TWO THEOREMS ON MOSAICS

B. GORDON AND M. M. ROBERTSON

1. Introduction. The concept of a mosaic was recently introduced by A. A. Mullin **(1)**. By the fundamental theorem of arithmetic, every integer $n > 1$ can be uniquely expressed in the form

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r},$$

where the p_i are primes satisfying $p_1 < p_2 < \ldots < p_r$. We then express any exponents α_j which are greater than unity in the same manner, and continue in this way until the process terminates. The resulting planar configuration of primes is called the *mosaic* of n. We denote by $\psi(n)$ the product of all the primes occurring in the mosaic of n; by convention, $\psi(1) = 1$. Then $\psi(n)$ is a multiplicative mapping of the set of natural numbers onto itself. Clearly $\psi(n)$ tends to infinity with n, and hence for fixed k, the equation $\psi(n) = k$ has only a finite number of solutions, which we denote by $\xi(k)$. Our first result is

THEOREM 1. *If $k = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ is the prime decomposition of k, then*

$$\xi(k) = \beta^{-1} \prod_{j=1}^{r} \binom{\beta}{\alpha_j}, \qquad \text{where } \beta = 1 + \sum_{j=1}^{r} \alpha_j.$$

To state the second theorem we require some more notation. We define the iterates ψ_ν of ψ in the usual way, i.e., $\psi_0(n) = n$, and $\psi_\nu(n) = \psi(\psi_{\nu-1}(n))$ for $\nu > 0$. It is easily seen that $\psi(n) \leqslant n$. The equality holds if and only if either n is square-free or $n = 4m$ where m is odd and square-free. Hence for any n there exists a smallest non-negative integer $\nu = \nu(n)$ such that $\psi_{\nu+1}(n) = \psi_\nu(n)$. If $k \geqslant 0$, we let $\theta(k) = \min\{n: \nu(n) = k\}$; for example, $\theta(0) = 1$, $\theta(1) = 8$, $\theta(2) = 16$, $\theta(3) = 36$, $\theta(4) = 72$.

THEOREM 2. (1) *For any constant $c > 1$, there exists a constant $A = A(c) > 0$ such that $\theta(k) \geqslant Ac^k$ for all $k \geqslant 0$.*

(2) *There exists a function $\mu(k) \geqslant \theta(k)$ satisfying $\mu(0) = 1$, $\mu(1) = 8$, and*

$$\mu(k+1) < (5 \log \mu(k) \log \log \mu(k))^{\sqrt{\mu(k)} \log \mu(k) / \log 2}$$

for $k \geqslant 1$.

2. Proof of Theorem 1. By definition, $\xi(p_1^{\alpha_1} \ldots p_r^{\alpha_r})$ is the number of different mosaics which can be formed with α_j primes p_j $(j = 1, \ldots, r)$. We may write $\xi(p_1^{\alpha_1} \ldots p_r^{\alpha_r}) = \eta(\alpha_1, \ldots, \alpha_r)$ because ξ does not depend upon the particular primes p_j, but only on their multiplicities α_j. Since a mosaic cannot have two equal primes on the "first stratum," it follows that

Received July 20, 1964.

https://doi.org/10.4153/CJM-1965-094-9 Published online by Cambridge University Press

$$\eta(\alpha_1, \ldots, \alpha_r) = \sum_{s=1}^{r} \sum{}' \sum{}'' \prod_{t=1}^{s} \eta(\alpha_1^{(t)}, \ldots, \alpha_r^{(t)}). \tag{2.1}$$

Here the sum $\sum'$ is extended over the $\binom{r}{s}$ distinct r-partite numbers $(\epsilon_1, \ldots, \epsilon_r)$ in which s of the ϵ_j are equal to 1, and the remaining $r - s$ of the ϵ_j are 0; the sum $\sum''$ is extended over all ordered partitions of $(\alpha_1 - \epsilon_1, \ldots, \alpha_r - \epsilon_r)$ into s parts, in which $(0, \ldots, 0)$ may be counted as a part.

For $r \geqslant 2$ we consider the function

$$g(z) = g(z; x_1, \ldots, x_r) = \prod_{j=1}^{r} (1 + x_j z) - z,$$

where $0 < x_j < x$ for $1 \leqslant j \leqslant r$. Clearly $g''(z) > 0$ for z real and positive; moreover $g(0) = 1$, and $g(z)$ is positive for z sufficiently large. If x is sufficiently small (in fact if $x < \frac{1}{2}(2^{1/r} - 1)$), $g(2) < 0$, and so for such x, $g(z)$ has exactly two positive roots $\gamma_1, \gamma_2 (\gamma_1 < \gamma_2)$, which depend on the x_j. When $\gamma_1 < z < \gamma_2$, $g(z) < 0$, and when $0 \leqslant z < \gamma_1$ or $z > \gamma_2$, $g(z) > 0$. Hence if z is complex and satisfies $\gamma_1 < |z| < \gamma_2$, then

$$\left| \prod_{j=1}^{r} (1 + x_j z) \right| \leqslant \prod_{j=1}^{r} (1 + x_j |z|) < |z|.$$

A simple application of Rouché's theorem now shows that $z = \gamma_1$ is the only solution of $g(z) = 0$ in $|z| < \gamma_2$. When $r = 1$, we let γ_1 be the solution of $g(z) = 1 + x_1 z - z = 0$, and we put $\gamma_2 = \infty$. Then for all $r \geqslant 1$, γ_1 is the only solution of $g(z) = 0$ in $|z| < \gamma_2$, and

$$\prod_{j=1}^{r} |1 + x_j z| < |z|$$

for $\gamma_1 < |z| < \gamma_2$.

We write

$$G(x_1, \ldots, x_r) = \sum_{\alpha_1=0}^{\infty} \ldots \sum_{\alpha_r=0}^{\infty} \delta(\alpha_1, \ldots, \alpha_r) x_1^{\alpha_1} \ldots x_r^{\alpha_r},$$

where

$$\delta(\alpha_1, \ldots, \alpha_r) = \beta^{-1} \prod_{j=1}^{r} \binom{\beta}{\alpha_j}.$$

Let C be any circle $|z| = R$, where $\gamma_1 < R < \gamma_2$. Then for

$$0 < x_j < x \ (1 \leqslant j \leqslant r),$$

we obtain from the residue theorem:

$$\begin{aligned} G(x_1, \ldots, x_r) &= \sum_{\beta=1}^{\infty} \int_C (2\pi i \beta z^\beta)^{-1} \prod_{j=1}^{r} (1 + x_j z)^\beta dz \\ &= -\frac{1}{2\pi i} \int_C \log\left\{1 - z^{-1} \prod_{j=1}^{r} (1 + x_j z)\right\} dz \\ &= \frac{1}{2\pi i} \int_C z \frac{d}{dz} \log\left\{1 - z^{-1} \prod_{j=1}^{r} (1 + x_j z)\right\} dz. \end{aligned}$$

https://doi.org/10.4153/CJM-1965-094-9 Published online by Cambridge University Press

Applying the residue theorem again, we get $G(x_1, \ldots, x_r) = \gamma_1$, which shows that

$$G = \prod_{j=1}^{r} (1 + x_j G).$$

Since this equation is an identity in $x_1, \ldots, x_r$, we may equate corresponding coefficients and obtain

$$\delta(\alpha_1, \ldots, \alpha_r) = \sum_{s=1}^{r} \sum{}' \sum{}'' \prod_{t=1}^{s} \delta(\alpha_1^{(t)}, \ldots, \alpha_r^{(t)}), \tag{2.2}$$

which is identical in form to (2.1). Now $\eta(0, \ldots, 0) = \xi(1) = 1$ and $\delta(0, \ldots, 0) = 1$. Since $\eta(0, \ldots, 0) = \delta(0, \ldots, 0)$, (2.1) and (2.2) show that $\eta(\alpha_1, \ldots, \alpha_r) = \delta(\alpha_1, \ldots, \alpha_r)$ for all non-negative $\alpha_1, \ldots, \alpha_r$. This completes the proof of Theorem 1.

3. Proof of Theorem 2. (1) Let $\lambda(n)$ be the Liouville function, that is, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, then $\lambda(n) = \alpha_1 + \alpha_2 + \ldots + \alpha_r$. Clearly

$$n \geqslant 2^{\alpha_1 + \cdots + \alpha_r} = 2^{\lambda(n)},$$

and therefore $\lambda(n) \leqslant \log_2 n$. From this it follows easily that $1 + \lambda(n) \leqslant n$.

We now assert that $\lambda(\psi(n)) \leqslant \lambda(n)$. This is obviously true for $n = 1$. Suppose that $n > 1$ and that $\lambda(\psi(m)) \leqslant \lambda(m)$ for all $m < n$. If

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r},$$

then

$$\psi(n) = p_1 p_2 \ldots p_r \psi(\alpha_1)\psi(\alpha_2) \ldots \psi(\alpha_r).$$

Hence

$$\lambda(\psi(n)) = r + \lambda(\psi(\alpha_1)) + \ldots + \lambda(\psi(\alpha_r)).$$

All the α_j are less than n, so by induction

$$\lambda(\psi(n)) \leqslant r + \lambda(\alpha_1) + \ldots + \lambda(\alpha_r) = (1 + \lambda(\alpha_1)) + \ldots + (1 + \lambda(\alpha_r))$$
$$\leqslant \alpha_1 + \ldots + \alpha_r = \lambda(n).$$

In particular, if p is a prime, and $\text{ord}_p\, m$ denotes the greatest integer β such that $p^\beta | m$, then $\text{ord}_p\, \psi(n) \leqslant \lambda(n)$.

For a fixed prime p, the sequence $f(k) = k/p^{k-1}$ $(k = 1, 2, 3, \ldots)$ decreases monotonically from 1 to 0. Hence there is a greatest integer δ_p such that $f(\delta_p) > 1/c$. A simple calculation shows that $\delta_p = 1$ for all $p \geqslant 2c$ and $\delta_p \geqslant \delta_q$ if $p < q$. Hence, if M is a fixed positive number, we have

$$g(M) = \sum_{p<M} \delta_p \leqslant \pi(M)\delta_2.$$

Since $\pi(M) \sim M/(\log M)$ as $M \to \infty$, we have $g(M) < M$ for all sufficiently large M. Let p_0 be the least prime $\geqslant 2c$ such that $g(p_0) < p_0$. Then let S be the (finite) set of integers n whose prime factors are all $< p_0$, and which satisfy

$\lambda(n) \leqslant g(p_0)$. Let A be the minimum of $n/c^{\nu(n)}$ for all $n \in S$. Since $1 \in S$, we have $A \leqslant 1/c^{\nu(1)} = 1$.

We shall now prove by induction on k that $\theta(k) \geqslant A \cdot c^k$ for all $k \geqslant 0$. Since $\theta(0) = 1 \geqslant A$, this is true for $k = 0$. Suppose that $k > 0$, and that it has already been shown that $\theta(k-1) \geqslant A \cdot c^{k-1}$. We have to show that if $n < A \cdot c^k$, then $\nu(n) < k$. Let p be a prime, and suppose that $n = p^\beta m$, where $p \nmid m$. Then

$$\frac{\psi(n)}{n} = \frac{\psi(p^\beta)}{p^\beta}\frac{\psi(m)}{m} \leqslant \frac{\psi(p^\beta)}{p^\beta} = \frac{p\psi(\beta)}{p^\beta} \leqslant \frac{\beta}{p^{\beta-1}}.$$

If $\beta > \delta_p$, then $\beta/p^{\beta-1} \leqslant 1/c$, by the definition of δ_p. Hence $\psi(n) \leqslant n/c < Ac^{k-1}$, and by induction, $\nu(\psi(n)) < k - 1$. Since $\nu(n) \leqslant \nu(\psi(n)) + 1$, this implies that $\nu(n) < k$, completing the induction. Hence we may suppose that for every prime p, $\text{ord}_p\, n \leqslant \delta_p$. Since $p_0 \geqslant 2c$, this means that for any prime $p \geqslant p_0$ we have $\text{ord}_p\, n \leqslant 1$. Thus we may write $n = n_1 n_2$, where all prime factors of n_1 are $< p_0$, and n_2 is a square-free integer all of whose prime factors are $\geqslant p_0$. Moreover

$$\lambda(n_1) \leqslant \sum_{p<p_0} \delta_p = g(p_0),$$

so that $n_1 \in S$.

We shall now prove that the set S is mapped into itself by the ψ-function. If $n_1 \in S$, then $\lambda(\psi(n_1)) \leqslant \lambda(n_1) \leqslant g(p_0)$. If q is a prime occurring on the "first stratum" of the mosaic of n_1, then $q < p_0$ by the definition of S. If q occurs on a higher stratum, then there is a prime p such that $\text{ord}_p\, n_1 \geqslant q$ Thus $q \leqslant \lambda(n_1) < p_0$.

It follows from these considerations that $\psi(n) = \psi(n_1) \cdot n_2$, $\psi_2(n) = \psi_2(n_1) \cdot n_2$, and in general $\psi_\nu(n) = \psi_\nu(n_1) \cdot n_2$ for any ν. Thus $\nu(n) = \nu(n_1)$. Since $n_1 \in S$, we see from the definition of A that $n_1/c^{\nu(n_1)} \geqslant A$. Hence

$$A \leqslant n/c^{\nu(n)} < (A \cdot c^k)/c^{\nu(n)},$$

which implies that $\nu(n) < k$, completing the proof of (1).

(2) In this section we denote the sth prime by $p(s)$. We consider the sequence $\mu(k)$, where

$$\mu(0) = 1, \quad \mu(1) = 8, \quad \mu(2) = 16, \quad \mu(3) = 36 = p(1)^{p(1)}p(2)^{p(1)},$$

and if $\mu(k) = p(1)^{\alpha_1} \ldots p(t)^{\alpha_t}$ $(k \geqslant 3)$, then

$$\mu(k+1) = [p(1) \ldots p(\alpha_t - 1)]^{p(t)}[p(\alpha_t) \ldots p(\alpha_t + \alpha_{t-1} - 2)]^{p(t-1)} \ldots \left[p\left(\sum_{j=2}^{t} \alpha_j - t + 2\right) \ldots p\left(\sum_{j=1}^{t} \alpha_j - t\right)\right]^{p(1)}.$$

It is easily seen that the α_j are primes satisfying $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_t = 2$. Furthermore

$$\psi(\mu(k+1)) = \mu(k)p(t+1) \ldots p\left(\sum_{j=1}^{t} \alpha_j - t\right).$$

https://doi.org/10.4153/CJM-1965-094-9 Published online by Cambridge University Press

An easy induction proves that $\nu(\mu(k)) = k$ for all $k \geqslant 3$. Hence for all k, $\theta(k) \leqslant \mu(k)$.

Suppose that $k \geqslant 3$, and put

$$\alpha = \sum_{j=1}^{t} \alpha_j.$$

Then $\mu(k) > 2^\alpha$, so that $\alpha < \log \mu(k)/\log 2$. Also,

$$\mu(k+1) < p(\alpha)^{\alpha p(t)} < p(\alpha)^{\alpha \mu(k)^{1/2}}.$$

It is known **(2)** that $p(s) < s \log s + 2s \log \log s$ for all $s \geqslant 4$. This implies that $p(s) < 3s \log s$ for all $s \geqslant 2$. Hence

$$\begin{aligned} \mu(k+1) &< \frac{3 \log \mu(k)}{\log 2} \log \left(\frac{\log \mu(k)}{\log 2}\right)^{\mu(k)^{1/2} \log \mu(k)/\log 2} \\ &< \{5 \log \mu(k) \log \log \mu(k)\}^{\mu(k)^{1/2} \log \mu(k)/\log 2}, \end{aligned}$$

which completes the proof of Theorem 2.

References

1. A. A. Mullin, *Some related number-theoretic functions*, Bull. Amer. Math. Soc., *69* (1963), 446–447.

2. J. B. Rosser, *The n'th prime is greater than n log n*, Proc. London Math. Soc., *45* (1939), 21–44.

University of California, Los Angeles