to Vaidyanathaswamy's identity (1.1), the following is another interesting special case of Theorem 1, and is kindly supplied by the referee.

Let L denote the set of the integers n with the property that each prime divisor of n has multiplicity at least 2, and let $\lambda(n)$ denote the characteristic function of L. It is easily observed that $\lambda(n)$ satisfies the associativity condition, (2.1). Theorem 1 becomes now, with f^{-1} representing the inverse of f with respect to the kernel λ,

$$f(mn) = \sum_{\substack{ad = m,\ b\delta = n \\ (d,\delta) \in L \\ (ab,\ d\delta) \in L}} f(d)f(\delta)f^{-1}(ab)C(a,b) .$$

4. <u>An identity for unitary products.</u> For Dirichlet products, $K((m, n)) = 1$ for all m and n, and the identity of Theorem 1 reduces to (1.1). However, in the case of unitary products, Theorem 1 reduces to a triviality. To see this, we require the

LEMMA. If f is a multiplicative function and if f^{-1} denotes the unitary inverse of f, then $f^{-1}(n) = (-1)^{L(n)} f(n)$ for all positive integers n.

<u>Proof.</u> The result is obvious if $n = 1$. For any prime p and any positive integer x,

$$0 = E_0(p^x) = \sum_{\substack{d \mid p^x \\ (d, p^x/d) = 1}} f^{-1}(d)\, f(p^x/d) = f(p^x) + f^{-1}(p^x) ,$$

or $f^{-1}(p^x) = (-1)^{L(p^x)} f(p^x)$. Since f^{-1} and f are multiplicative, the lemma follows for any n.

Now for the unitary product, $K((m, n)) = E_0((m,n))$; hence, if we write $m = m_1 m_2$ and $n = n_1 n_2$, where $w(m_1) = w(n_1)$ and $(m_1, m_2) = (n_1, n_2) = 1$ and $(m_i, n_j) = 1$ except for $i = j = 1$, we see that

$$K((mn/ab, ab))\ K((m/a,\ n/b))\ C(a, b)$$

vanishes unless $a = m_1$ and $b = n_1$. Using the lemma it is seen that the identity reduces to the obvious relation $f(mn) = f(m_2)\ f(m_1 n_1)\ f(n_2)$.

We will now give a non-trivial identity for the unitary product. We write $d \| n$ to mean that d is a unitary divisor of n, i.e. $d | n$ and $(d, n/d) = 1$. Let

$$\lambda(a,b) = \begin{cases} (-1)^{L(a)}, & \text{if } w(a) | w(b) \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 2. For arbitrary positive integers m and n and for any multiplicative function f,

$$f(mn) = \sum_{\substack{a \| m \\ b \| n \\ w(b) | w((m,n)) \\ w(a) | w((m,n))}} f(m/a)\, f(n/b)\, f^{-1}(ab)\, \lambda(a,b) .$$

Proof. Let $T(m,n) = \sum_{\substack{a \| m \\ b \| n \\ w(b) | w((m,n)) \\ w(a) | w((m,n))}} f(m/a)\, f(n/b)\, f^{-1}(ab)\, \lambda(a,b)$

$$= \sum_{\substack{a \| m \\ b \| n \\ w(a) | w(b) | w((m,n))}} f(m/a)\, f(n/b)\, f(ab)\, (-1)^{L(a)+L(b)}$$

Clearly, $T(1,n) = f(n)$ and $T(m,1) = f(m)$ for all m, n. If p is a prime and $x, y \geq 1$, for $m = p^x$ and $n = p^y$ we have

$$T(m,n) = f(m)\, f(n) + f(m)\, f^{-1}(n) - f^{-1}(mn) = f(mn) ,$$

using the above lemma. Therefore,

$$T\left(\prod_{i=1}^{t} p_i^{x_i}, \prod_{i=1}^{t} p_i^{y_i}\right)$$

$$= \sum_{\substack{a_1 \| p_1^{x_1} \\ \vdots \\ a_t \| p_t^{x_t}}} \sum_{\substack{b_1 \| p_1^{y_1} \\ \vdots \\ b_t \| p_t^{y_t}}} f\left(\prod \frac{p_i^{x_i}}{a_i}\right) f\left(\prod \frac{p_i^{y_i}}{b_i}\right) f\left(\prod a_i b_i\right)(-1)^{\Sigma[L(a_i)+L(b_i)]}$$

$$w(a_1) \mid w(b_1) \mid w((p_1^{x_1}, p_1^{y_1}))$$

$$\vdots$$

$$w(a_t) \mid w(b_t) \mid w((p_t^{x_t}, p_t^{y_t}))$$

$$= \prod_{i=1}^{t} T(p_i^{x_i}, p_i^{y_i})$$

$$= \prod_{i=1}^{t} f(p_i^{x_i+y_i}) = f(\Pi p_i^{x_i+y_i}) .$$

A restatement of theorem 2 would be as follows:

If f is multiplicative, then for arbitrary integers m, n,

$$f(mn) = \sum_{\substack{a \| m \\ b \| n \\ w(a) | w(b) | w((m,n))}} f(m/a)\, f(n/b)\, f(ab)\, (-1)^{L(a)+L(b)} .$$

REFERENCES

1. Eckford Cohen, Arithmetical functions associated with the

unitary divisors of an integer. Math. Zeit., 74 (1960), pages 66-80.

2. Eckford Cohen, Unitary functions (mod r). Duke Math. J., 28 (1961), pages 475 - 485.

3. A. A. Gioia, On an identity for multiplicative functions. Amer. Math. Monthly, 69 (1962), pages 988 - 991.

4. A. A. Gioia, The K-product of arithmetic functions. Can. J. Math., 17 (1965), pages 970-976.

5. A. A. Gioia and M. V. Subbarao, Generalized Dirichlet products of arithmetic functions (Abstract). Notices Amer. Math. Soc., 9 (1962), page 305.

6. R. Vaidyanathaswamy, The identical equation of the multiplicative functions. Bull. Amer. Math. Soc., 36 (1930), pages 762-772.

7. R. Vaidyanathaswamy, The theory of multiplicative arithmetic functions. Trans. Amer. Math. Soc., 33 (1931) pages 579-662.

University of Alberta,
University of Kerala and
Texas Technological College

https://doi.org/10.4153/CMB-1967-007-x Published online by Cambridge University Press