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REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE II

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We consider a certain real hypersurface M of a complex projective space. The purpose of this paper is to characterize M in terms of Ricci curvatures.

0. Introduction

Let $P_n(\mathbb{C})$ be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4. We consider the Hopf fibration $\bar{\pi}$:

$$S^1 \to S^{2n+1} \xrightarrow{\bar{\pi}} P_n(\mathbb{C}) ,$$

where S^k denotes the Euclidean sphere of curvature 1. In S^{2n+1} we have the family of generalized Clifford surfaces whose fibres lie in complex subspaces (see [2]):

$$M_{2p+1,2q+1} = S^{2p+1}\left(\frac{2n}{2p+1}\right) \times S^{2q+1}\left(\frac{2n}{2q+1}\right) ,$$

where $p + q = n-1$. Then we have a fibration π :

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$$S^1 \to M_{2p+1,2q+1} \xrightarrow{\pi} M^{\mathbb{C}}_{p,q} ,$$

which is compatible with $\bar{\pi}$.

These manifolds $M^{\mathbb{C}}_{p,q}$ thus obtained have various beautiful properties (cf. [3], [4]). In the special case of $p = 0$, $M^{\mathbb{C}}_{0,n-1}$ is called the geodesic minimal hypersphere of $P_n(\mathbb{C})$ (see [5]).

Kon ([1]) characterized $M^{\mathbb{C}}_{0,n-1}$ in terms of sectional curvatures.

The purpose of this paper is to prove a pinching theorem in terms of Ricci curvatures. We have the following

THEOREM. *Let M be a connected real minimal hypersurface of $P_n(\mathbb{C})$. If $n \geq 3$ and the Ricci curvature S of M satisfies $2n - 2 \leq S \leq 2n$, then M is locally congruent to $M^{\mathbb{C}}_{p,p}$ $(2p = n - 1)$.*

1. Preliminaries

Let M be a real hypersurface of $P_n(\mathbb{C})$. In a neighbourhood of each point, we choose a unit normal vector field N in $P_n(\mathbb{C})$. The Riemannian connections $\tilde{\nabla}$ in $P_n(\mathbb{C})$ and ∇ in M are related by the following formulas for arbitrary vector fields X and Y on M:

(1.1) $$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N ,$$

(1.2) $$\tilde{\nabla}_X N = -AX ,$$

where g denotes the Riemannian metric on M induced from the Fubini-Study metric G on $P_n(\mathbb{C})$ and A is the (local) second fundamental form of M in $P_n(\mathbb{C})$. An eigenvector X of the second fundamental form A is called a *principal curvature vector*. Also an eigenvalue r of A is called a *principal curvature*. In what follows, we denote by V_r the eigenspace of A with eigenvalue r. It is known that M has an almost contact metric structure induced from the complex structure J of $P_n(\mathbb{C})$ (cf. [4]) i.e. we have a tensor field ϕ of type $(1,1)$ on M, given by

$g(\phi X,Y) := G(JX,Y)$ i.e. $\phi(X) = JX - G(JX,N)N$ for all tangent vectors X,Y of M, and depending on the local choice of N - one has the unit tangent vector field ξ and the 1-form η of M defined by

$$\xi := -JN \text{ respectively } \eta(X) := g(\xi,X) = G(JX,N) .$$

Then we have

$$\phi^2(X) = -X + \eta(X)\xi , \ g(\xi,\xi) = 1, \ \phi\xi = 0, \ \eta(\xi) = 1 . \tag{1.3}$$

From the above remark and (1.1), we have easily

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi , \tag{1.4}$$

$$\nabla_Y\xi = \phi AY . \tag{1.5}$$

Let $\tilde{R}$ and R be the curvature tensors of $P_n(\mathbb{C})$ and M, respectively. Since the curvature tensor $\tilde{R}$ has a nice form, we have the following Gauss and Codazzi equations:

$$(1.6) \quad \begin{cases} g(R(X,Y)Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(\phi Y,Z)g(\phi X,W) \\ \qquad -g(\phi X,Z)g(\phi Y,W) - 2g(\phi X,Y)g(\phi Z,W) \\ \qquad + g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W) \end{cases}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi . \tag{1.7}$$

Using (1.3) and (1.6), we get

$$R_0(X,Y) = (2n+1)g(X,Y) - 3\eta(X)\cdot\eta(Y)+(\text{trace } A)g(AX,Y)-g(AX,AY) , \tag{1.8}$$

where R_0 denotes the Ricci tensor of M.

Moreover, from (1.3) and (1.7) we obtain

$$g((\nabla_X A)Y,\xi) - g((\nabla_Y A)X,\xi) = -2g(\phi X,Y) . \tag{1.9}$$

2. Proof of Theorem

It follows from the assumption that the immersion is minimal and (1.3) that the equation (1.8) implies

(2.1) $$R_0(\xi,\xi) = 2n - 2 - g(A\xi,A\xi) .$$

This, together with $R_0(\xi,\xi) \geqq 2n - 2$, shows

(2.2) $$A\xi = 0 .$$

Now, differentiating (2.2) covariantly along X and making use of (1.5), for any Y we get

(2.3) $$g((\nabla_X A)Y,\xi) + g(A\phi AX,Y) = 0 .$$

Exchanging X and Y in (2.3), we see

(2.4) $$g((\nabla_Y A)X,\xi) + g(A\phi AY,X) = 0 .$$

From (1.9), (2.3) and (2.4), we find $g(A\phi AX - \phi X,Y) = 0$ so that

(2.5) $$A\phi AX = \phi X .$$

Here and in the sequel, let $X(\perp\xi)$ be a principal curvature vector with eigenvalue r i.e. $X \in V_r$. From (2.5) we obtain $r\cdot A(\phi X) = \phi X$, that is,

(2.6) $$A(\phi X) = 1/r\cdot \phi X \text{ ; that is, } \phi X \in V_{1/r} .$$

(In fact, if $r = 0$, then $\phi X = 0$, which is a contradiction.)

On the other hand the equation (1.8) shows $R_0(X,X) = 2n + 1 - r^2 \leqq 2n$ so that $r^2 \geqq 1$.

Similarly we have $R_0(\phi X,\phi X) = 2n + 1 - 1/r^2 \leqq 2n$ so that $r^2 \leqq 1$.

So we find that $r = \pm 1$.

This, together with the assumption that the immersion is minimal, implies that our real hypersurface M has three constant principal curvatures $\{0,\pm 1\}$ at each point.

Here we recall Takagi's work [5]. He determined all real hypersurfaces in $P_n(\mathbb{C})$ $(n \geqq 3)$ with three constant principal curvatures.

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Due to his work, we conclude that our real hypersurface M is locally congruent to $M^{\mathbb{C}}_{p,p}$ $(2p = n - 1)$.

Of course the manifold $M^{\mathbb{C}}_{p,p}$ satisfies the assumption of our Theorem.

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