Canad. Math. Bull. Vol. 51 (3), 2008 pp. 359–371

Real Hypersurfaces in Complex Space Forms with Reeb Flow Symmetric Structure Jacobi Operator

Jong Taek Cho and U-Hang Ki

Abstract. Real hypersurfaces in a complex space form whose structure Jacobi operator is symmetric along the Reeb flow are studied. Among them, homogeneous real hypersurfaces of type (A) in a complex projective or hyperbolic space are characterized as those whose structure Jacobi operator commutes with the shape operator.

1 Introduction

Let $(\widetilde{M}_n(c), J, \widetilde{g})$ be an n-dimensional complex space form with Kähler structure $(J, \widetilde{g})$ of constant holomorphic sectional curvature c and let M be an orientable real hypersurface in $\widetilde{M}_n(c)$. Then M has an almost contact metric structure (η, ϕ, ξ, g) induced from $(J, \widetilde{g})$ (see Section 1).

The second author [7] proved that there are no real hypersurfaces with parallel Ricci tensor in a non-flat complex space form $\widetilde{M}_n(c)$, $(c \neq 0)$ when $n \geq 3$. Recently, U. K. Kim [10] proved that this is also true when $n = 2$. These results imply, in particular, that there do not exist locally symmetric real hypersurfaces in a non-flat complex space form.

The structure Jacobi operator $R_\xi = R(\,\cdot\,, \xi)\xi$ has a fundamental role in almost contact geometry. It is notable that R_ξ is a self-adjoint operator. The present authors start the study on real hypersurfaces in a complex space form by using the operator R_ξ [5, 6, 9]. Recently, Ortega, Pérez and Santos [15] proved that there are no real hypersurfaces in the n-dimensional complex projective space $P_n\mathbb{C}$, $n \geq 3$ with parallel structure Jacobi operator $\nabla R_\xi = 0$. In a continuing work [16], Pérez, Santos, and Suh considered a weaker condition, called D-parallelness, that is, $\nabla_V R_\xi = 0$ for any vector field V orthogonal to ξ. But, it was proved further that there exist no real hypersurfaces in $P_n\mathbb{C}$, $n \geq 3$ with the D-parallel structure Jacobi operator. We may refer to a different literature [14] for the above two results.

This situation naturally leads to consider another weaker condition ξ-parallelness, that is, $\nabla_\xi R_\xi = 0$. This symmetry condition along the structure flow (or the Reeb flow) ξ also means that R_ξ is diagonalizable by a parallel orthonormal frame field $\{E_i\}$ along each flow ξ and its corresponding eigenvalues λ_i are constant along ξ, that is, $R_\xi E_i = \lambda_i E_i$ with $\nabla_\xi E_i = 0$ and $\xi\lambda_i = 0$ for $i = 1, 2, \dots, 2n-1$ (see [2, 4]).

Received by the editors April 4, 2006; revised January 10, 2007.
AMS subject classification: Primary: 53B20; secondary: 53C15, 53C25.
Keywords: complex space form, real hypersurface, structure Jacobi operator.

https://doi.org/10.4153/CMB-2008-036-7 Published online by Cambridge University Press

Takagi [17,18] classified the homogeneous real hypersurfaces of $P_n\mathbb{C}$ into six types. On the other hand, Cecil and Ryan [3] extensively studied Hopf hypersurfaces (whose Reeb vector field ξ is a principal curvature vector field), which are realized as tubes over certain submanifolds in $P_n\mathbb{C}$. By making use of those results and the aforementioned work of Takagi, Kimura [11] proved the local classification theorem for Hopf hypersurfaces of $P_n\mathbb{C}$ all of whose principal curvatures are constant. For the case of the n-dimensional complex hyperbolic space $H_n\mathbb{C}$, Berndt [1] proved the classification theorem for Hopf hypersurfaces all of whose principal curvatures are constant. Among the several types of real hypersurfaces appearing in Takagi's list or Berndt's list, a particular type of tubes over totally geodesic $P_k\mathbb{C}$ or $H_k\mathbb{C}$ $(0 \leq k \leq n-1)$ adding a horosphere in $H_n\mathbb{C}$, which is called type (A), has a many nice geometric properties. For example, Okumura [13] (resp. Montiel and Romero [12]) showed that a real hypersurface in $P_n\mathbb{C}$ (resp. $H_n\mathbb{C}$) is locally congruent to a real hypersurface of type (A) if and only if the Reeb flow ξ is isometric, or equivalently the structure operator ϕ commutes with the shape operator A $(\phi A = A\phi)$.

The main purpose of this paper is to prove the following.

Theorem 1 *Let M be a connected real hypersurface of $\widetilde{M}_n(c)$, $c \neq 0$, whose shape operator A commutes with R_ξ, that is, $R_\xi A = AR_\xi$. Then M satisfies $\nabla_\xi R_\xi = 0$ if and only if M is locally congruent to one of the following:*

(i) *in case that $\widetilde{M}_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,*

A_1 *a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{4}$,*

A_2 *a tube of radius r over a totally geodesic $P_k\mathbb{C}$, $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$ and $r \neq \frac{\pi}{4}$;*

(ii) *in case that $\widetilde{M}_n(c) = H_n\mathbb{C}$,*

A_0 *a horosphere,*

A_1 *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,*

A_2 *a tube over a totally geodesic $H_k\mathbb{C}$, $(1 \leq k \leq n-2)$.*

For the case of $P_n\mathbb{C}$, we need the technical assumption $\eta(A\xi) \neq 0$ in order to determine real hypersurfaces of type (A). Actually, there is a non-homogeneous tube with $A\xi = 0$ (of radius $\frac{\pi}{4}$) over a certain Kähler submanifold in $P_n\mathbb{C}$, when its focal map has constant rank on M [3]. However, for Hopf hypersurfces in $H_n\mathbb{C}$, it is known that the associated principal curvature of ξ never vanishes [1].

We note that the commutativity condition $R_\xi A = AR_\xi$ is a much weaker condition than $\phi A = A\phi$. Indeed, every Hopf hypersurface always satisfies it (see Remark 1 in Section 2).

2 Preliminaries

All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented. First, we review several fundamental facts on a real hypersurface of a complex space form. Let M be a real hypersurface of a non-flat complex space form $\widetilde{M}_n(c)$, $c \neq 0$, and N be a unit normal vector on M. We denote by $\widetilde{\nabla}$ the Levi–Civita connection with respect to the Kähler metric $\widetilde{g}$. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX,$$

for any vector fields X and Y on M, where g denotes the Riemannian metric of M induced from $\widetilde{g}$. An eigenvector (resp. eigenvalue) of the shape operator A is called a principal curvature vector (resp. principal curvature). For any vector field X tangent to M, we put $JX = \phi X + \eta(X)N$, $JN = -\xi$. We call ξ the *structure vector field* (or the *Reeb vector field*) and its flow also denoted by the same ξ. Then we may see that the structure (η, ϕ, ξ, g) is an almost contact metric structure on M, that is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

From this, we get easily $\phi\xi = 0$, $\eta \circ \phi = 0$, and $\eta(X) = g(X, \xi)$. From the Kähler condition $\widetilde{\nabla} J = 0$, making use of the Gauss and Weingarten formulas, we have

$$(2.1) \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi,$$

$$(2.2) \qquad \nabla_X \xi = \phi AX.$$

Since the ambient space is of constant holomorphic sectional curvature c, we have the following Gauss and Codazzi equations

$$(2.3) \qquad \begin{aligned} R(X, Y)Z = \frac{c}{4}\Big\{ & g(Y, Z)X - g(X, Z)Y \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \Big\} \\ & + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

$$(2.4) \qquad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}.$$

The curvature equation (2.3) gives the structure Jacobi operator R_ξ:

$$(2.5) \qquad R_\xi(X) = R(X, \xi)\xi = \frac{c}{4}\{X - \eta(X)\xi\} + \alpha AX - \eta(AX)A\xi.$$

From this, we have

$$(2.6) \qquad (R_\xi A - AR_\xi)(X) = \eta(AX)A^2\xi - \eta(A^2X)A\xi + \frac{c}{4}\{\eta(X)A\xi - \eta(AX)\xi\}.$$

Remark 1. From the above formula, we easily see that every Hopf hypersurface satisfies the commutativity condition $R_\xi A = AR_\xi$.

In the sequel, to write our formulas in conventional forms, we let $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$, and for a function f we denote by ∇f the gradient vector field of f. If we put $U = \nabla_\xi \xi$, then U is orthogonal to the structure vector ξ. From (2.2), we get

$$(2.7) \qquad \phi U = -A\xi + \alpha\xi,$$

which leads to $g(U,U) = \beta - \alpha^2$. If we put

$$A\xi = \alpha\xi + \mu W, \tag{2.8}$$

where W is a unit vector field orthogonal to ξ, then we get $U = \mu\phi W$, which says that W is also orthogonal to U. Further we have

$$\mu^2 = \beta - \alpha^2. \tag{2.9}$$

Thus we see that ξ is a principal curvature vector, that is, $A\xi = \alpha\xi$ if and only if $\beta - \alpha^2 = 0$, or equivalently ξ is a geodesic flow.

We set $\Omega = \{p \in M : \mu(p) \neq 0\}$ and suppose that Ω is non-empty, that is, ξ is not a principal curvature vector on M. Hereafter, unless otherwise stated, we discuss our arguments on the open subset Ω of M. And then we basically use the technical computations with the orthogonal triplet $\{\xi, U, W\}$ and their associated scalar functions α, β and μ.

By using (2.2), it follows that

$$g(\nabla_X\xi, U) = \mu g(AW, X) \tag{2.10}$$

$$\mu g(\nabla_X W, \xi) = g(AU, X) \tag{2.11}$$

for any vector field X on Ω.

Differentiating (2.7) covariantly along M and making use of (2.2), we find

$$(\nabla_X A)\xi = -\phi\nabla_X U + g(AU + \nabla\alpha, X)\xi - A\phi AX + \alpha\phi AX, \tag{2.12}$$

which enables us to obtain

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha, \tag{2.13}$$

where we have used (2.4). From (2.1) and (2.13), it is verified that

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha. \tag{2.14}$$

3 U is a Principal Curvature Vector Field on Ω

In this section, we prove that the condition $\nabla_\xi R_\xi = 0$ implies that U is a principal curvature vector field on Ω. Differentiating (2.5) covariantly with respect to ξ and taking account of (2.13) we get

$$\begin{aligned} g((\nabla_\xi R_\xi)Y, Z) = &-\frac{c}{4}\{u(Y)\eta(Z) + u(Z)\eta(Y)\} + (\xi\alpha)g(AY, Z) \\ &+ \alpha g((\nabla_\xi A)Y, Z) - \eta(AZ)\{3g(AU, Y) + Y\alpha\} \\ &- \eta(AY)\{3g(AU, Z) + Z\alpha\}, \end{aligned} \tag{3.1}$$

where u is a 1-form dual to U with respect to g, that is, $u(X) = g(U, X)$.

https://doi.org/10.4153/CMB-2008-036-7 Published online by Cambridge University Press

We assume that $\nabla_\xi R_\xi = 0$. Then we have from (3.1)

$$(3.2)\quad \alpha(\nabla_\xi A)X + (\xi\alpha)AX = \frac{c}{4}\{u(X)\xi + \eta(X)U\} + \eta(AX)(3AU + \nabla\alpha) + \{3g(AU, X) + X\alpha\}A\xi.$$

If we put $X = \xi$ in this and making use of (2.13), we find

$$(3.3)\quad \alpha AU + \frac{c}{4}U = 0,$$

which shows that $\alpha \neq 0$ on Ω, that is, U is a principal curvature vector field on Ω.

If we differentiate (3.3) covariantly along Ω, and use (3.3) again, then we obtain

$$-\frac{c}{4}(X\alpha)U + \alpha^2(\nabla_X A)U + \alpha^2 A\nabla_X U + \frac{c}{4}\alpha\nabla_X U = 0,$$

which, together with (2.4) and (2.7), implies that

$$(3.4)\quad \frac{c}{4}\{(Y\alpha)u(X) - (X\alpha)u(Y)\} + \frac{c}{4}\alpha^2\mu(\eta(X)w(Y) - \eta(Y)w(X)) + \alpha^2\{g(A\nabla_X U, Y) - g(A\nabla_Y U, X)\} + \frac{c}{4}\alpha du(X, Y) = 0,$$

where w is a dual 1-form of W with respect to g, that is $w(X) = g(W, X)$. Here, du is the exterior derivative of a 1-form u given by $du(X, Y) = Xu(Y) - Yu(X) - u([X, Y])$. If we replace X by U, then it follows that

$$(3.5)\quad \frac{c}{4}(\mu^2\nabla\alpha - (U\alpha)U) + \alpha^2 A\nabla_U U + \frac{c}{4}\alpha\nabla_U U = 0,$$

because U and W are mutually orthogonal. Combining (2.12) with (3.2) and using (2.4), we obtain

$$\begin{aligned}\alpha^2\phi\nabla_X U = {} & \alpha^2(X\alpha)\xi - \frac{c}{4}\alpha u(X)\xi + \alpha(\xi\alpha)AX + \frac{c}{4}\alpha^2\phi X \\ & - \eta(AX)(\alpha\nabla\alpha - \frac{3}{4}cU) - (\alpha(X\alpha) - \frac{3}{4}cu(X))A\xi \\ & - \frac{c}{4}\{u(X)\xi + \eta(X)U\} - \alpha^2 A\phi AX + \alpha^3\phi AX.\end{aligned}$$

Applying ϕ to this and using (2.10), we have

$$\begin{aligned}(3.6)\quad & \alpha^2\nabla_X U + \alpha^2\mu g(AW, X)\xi - \alpha\eta(AX)\phi\nabla\alpha \\ & = -\alpha(\xi\alpha)\phi AX + \frac{c}{4}\alpha^2(X - \eta(X)\xi) + \frac{3}{4}c\mu\eta(AX)W + \alpha(X\alpha)U \\ & \quad - \frac{3}{4}cu(X)U + \alpha^3 AX - \frac{c}{4}\alpha\mu\eta(X)W - \alpha^3\eta(X)A\xi + \alpha^2\phi A\phi AX.\end{aligned}$$

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Putting $X = U$ in (3.6) and using (2.7), (2.8) and (3.3), we get

$$\alpha^2\nabla_U U = -\frac{c}{4}\mu(\xi\alpha)W + \Big\{\alpha(U\alpha) - \frac{3}{4}c\mu^2\Big\}U + \frac{c}{4}\mu\alpha\phi AW, \tag{3.7}$$

which yields that

$$\alpha^2 A\nabla_U U = -\frac{c}{4}\mu(\xi\alpha)AW + \Big\{\alpha(U\alpha) - \frac{3}{4}c\mu^2\Big\}AU + \frac{c}{4}\mu\alpha A\phi AW. \tag{3.8}$$

4 $\nabla\alpha$ Is Proportional to U on Ω

In what follows, we will continue our discussions on Ω in M which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi A = AR_\xi$.

Then from the condition $R_\xi A = AR_\xi$ and (2.6) we get

$$\alpha A^2\xi = \Big(\beta - \frac{c}{4}\Big)A\xi + \frac{c}{4}\alpha\xi,$$

which together with $\alpha \neq 0$ gives

$$\alpha A^2\xi = \rho A\xi + \frac{c}{4}\,\xi, \tag{4.1}$$

where we have put

$$\alpha\rho = \beta - \frac{c}{4}. \tag{4.2}$$

Using (2.8), (2.9), (4.1) and (4.2), we get

$$AW = \mu\xi + (\rho - \alpha)W \tag{4.3}$$

because of $\mu \neq 0$. Substituting (3.7) and (3.8) into (3.5) and making use of (2.7), (3.3) and (4.1), we obtain

$$\alpha\mu^2\nabla\alpha = \alpha(U\alpha)U + \alpha\mu^2(\xi\alpha)\xi + \mu\Big\{\alpha(\rho - \alpha) + \frac{c}{4}\Big\}(\xi\alpha)W, \tag{4.4}$$

where we have used the relation $\mu W = -\phi U$. This, together with (2.9) and (4.2), imply that

$$\alpha(W\alpha) = \mu(\xi\alpha). \tag{4.5}$$

Thus, (4.4) turns out to be

$$\alpha\nabla\alpha = \frac{\alpha(U\alpha)}{\mu^2}U + (\xi\alpha)A\xi. \tag{4.6}$$

On the other hand, from (2.14) we have

$$\xi\mu = W\alpha, \tag{4.7}$$

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and hence with (4.5) it follows that $\alpha(\xi\mu) = \mu(\xi\alpha)$. Since $\mu^2 = \beta - \alpha^2$, together with the above equation we get further that

$$\alpha(\xi\beta) = 2\beta(\xi\alpha). \tag{4.8}$$

Differentiating (4.1) covariantly along Ω and making use of (2.2), we then have

$$\begin{aligned} g((\nabla_X A)A\xi, Y) + g(A(\nabla_X A)\xi, Y) + g(A^2\phi AX, Y) - \rho g(A\phi AX, Y) \\ = (X\rho)\eta(AY) + \rho g((\nabla_X A)\xi, Y) + \frac{c}{4}g(\phi AX, Y), \end{aligned} \tag{4.9}$$

which together with (2.4) and (2.13) give

$$(\nabla_\xi A)\xi = \rho AU - \frac{c}{4}U + \frac{1}{2}\nabla\beta. \tag{4.10}$$

If we put $X = \xi$ in (4.9) and take account of (2.13) and the above equation, we obtain

$$\frac{1}{2}\nabla\beta = -A\nabla\alpha + \rho\nabla\alpha + (\xi\rho)A\xi - 3A^2U + 2\rho AU + \frac{c}{2}U, \tag{4.11}$$

which together with (2.8) and (4.2) imply that

$$(\rho - 2\alpha)(\xi\alpha) + \alpha(\xi\rho) = 2\mu(W\alpha). \tag{4.12}$$

If we take an inner product (4.11) with W and make use of (4.2) and (4.3), then we obtain

$$\alpha(W\rho) = (2\alpha - \rho)(W\alpha) + 2\mu(\xi\rho - \xi\alpha). \tag{4.13}$$

From (4.13), together with (4.5) and (4.12) we get

$$\alpha^3(W\rho) = \mu(\rho\alpha + c)(\xi\alpha). \tag{4.14}$$

Since $W\beta = (W\alpha)\rho + \alpha(W\rho)$, using (4.5) and (4.14) we get

$$\alpha^2(W\beta) = \mu(2\rho\alpha + c)(\xi\alpha). \tag{4.15}$$

From the relation $\mu^2 = \rho\alpha + \frac{c}{4} - \alpha^2$, it is also seen that

$$2\mu(W\mu) = (\rho - 2\alpha)(W\alpha) + \alpha(W\rho),$$

and then using (4.5) and (4.14) we obtain

$$\alpha^2(W\mu) = \left(\rho\alpha - \alpha^2 + \frac{c}{2}\right)(\xi\alpha). \tag{4.16}$$

We are now to prove that $\xi\alpha = 0$ on Ω. First, from (3.3) we get

$$\alpha\phi AU = \frac{c}{4}\mu W \tag{4.17}$$

and from (4.3) we get also

$$\alpha(\phi A\phi AW - A\phi A\phi W) = -\frac{c}{4}\mu\xi, \tag{4.18}$$

where we have used the relation $\phi U = -\mu W$.

On the other hand, from (3.6) we get

$$\begin{aligned}
\alpha^2(\nabla_X u)(Y) + \alpha^2\mu\eta(Y)g(AX, W) - \alpha\eta(AX)g(\phi\nabla\alpha, Y)\\
= \alpha(\xi\alpha)g(AX, \phi Y) + \frac{c}{4}\alpha^2(g(X, Y) - \eta(X)\eta(Y))\\
+ \frac{3}{4}c\mu w(Y)\eta(AX) + \alpha u(Y)g(\nabla\alpha, X)\\
- \frac{3}{4}cu(X)u(Y) + \alpha^3 g(AX, Y) - \frac{c}{4}\alpha\mu\eta(X)w(Y)\\
- \alpha^3\eta(Y)\eta(AX) + \alpha^2 g(\phi A\phi AX, Y).
\end{aligned}$$

From this, we have a Codazzi-type formula for u:

$$\begin{aligned}
\alpha\big((\nabla_X u)(Y) - (\nabla_Y u)(X)\big)\\
= \frac{2}{\alpha}(\xi\alpha)(\eta(AX)u(Y) - \eta(AY)u(X)) - (\xi\alpha)g((\phi A + A\phi)X, Y)\\
+ \alpha g((\phi A\phi A - A\phi A\phi)X, Y)\\
+ \left\{\mu\left(\rho\alpha + \frac{c}{2}\right) - \frac{\alpha}{\mu}(U\alpha)\right\}\big(\eta(X)w(Y) - \eta(Y)w(X)\big),
\end{aligned} \tag{4.19}$$

where we have used (2.8) and (4.6). Putting $X = \xi$, and using (4.17), we have

$$\alpha du(\xi, X) = (\xi\alpha)u(X) + \left\{\mu\left(\rho\alpha + \frac{3}{4}c\right) - \frac{\alpha}{\mu}(U\alpha)\right\}w(X). \tag{4.20}$$

Putting $X = W$ in (4.19) this time, and using (3.3) and (4.18), we obtain

$$\begin{aligned}
\alpha du(W, X) = \left\{\frac{\alpha}{\mu}(U\alpha) - \mu\left(\rho\alpha + \frac{3}{4}c\right)\right\}\eta(X)\\
+ (\xi\alpha)\left\{2\frac{\mu}{\alpha} - \frac{\rho - \alpha}{\mu} + \frac{c}{4\alpha\mu}\right\}u(X).
\end{aligned} \tag{4.21}$$

Combining (4.10) with (2.13) and using (2.8), we then find

$$\mu(\nabla_\xi A)W = (\rho - 2\alpha)AU - \frac{c}{4}U + \frac{1}{2}\nabla\beta - \alpha\nabla\alpha. \tag{4.22}$$

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If we replace X by W in (3.2) and take account of (4.3) and (4.22), we then have

$$\frac{\alpha}{\mu}\Big\{(\rho-2\alpha)AU-\frac{c}{4}U+\frac{1}{2}\nabla\beta-\alpha\nabla\alpha\Big\}+(\xi\alpha)AW=(W\alpha)A\xi+\mu(3AU+\nabla\alpha).$$

This, together with (2.8), (3.3), (4.2) and (4.5), imply that

$$\alpha\Big(\frac{1}{2}\alpha\nabla\beta-\beta\nabla\alpha\Big)+\frac{c}{4}(3\beta-2\alpha^2-\rho\alpha)U=\mu(\xi\alpha)(\mu A\xi-\alpha AW). \tag{4.23}$$

By using (2.8) (4.2) and (4.3), this is rewritten as

$$\alpha^2\nabla\beta-\beta\nabla\alpha^2+c\Big(\mu^2+\frac{c}{8}\Big)U=\frac{c}{2}(\xi\alpha)(A\xi-\alpha\xi),$$

or for any vector field Y we get

$$\alpha^2(Y\beta)-\beta(Y\alpha^2)+c\Big(\mu^2+\frac{c}{8}\Big)u(Y)=\frac{c}{2}(\xi\alpha)(\eta(AY)-\alpha\eta(Y)).$$

Differentiating this with respect to a vector field X again, and taking the skew-symmetric part for X and Y, then we eventually have

$$\begin{aligned}\frac{8}{c}\alpha^2\big((X\alpha)(Y\beta)-(Y\alpha)(X\beta)\big)&+4\alpha\mu\big((X\mu)u(Y)-(Y\mu)u(X)\big)\\&+\Big(2\mu^2+\frac{c}{4}\Big)\alpha\big((\nabla_Xu)(Y)-(\nabla_Yu)(X)\big)\\&=\mu\alpha\big(X(\xi\alpha)w(Y)-Y(\xi\alpha)w(X)\big)\\&+(\xi\alpha)\Big\{\frac{c}{4}\mu\alpha\big(\eta(X)w(Y)-\eta(Y)w(X)\big)\\&\quad+2\alpha g(\phi AX,AY)+\alpha\big((Y\alpha)\eta(X)\\&\quad-(X\alpha)\eta(Y)\big)-\alpha^2\big(g(\phi AX,Y)-g(\phi AY,X)\big)\Big\}.\end{aligned} \tag{4.24}$$

If we put $Y=W$ in (4.24), and make use of (4.5), (4.15), (4.16) and (4.21), then we find

$$\begin{aligned}\mu\alpha(X(\xi\alpha))&=\mu\alpha(W(\xi\alpha))w(X)\\&+\Big[\Big(2\mu^2+\frac{c}{4}\Big)\Big\{\mu\Big(\rho\alpha+\frac{3}{4}c\Big)-\frac{\alpha}{\mu}(U\alpha)\Big\}-(\xi\alpha)\mu\Big((\xi\alpha)+\frac{c}{4}\alpha\Big)\Big]\eta(X)\\&+\frac{8}{c}\mu(\xi\alpha)\{(2\rho\alpha+c)(X\alpha)-\alpha(X\beta)\}+f_1u(X)\end{aligned}$$

for some smooth function f_1. Substituting this into (4.24), we then have

(4.25)

$$\frac{8}{c}\alpha^2\big((X\alpha)(Y\beta)-(Y\alpha)(X\beta)\big)+4\alpha\mu\big((X\mu)u(Y)-(Y\mu)u(X)\big)$$
$$+\Big(2\mu^2+\frac{c}{4}\Big)\alpha((\nabla_X u)(Y)-(\nabla_Y u)(X))$$
$$=\Big[\Big(2\mu^2+\frac{c}{4}\Big)\{\mu(\rho\alpha+\frac{3}{4}c)-\frac{\alpha}{\mu}(U\alpha)\}-\mu(\xi\alpha)^2\Big]\big(\eta(X)w(Y)-\eta(Y)w(X)\big)$$
$$+\frac{8}{c}\mu(\xi\alpha)\big\{(2\rho\alpha+c)\big((X\alpha)w(Y)-(Y\alpha)w(X)\big)-\alpha\big((X\beta)w(Y)-(Y\beta)w(X)\big)\big\}$$
$$+(\xi\alpha)\Big\{\alpha\big((Y\alpha)\eta(X)-(X\alpha)\eta(Y)\big)+2\alpha g(\phi AX,AY)$$
$$-\alpha^2\big(g(\phi AX,Y)-g(\phi AY,X)\big)\Big\}+f_1\big(u(X)w(Y)-u(Y)w(X)\big).$$

If we put $X=\xi$ in (4.25), and use (3.3), (4.8) and (4.20), then we obtain

$$\frac{8}{c}(\xi\alpha)(\alpha^2\nabla\beta-\beta\nabla\alpha^2-\frac{c}{8}\alpha\nabla\alpha)+f_2U$$
$$=-\alpha(\xi\alpha)^2\xi-\mu(\xi\alpha)\Big\{\frac{8}{c}(2\rho\alpha+c)((W-\xi)(\alpha))-(\xi\alpha)\Big\}W,$$

for some smooth function f_2.

Now we suppose that $\xi\alpha\neq 0$ on Ω, and then we restrict the arguments on such a place. Taking the inner product with W in the above equation, and using (4.5) and (4.15), we can then deduce that $\alpha=\frac{4}{c}(\mu-\alpha)\big(\beta+\frac{c}{4}\alpha\big)$, where we have used $\beta=\mu^2+\alpha^2$. Differentiating this equation covariantly with respect to ξ, making use of (4.7) and (4.8), then we get again $\alpha=\frac{4}{c}(\mu-\alpha)\big(3\beta+\frac{c}{2}\alpha\big)$. Combining the last two equations, we have $(\mu-\alpha)\big(2\beta+\frac{c}{4}\alpha\big)=0$, and then it gives that $\mu=\alpha$ or $2\beta=-\frac{c}{4}\alpha$. But, both give that $\alpha=0$, a contradiction. Thus, we have proved the following.

Lemma 1 *$\xi\alpha=W\alpha=0$ on Ω.*

By Lemma 1, (4.6) and (4.21) reduce respectively to

$$\mu^2\nabla\alpha=(U\alpha)U, \tag{4.26}$$

$$\alpha du(W,X)=\Big\{\frac{\alpha}{\mu}(U\alpha)-\mu\Big(\rho\alpha+\frac{3}{4}c\Big)\Big\}\eta(X). \tag{4.27}$$

In the next step, we prove the following.

Lemma 2 *$\alpha\nabla\alpha=(\rho\alpha+\frac{3}{4})U$ on Ω.*

Proof If we replace $Y = W$ in (3.4) and make use of (2.14), (4.3) and Lemma 1, then we find

$$\frac{c}{4}\alpha^2\eta(X) - \alpha\mu^2 g(AX,W) + \alpha^2\{(\rho-\alpha)g(\nabla_X U,W) - g(A\nabla_W U,X)\} + \frac{c}{4}\alpha du(X,W) = 0,$$

which combining with (4.27) shows that

$$\alpha^2 g(A\nabla_W U,X) = \frac{c}{4}\Big\{\frac{\alpha}{\mu}(U\alpha) + \mu(\alpha^2 - \rho\alpha - \frac{3}{4}c)\Big\}\eta(X) - \alpha^2\mu^2 g(AX,W) + \alpha^2(\rho-\alpha)g(\nabla_X U,W). \tag{4.28}$$

By the way, if we put $X = W$ in (3.6) and take account of (2.8), (3.3), (4.3), (4.26) and Lemma 1, then we have

$$\alpha^2 A\nabla_W U = \mu\alpha^2(\alpha-\rho)A\xi + \Big\{\frac{3}{4}c\mu^2 + \frac{c}{4}\rho\alpha + \alpha^3(\rho-\alpha) - \alpha(U\alpha)\Big\}AW. \tag{4.29}$$

In addition, putting $X = \xi$ in (3.6) and taking an inner product with W, we then obtain

$$\alpha^2 g(\nabla_\xi U,W) = \alpha\Big(\alpha^2 + \frac{3}{4}c\Big)\mu - \alpha^2\frac{(U\alpha)}{\mu}. \tag{4.30}$$

If we put $X = \xi$ in (4.28) and use (2.8), (4.3), (4.29), and (4.30), we then have

$$\alpha(U\alpha) = \Big(\rho\alpha + \frac{3}{4}c\Big)\mu^2.$$

Thus, together with (4.26) we have proved Lemma 2. ■

5 Proof of Theorem 1

By making use of the results (Lemmas 1 and 2) obtained in the previous section, we want to prove that the open subset $\Omega = \{p \in M : \mu(p) \neq 0\}$ must be empty. Otherwise, since $\xi\alpha = 0$(Lemma 1), the equation (4.23) becomes

$$\alpha\Big(\frac{1}{2}\alpha\nabla\beta - \beta\nabla\alpha\Big) + \frac{c}{4}(3\beta - 2\alpha^2 - \rho\alpha)U = 0. \tag{5.1}$$

Taking into account that $\beta = \rho\alpha + \frac{c}{4}$, by Lemma 2 it follows that

$$\frac{1}{2}\alpha\nabla\beta = \Big\{\rho^2\alpha + \frac{c}{2}(\rho+\alpha)\Big\}U. \tag{5.2}$$

https://doi.org/10.4153/CMB-2008-036-7 Published online by Cambridge University Press

By using the relation $\nabla\beta = \rho\nabla\alpha + \alpha\nabla\rho$ and Lemma 2 again, we also get

$$\alpha^2\nabla\rho = \left(\rho^2\alpha + \frac{c}{4}\rho + c\alpha\right)U,$$

from which we can see that $\xi\rho = W\rho = 0$.

Now we differentiate (4.3) covariantly along Ω. Then it follows that

$$(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\nabla_X\xi + X(\rho-\alpha)W + (\rho-\alpha)\nabla_X W.$$

By taking an inner product with W, and making use of (2.8) and (2.11), we obtain

$$g((\nabla_X A)W, W) = -2g(AX, U) + X(\rho-\alpha). \tag{5.3}$$

This time we differentiate (4.1) covariantly and use (2.2) and and (2.8). Then we find

$$A(\nabla_X A)\xi + (\alpha-\rho)(\nabla_X A)\xi + \mu(\nabla_X A)W = (X\rho)A\xi + \frac{c}{4}\phi AX + \rho A\phi AX - A^2\phi AX.$$

Replacing X by $\alpha\xi + \mu W$ in this equation and making use of (2.4), (2.8), (2.10), (4.22), and (5.3), we then have

$$\begin{aligned} 2\rho A^2U + 2\left(\alpha\rho - \beta - \rho^2 - \frac{c}{4}\right)AU + (\alpha\rho^2 - \beta\rho + \frac{c}{2}\rho - \frac{3}{4}c\alpha)U \\ = g(A\xi, \nabla\rho)A\xi - \frac{1}{2}A\nabla\beta + \frac{1}{2}(\rho - 2\alpha)\nabla\beta + \beta\nabla\alpha - \mu^2\nabla\rho. \end{aligned} \tag{5.4}$$

Since $\xi\rho = W\rho = 0$, we see that $g(A\xi, \nabla\rho) = 0$. Thus, (5.4) becomes

$$\begin{aligned} &2\rho A^2U - (2\rho^2 + c)AU + \frac{c}{4}(\rho - 3\alpha)U \\ &\qquad = -\frac{1}{2}A\nabla\beta + \frac{1}{2}(\rho - 2\alpha)\nabla\beta + \beta\nabla\alpha - \left(\rho\alpha - \alpha^2 + \frac{c}{4}\right)\nabla\rho, \end{aligned}$$

where we have used $\beta = \rho\alpha + \frac{c}{4}$. Multiplying by α^2 and using (3.3), (5.1) and (5.2), direct computations lead us to $(\rho + 3\alpha)(\beta - \alpha^2)U = 0$, which yields $\rho + 3\alpha = 0$ on Ω. Differentiating it and multiplying by α^2, and using (5.1) and (5.2) once again, we meet with $\alpha = 0$ on Ω. This is impossible. Finally, we conclude that $\Omega = \varnothing$, that is, $A\xi = \alpha\xi$ on M. We see in addition that α is constant [8]. Thus, from (3.2) we get $\alpha(\nabla_\xi A) = 0$. Using (2.2) and the Codazzi equation (2.4), we have

$$\alpha(\phi A - A\phi) = 0.$$

Here, we note the case $\alpha = 0$ corresponds to the case of the tube of radius $\frac{\pi}{4}$ in $P_n\mathbb{C}$ [3]. But, in the case of $H_n\mathbb{C}$, it is known that α never vanishes for Hopf hypersurfaces [1]. Due to Okumura's work for $P_n\mathbb{C}$ or Montiel and Romero's work for $H_n\mathbb{C}$ mentioned in Introduction, we have completed the proof of our theorem.

Acknowledgement The authors thank the referee for careful reading of the manuscripts and for useful comments and suggestions for the revised version.

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