ON CONTINUOUS REGULAR RINGS

Y. Utumi

(received January 4, 1961)

1. A subset K of a lattice is said to be directed if for any $a, b \in K$ there is $c \in K$ with $c \geq a, b$. A complete lattice L is called upper continuous if $\cup(a_\alpha \cap b) = (\cup a_\alpha) \cap b$ for every directed subset (a_α) and every element b.

The following is a slight improvement of [4; Anmerkung 1.11, p.11].

THEOREM 1. Let L be a complete complemented modular lattice. Then L is upper continuous if and only if it satisfies the following

CONDITION (A). Let T be a subset of L. If there is a nonzero element b of L such that $(\cup_{\alpha \in F} a) \cap b = 0$ for every finite subset F of T, then there exists an element c satisfying $1 \neq c \geq a$ for every $a \in T$.

Proof. First, assume that L is upper continuous, and let T be a subset of L. Set $a_F = \cup_{a \in F} a$ for every finite subset F of T. Then evidently the set of all a_F is directed, and so by assumption we have $(\cup_F a_F) \cap b = \cup_F (a_F \cap b)$ for any $b \in L$. Hence, if we assume moreover that $b \neq 0$ and $a_F \cap b = 0$ for every F, then $(\cup_F a_F) \cap b = 0$, and so $1 \neq \cup_F a_F = \cup_{a \in T} a$, as desired. Next, to prove the if part of the theorem we assume that L satisfies (A), and let T be a directed subset of L. Then for any finite subset F of T there is $a^F \in T$ such that $a^F \geq a$ for every $a \in F$. Since evidently $(\cup_{a \in T} a) \cap x \geq \cup_{a \in T}(a \cap x)$ for any $x \in L$, there is c with the properties that $(\cup_{a \in T} a) \cap x = (\cup_{a \in T} (a \cap x)) \cup c$ and $(\cup_{a \in T} (a \cap x)) \cap c = 0$. Now, denote a complement of $\cup_{a \in T} a$ by d. Then, $(\cup_{a \in F}(a \cup d)) \cap c \leq (a^F \cup d) \cap c = (a^F \cup d) \cap (\cup_{a \in T} a) \cap c = (((\cup_{a \in T} a) \cap d) \cup a^F) \cap c = a^F \cap c \leq a^F \cap x \cap c \leq (\cup_{a \in T} (a \cap x)) \cap c = 0$, and so

Canad. Math. Bull. vol. 4, no. 1, January 1961

$(\cup_{a \in F}(a \cup d)) \cap c = 0$. If $c \neq 0$, then by (A) we have $\cup_{a \in T}(a \cup d) \neq 1$, that is, $(\cup_{a \in T} a) \cup d \neq 1$ which is a contradiction. Therefore, $c = 0$ and $(\cup_{a \in T} a) \cap x = \cup_{a \in T}(a \cap x)$, completing the proof.

A ring with unit is called regular if for any element x there is y with $xyx = x$. It is well known that the set of all principal left (right) ideals of a regular ring S forms a complemented modular lattice $L(S)$ $(R(S))$. We shall say that a regular ring S is left (right) continuous if $L(S)$ $(R(S))$ is upper continuous.

The purpose of this note is to present some conditions for a regular ring to be left (right) continuous. The result may be regarded as a supplement to that in our earlier [7].

2. A module M is said to be an essential extension of a of a submodule N if $N \cap N' \neq 0$ for every nonzero submodule N' of M. Notation: $N \subset' M$.

An injective module is a module which is a direct summand of every extension module.

Let M be an injective module and N a submodule. Then it is well known that some maximal essential extensions of N are submodules of M, and each of them is injective and hence is a direct summand of M. Thus, every injective module M satisfies the following

CONDITION (B). For every submodule N of a module M there is a direct summand N' of M such that $N \subset' N'$.

THEOREM 2. A regular ring S is left continuous if and only if the left S-module S satisfies (B).

Proof. Suppose that the module S fulfils (B), and let (Se_α) be a subset of $L(S)$. Then $\Sigma Se_\alpha \subset' Se$ for some e. We shall show that $Se = \cup Se_\alpha$. To this end we let $Sf \supset Se_\alpha$ for every α and set $Se = (Se \cap Sf) \oplus Sg$. Then, $(\Sigma Se_\alpha) \cap Sg \subset (Se \cap Sf) \cap Sg = 0$, and so $Sg = Sg \cap Se = 0$, since $\Sigma Se_\alpha \subset' Se$. Hence $Se = Se \cap Sf \subset Sf$, which implies that $Se = \cup Se_\alpha$, as desired. Therefore $L(S)$ is complete. Next, in order to see that $L(S)$ satisfies (A) we let $(\cup Se_i) \cap Sh = 0$ for every finite subset (Se_i)

of (Se_α) and for some $h \neq 0$. Then $(\Sigma Se_\alpha) \cap Sh = 0$, and therefore $(\cup Se_\alpha) \cap Sh = 0$ since $\Sigma Se_\alpha \subset' Se = \cup Se_\alpha$. Thus, $\cup Se_\alpha \neq S$ and L(S) satisfies (A). This implies by theorem 1 that L(S) is upper continuous, as desired. The only if part of the theorem also may be proved easily. See [7; lemma 2].

A ring S with unit is called <u>left</u> (<u>right</u>) <u>self</u> <u>injective</u> if the left (right) S-module S is injective.

Now we consider the following

CONDITION (C). Let S be a ring. For any left ideals L_1 and L_2 with $L_1 \cap L_2 = 0$ there is an element x of S such that the right multiplication of x gives the projection $L_1 + L_2 \to L_1$.

It is easy to see that every left self injective ring satisfies this condition.

THEOREM 3. A regular ring S is left continuous if and only if it satisfies (C).

Proof. If S is left continuous, and if L_1 and L_2 are left ideals of S such that $L_1 \cap L_2 = 0$, then by theorem 2 there are principal left ideals Se_i, $i = 1, 2$, with $L_i \subset' Se_i$. Since evidently $Se_1 \cap Se_2 = 0$ we may assume with no loss in generality that e_1 and e_2 are orthogonal idempotents. It is then evident that the projection $L_1 \oplus L_2 \to L_1$ is given by the right multiplication of e_1, as desired. Next, to prove the if part of the theorem let L_1 be a left ideal, and denote by L_2 a maximal left ideal disjoint to L_1. Then by (C) there is an element x such that $L_1(1 - x) = 0$, and $L_2x = 0$. If $L_1 \cap L_3 = 0$ and $L_3(1 - x) = 0$, the sum of L_1, L_2 and L_3 is direct, and hence $L_3 = 0$ by the maximality of L_2. Therefore the left annihilator ideal of $1 - x$ which is a direct summand of S, is an essential extension of L_1. Thus the left S-module S satisfies (B), and so S is left continuous by theorem 2, completing the proof.

From theorem 2 or theorem 3 we re-obtain immediately

THEOREM 4. Every semi-simple left self injective ring is left continuous. (See [7; theorem 1].)

In fact, any semi-simple left self injective ring is regular. (See, for instance, [6; lemma 8].)

We have proved in [7; lemma 8] the only if part of the following

THEOREM 5. A regular ring S is left continuous if and only if there is an extension ring T such that (i) T is semi-simple left self injective and (ii) every idempotent of T is contained in S.

Proof. If S has such an extension ring T, L(T) is upper continuous by theorem 4. Since it is easily verified that $L(S) \simeq L(T)$, L(S) is also upper continuous, and hence S is left continuous, as desired.

3. R. E. Johnson [1] considered the following

CONDITION (D). Let S be a ring. If $L \subset' S$ for a left ideal L, then $r(L) = 0$.[1]

Now we consider in connection with this the following

CONDITION (E). Let S be a ring. If S is not an essential extension of a left ideal L, the $r(L) \neq 0$.

THEOREM 6. Let S be a regular ring. Then S satisfies (E) if and only if L(S) satisfies (A).

Proof. We assume that S satisfies (E). Let (Se_α) be a subset of L(S), and suppose that there is $0 \neq h \in S$ with $(\cup Se_i) \cap Sh = 0$ for every finite subset (Se_i) of (Se_α). Set $L = \Sigma Se_\alpha$. Then $L \cap Sh = 0$, and so $r(L) \neq 0$ by (E). Let $r(L) \ni e = e^2 \neq 0$. We have then $S \neq S(1 - e) \supset L$, and therefore L(S) satisfies (A). Conversely, if L(S) satisfies (A), and if S is not an essential extension of a left ideal L, then $L \cap Sh = 0$ for some $h \neq 0$, and so $(\cup Sx_i) \cap Sh = 0$ for every finite subset (x_i) of L. Hence by (A) there is an idempotent f such that $S \neq Sf \supset L$. Thus $r(L) \supset (1 - f)S \neq 0$, which shows that S satisfies (E), completing the proof.

---

[1] By r(*) we denote the right annihilator ideal of * in the ring S.

As an immediate consequence of theorems 1 and 6 we obtain

THEOREM 7. Let S be a regular ring. Then S is left continuous if and only if S satisfies (E) and L(S) is complete.

R. E. Johnson [1] defined the <u>maximal left quotient ring</u> for every ring satisfying (D).

A ring is called a <u>semi-simple I-ring</u> if every nonzero one-sided ideal contains a non-zero idempotent. It is known that every semi-simple I-ring fulfils (D). (See [5; (4.10)].)

THEOREM 8. Let S be a semi-simple I-ring and $\overline{S}$ the maximal left quotient ring of S. Then S satisfies (E) if and only if every nonzero right ideal of $\overline{S}$ has a nonzero intersection with S.

Proof. Assume that S satisfies (E), and let $0 \neq x \in \overline{S}$. Then the left annihilator ideal L of x in $\overline{S}$ is a principal left ideal of $\overline{S}$. Since $x \neq 0$, $L \neq \overline{S}$ and $L \cap S \neq S$. Hence S is not an essential extension of $L \cap S$, and so $r(L \cap S) \neq 0$ by (E), whence $r(L \cap S) \ni f = f^2 \neq 0$ for some f. Then $L \cap S \subset \overline{S}(1 - f)$, and $L \subset \overline{S}(1 - f)$. Therefore $x\overline{S} \supset f\overline{S}$ and $0 \neq f \in x\overline{S} \cap S$, proving the only if part of the theorem. In order to see the if part, we assume that a regular ring S satisfies the condition of the theorem. Let L be a left ideal of S, and suppose that S is not an essential extension of L. Then there is $e \in \overline{S}$ such that $L \subset' \overline{S}e \cap S$. Clearly $\overline{S}e \cap S \neq S$, and so $e \neq 1$. By assumption $(1 - e)\overline{S} \cap S \neq 0$. Hence $(1 - e)\overline{S} \cap S$ contains a nonzero idempotent f. We have $\overline{S}e \subset \overline{S}(1 - f)$, and hence $L \subset \overline{S}(1 - f)$, whence $0 \neq f \in r(L)$, completing the proof.

4. Levitzki [2] introduced the concept of <u>weakly reducible rings</u>, imposing an additional postulate upon that of I-rings.

THEOREM 9. A semi-simple I-ring is weakly reducible if and only if every nonzero two-sided ideal contains a nonzero two-sided ideal of bounded index.

Proof. Let S be a semi-simple weakly reducible ring and A a nonzero two-sided ideal. Then A contains a matrix ideal M by [2; definition 3.1]. From [3; theorem 4] or [5; theorem 6] it

https://doi.org/10.4153/CMB-1961-011-6 Published online by Cambridge University Press

follows that M is of bounded index. Conversely, let B be a nonzero two-sided ideal of a semi-simple I-ring S, and suppose that B contains a nonzero two-sided ideal C of bounded index. By [2; theorem 3.3] C is semi-simple weakly reducible. Hence there is a matrix ideal N of C. Since N has a unit, N is a two-sided ideal of S, as desired.

We proved in [5; theorem 5] that the maximal left quotient ring of a semi-simple weakly reducible ring S is a right quotient ring of S. Hence by theorem 8 we obtain

THEOREM 10. Every semi-simple weakly reducible ring satisfies (E).

The following is an immediate consequence of theorems 7 and 10.

THEOREM 11. A regular weakly reducible ring S is (both left and right) continuous if and only if L(S) (or R(S)) is complete.

REFERENCES

1. R. E. Johnson, The extended centralizer of a ring over a module, Proc. Amer. Math. Soc. 2 (1951), 891-895.

2. J. Levitzki, Structure of algebraic algebras and related rings, Trans. Amer. Math. Soc. 74 (1953), 384-409.

3. ----------, Some theorems concerning associative Zorn rings, Bull. Res. Council Israel 3 (1954), 380-384.

4. F. Maeda, Kontinuierliche Geometrien (Springer, 1958).

https://doi.org/10.4153/CMB-1961-011-6 Published online by Cambridge University Press

5. Y. Utumi, On quotient rings, Osaka Math. J. 8 (1956), 1-18.

6. ----------, On a theorem on modular lattices, Proc. Japan Acad. 35 (1959), 16-21.

7. ----------, On continuous regular rings and semi-simple self injective rings, Canad. J. Math. 12 (1960), 597-605.

Osaka Women's University

and

McGill University