

## THE STRONG $c$ -SYMMETRIC DISTRIBUTION

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### Abstract

In this paper, we consider a class of strong symmetric distributions, which we refer to as the strong  $c$ -symmetric distributions. We provide, as the main result of this paper, conditions satisfied by the recurrence relations of certain polynomials associated with these distributions.

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### 1. Introduction

A strong distribution,  $d\psi(t)$ , is one for which the moments

$$\mu_n = \int_{\mathbf{R}} t^n d\psi(t),$$

exist and are finite for  $n = 0, \pm 1, \pm 2, \dots$ . We believe that the first to consider such distributions were Jones et al. [1], in relation to the moment problem which they referred to as the “strong moment problem.” Recently, in Sri Ranga [3] and in Sri Ranga and McCabe [4], extensive studies have been made on certain polynomials associated with such distributions, and in particular, on the recurrence relations satisfied by these polynomials. These polynomials are very much related to the Laurent polynomials considered for example in [2].

Here, first we present some results, a good part of which are given in [4], and the rest easily follow from these.

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For any strong distribution,  $d\psi(t)$ , the relations

$$(1.1) \quad \begin{aligned} & \text{(i) } Q_n(z) \text{ is a monic polynomial of degree } n, \text{ for } n \geq 0, \\ & \text{(ii) } \int_{\mathbf{R}} t^{-2[n/2]+s} Q_n(t) d\psi(t) = 0, \quad 0 \leq s \leq n-1, \text{ for } n \geq 1, \end{aligned}$$

uniquely define the (monic) polynomials  $Q_n(z)$ , for  $n \geq 0$ . Here,  $[x]$  represents the integral part of  $x$ .

If the Hankel determinants  $H_s^{(k)} = \det(\mu_{k+j+m})_{j,m=0}^{s-1}$ , associated with the moments of this distribution satisfy

$$H_{2n}^{(-2n+1)} \neq 0, \quad n \geq 0,$$

then the polynomials  $Q_n(z)$  satisfy the recurrence relations

$$(1.2) \quad \begin{aligned} Q_{2n}(z) &= (z - \beta_{2n})Q_{2n-1}(z) - \alpha_{2n}Q_{2n-2}(z), \\ Q_{2n+1}(z) &= \{(1 + \alpha_{2n+1})z - \beta_{2n+1}\}Q_{2n}(z) - \alpha_{2n+1}z^2Q_{2n-1}(z), \end{aligned}$$

for  $n \geq 1$ , where  $Q_0(z) = 1$ ,  $Q_1(z) = (z - \beta_1)$ ,  $\alpha_{n+1} > 0$  and  $\beta_n \in \mathbb{R}$ .

If this distribution is symmetric (about the origin), i.e.,  $d\psi(t) = -d\psi(-t)$ , then its Hankel determinants satisfy

$$H_{2r+1}^{(2m+1)} = 0 \quad \text{and} \quad (-1)^r H_{2r}^{(2m+1)} > 0,$$

for  $r \geq 0$  and  $m = 0, \pm 1, \pm 2, \dots$ . As a result, the polynomials  $Q_n(z)$  satisfy the recurrence relations (1.2), and in these recurrence relations, the coefficients  $\beta_n$  are zero for all  $n \geq 1$ . Hence, for example, we have for any strong symmetric distribution

$$(1.3) \quad Q_{2n}(0) \neq 0, \quad n \geq 0,$$

and

$$(1.4) \quad \int_{\mathbf{R}} t^{-2n-1} Q_{2n}(t) d\psi(t) = 0, \quad n \geq 0.$$

In the three examples of strong symmetric distributions given in [4], namely, the strong Tchebycheff, the strong Legendre and the strong Hermite distributions, we note that the coefficients  $\alpha_n$  and  $\beta_n$  of the associated recurrence relations satisfy, in addition to the expected condition  $\beta_n = 0, n \geq 1$ , the condition that  $\alpha_{2n}$  are constant for all  $n \geq 1$ . We show here that this is due to these distributions being members of a "special" class of strong symmetric distributions.

### 2. The strong $c$ -symmetric distributions

We call any strong symmetric distribution,  $d\psi(t)$ , for which there corresponds a constant  $c$  such that

$$(2.1) \quad d\psi(t)/t = -d\psi(c/t)/(c/t),$$

for all  $t \in \mathbb{R}$ , a “strong  $c$ -symmetric distribution.”

For a strong  $c$ -symmetric distribution  $d\psi(t)$ , the following two conditions hold.

(a) If one can write the increment of  $\psi(t)$  at the point  $t$  as  $\rho(t)$ , then  $\rho(t)/t = \rho(c/t)/(c/t)$ . This relation characterizes the strong  $c$ -symmetric distribution in all the points where  $\psi(t)$  has step increments.

(b) If  $d\psi(t)$  can be given as  $w(t)dt$  in the neighbourhood of some point  $\tilde{t}$  (possible wherever  $\psi(t)$  is continuous and differentiable), then the same holds in the neighbourhood of  $c/\tilde{t}$ , and the function  $w(t)$  satisfies  $w(t) = w(c/t)$ .

It is easily verified that the examples of strong symmetric distributions given in [4] are strong  $c$ -symmetric distributions (with  $c = ab$  in the first two examples and  $c = a$  in the other).

For any strong  $c$ -symmetric distribution  $d\psi(t)$ , we have from (2.1), with the substitution  $t = c/u$ , that

$$\int_0^{\sqrt{c}} t^m d\psi(t) = c^{m+1} \int_{\sqrt{c}}^{\infty} u^{-m-2} d\psi(u), \quad m = 0, \pm 1, \pm 2, \dots$$

This result together with the symmetry of  $d\psi(t)$  yields for the moments

$$(2.2) \quad \mu_m = c^{m+1} \mu_{-m-2}, \quad m \geq 0.$$

For any polynomial  $Q(t) = q_0 + q_1t + \dots + q_l t^l$ , we can write

$$\int_{\mathbb{R}} t^m Q(t) d\psi(t) = \sum_{r=0}^l q_r \mu_{m+r}, \quad m = 0, \pm 1, \pm 2, \dots$$

Hence, the relation (2.2) gives immediately that

$$(2.3) \quad \int_{\mathbb{R}} t^m Q(t) d\psi(t) = c^{m+1} \int_{\mathbb{R}} t^{-m-2} Q(c/t) d\psi(t), \quad m = 0, \pm 1, \pm 2, \dots$$

We now give the principal result of this article.

**THEOREM 3.1.** *For any strong  $c$ -symmetric distribution  $d\psi(t)$ , the polynomials  $Q_n(z)$ ,  $n \geq 0$ , defined by (1.1), satisfy the recurrence relation (1.2), with*

$$\beta_n = 0, \quad \alpha_{2n} = c, \quad \text{for } n \geq 1.$$

**PROOF.** As we have already mentioned, the result that  $Q_n(z)$  satisfy the recurrence relations (1.2) and that  $\beta_n = 0$ , for  $n \geq 1$ , follows from the symmetry of  $d\psi(t)$ . To prove the other, we have from (2.3) that

$$\int_{\mathbb{R}} t^{-2n+s} Q_{2n}(t) d\psi(t) = c^{-2n+s+1} \int_{\mathbb{R}} t^{2n-s-2} Q_{2n}(c/t) d\psi(t),$$

for  $s = 0, \pm 1, \pm 2, \dots$ . Hence, from (1.1) and (1.4), one can obtain

$$\int_{\mathbf{R}} t^{-s-2} \{t^{2n} Q_{2n}(c/t)\} d\psi(t) = 0, \quad -1 \leq s \leq 2n - 2.$$

This relation is also true for  $s = 2n - 1$ . However, as we do not need this for the rest of our analysis, we have omitted it in the proof.

From (1.3), since  $Q_{2n}(0) \neq 0$ , dividing the above relation by  $Q_{2n}(0)$  and then replacing  $s$  by  $2n - 2 - s$ , we obtain

$$\int_{\mathbf{R}} t^{-2n+s} \{t^{2n} Q_{2n}(c/t)/Q_{2n}(0)\} d\psi(t) = 0, \quad 0 \leq s \leq 2n - 1.$$

This relation is equivalent to (1.1), with even values of  $n$  only, and where, in place of the monic polynomials  $Q_{2n}(z)$ , one finds the monic polynomials  $z^{2n} Q_{2n}(c/z)/Q_{2n}(0)$ . Hence, as (1.1) uniquely defines its polynomials, we have

$$(2.4) \quad Q_{2n}(z) = z^{2n} Q_{2n}(c/z)/Q_{2n}(0), \quad n \geq 0.$$

An immediate consequence of this result is that if  $\tilde{z}$  is a zero of  $Q_{2n}(z)$  then so are  $c/\tilde{z}$ ,  $-\tilde{z}$  and  $-c/\tilde{z}$ . Here, we have also considered the evenness of  $Q_{2n}(z)$ .

With  $\beta_n = 0, n \geq 1$ , we obtain from the recurrence relation (1.2)

$$(2.5) \quad Q_{2n}(z) = (z^2 - \alpha_{2n})Q_{2n-2}(z) - \alpha_{2n-2}\alpha_{2n-1}z^2Q_{2n-4}(z),$$

for  $n \geq 2$ , with  $Q_0(z) = 1, Q_2(z) = z^2 - \alpha_2$ .

In (2.5), letting  $z = 0$  gives

$$(2.6) \quad Q_{2n}(0) = -\alpha_{2n}Q_{2n-2}(0) = (-1)^n \prod_{r=1}^n \alpha_{2r}, \quad n \geq 1.$$

Again in (2.5), replacing  $z$  by  $c/z$ , and then multiplying the resulting equation by  $z^{2n}/Q_{2n}(0)$ , one obtains

$$\left\{ \frac{z^{2n} Q_{2n}(c/z)}{Q_{2n}(0)} \right\} = \left( z^2 - \frac{c^2}{\alpha_{2n}} \right) \left\{ \frac{z^{2n-2} Q_{2n-2}(c/z)}{Q_{2n-2}(0)} \right\} - \frac{\alpha_{2n-1} c^2 z^2}{\alpha_{2n}} \left\{ \frac{z^{2n-4} Q_{2n-4}(c/z)}{Q_{2n-4}(0)} \right\},$$

for  $n \geq 1$ , with  $z^0 Q_0(c/z)/Q_0(0) = 1, z^2 Q_2(c/z)/Q_2(0) = z - c^2/\alpha_2$ .

Therefore, with result (2.4), we obtain from this recurrence relation

$$Q_{2n}(0) = -\frac{c^2}{\alpha_{2n}} Q_{2n-2}(0), \quad n \geq 1.$$

Comparing this with (2.6) we obtain  $\alpha_{2n} = c$ , for  $n \geq 1$ , and this completes the proof of the theorem.

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