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REPRESENTATION OF ALGEBRAS WITH INVOLUTION

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Introduction. Let K be a field with an involution J. A $*$-algebra over K is an associative algebra A with an involution $*$ satisfying $(\alpha.a)^* = \alpha^J.a^*$. A large class of examples may be obtained as follows. Let (V, φ) be an hermitian space over K consisting of a vector space V and a left hermitian (w.r.t. J) form φ on V which is nondegenerate in the sense that $\varphi(V, v) = 0$ implies $v = 0$. An endomorphism f of V may have an adjoint f^* w.r.t. φ, defined by $\varphi(f(u), v) = \varphi(u, f^*(v))$; due to the nondegeneracy of φ, f^* is unique if it exists. The set $B(V, \varphi)$ of all endomorphisms of V which do have an adjoint is easily verified to be a $*$-algebra.

We shall prove, conversely, that every $*$-algebra A satisfying the mild restriction

$$(1) \qquad Aa = 0 \text{ implies } a = 0$$

can be imbedded as a $*$-subalgebra of $B(V, \varphi)$ for some hermitian space (V, φ). Secondly, we shall investigate which $*$-algebras can still be imbedded in $B(V, \varphi)$ if φ is assumed to be "positive" in a certain sense.

Results of this type are well-known in the context of Banach algebras with involution; e.g., Gelfand and Naimark [**2**], Schatz [**5**]. Our methods of proof owe much to these sources.

1. The general case. Suppose A is a $*$-algebra over K. The dual space $A^\wedge$ also has an "involution" $s \mapsto s^*$, where $s^*(a) = s(a^*)^J$. If s is hermitian w.r.t. this involution, $(a, b) \mapsto s(a^*b)$ is a left hermitian form on A. Its radical is the left ideal

$$(2) \qquad I_s = \{b \in A \mid s(ab) = 0 \text{ for all } a \in A\}$$

of A, so that it induces a nondegenerate left hermitian form φ_s on the left A-module A/I_s. Since $s((xa)^*b) = s(a^*(x^*b))$, we have $\varphi_s(x.a, b) = \varphi_s(a, x^*.b)$. In other words, left multiplication by x has an adjoint w.r.t. φ_s equal to left multiplication by x^*.

Suppose J is nontrivial; let F be the fixed field of J and $\theta \in K$ such that $\theta^J \neq \theta$. The set A^h of hermitian elements of A is clearly a vector space over F. Every $a \in A$ can be written uniquely in the form

$$(3) \qquad a = a_1 + \theta.a_2,$$

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where a_1 and $a_2 \in A^h$, by taking

$$a_1 = (\theta.a^* - \theta^J.a)/(\theta - \theta^J),\ a_2 = (a - a^*)/(\theta - \theta^J).$$

An hermitian functional of A maps A^h into F and, conversely, every functional t of A^h induces an hermitian functional s of A, by defining $s(a)$ to be $t(a_1) + \theta t(a_2)$, relative to the decomposition (3). Symbolically, we have shown that $A^{\wedge h} \cong A^{h\wedge}$.

If J is trivial, hermitian functionals are those which vanish on the subspace $A^s = \{a - a^* \mid a \in A\}$ of A. In this case, $A^{\wedge h} \cong (A/A^s)^\wedge$.

PROPOSITION 1. *Let $I(A) = \bigcap I_s$, taken over all hermitian functionals s of A. If J is nontrivial, $A.I(A) = 0$. If J is trivial, $A.I(A)$ is a $*$-ideal of A contained in A^s and such that $(A.I(A))^3 = 0$.*

Proof. Suppose J is nontrivial. Let $a \in A$ be such that $ba \neq 0$ for some $b \in A$. We can write $ba = c_1 + \theta.c_2$ with $c_1, c_2 \in A^h$. There exists a functional t of A^h for which either $t(c_1)$ or $t(c_2)$ is nonzero. Extending t to an hermitian functional s of A, we conclude that $s(ba) \neq 0$ so that $a \in I_s$. Hence $A.I(A) = 0$.

Suppose J is trivial. If $x \in I(A)$ and $a \in A$, we have $ax \in A^s$ since, otherwise, we could find an hermitian functional s such that $s(ax) \neq 0$. In particular, $(ax)^* = -ax$; since $A.I(A)$ is already a left ideal, this shows that it is in fact a $*$-ideal. If $x \in A.I(A)$ and $a \in A$ we have, as before, $(ax)^* = -ax$ or $xa^* = ax$ since now $x^* = -x$. Suppose $x, y \in A.I(A)$ and $a \in A$. Then

$$(xy)a^* = a(xy) = (ax)y = (xa^*)y = x(a^*y) = x(ya) = (xy)a$$

so that $xy(a - a^*) = 0$. Since $A.I(A) \subset A^s$, this implies that $(A.I(A))^3 = 0$.

When J is trivial, it may happen that $A.I(A) \neq 0$. For example, suppose $\mathrm{char}(K) \neq 2$ and let A be the algebra $K[T]/(T^2)$ with the involution $(\alpha + \beta T)^* = \alpha - \beta T$. Then $I(A)$ consists of all multiples of T.

To rectify this difficulty, we turn to the skew-hermitian functionals of A. If t is such a functional, one can verify that

$$((x_1, x_2), (y_1, y_2)) \mapsto t(x_1^*y_2 - x_2^*y_1)$$

is an hermitian form on $A \oplus A$ with radical $I_t \oplus I_t$, where I_t is given by (2), and therefore induces a nondegenerate hermitian form φ_t on the left A-module $A/I_t \oplus A/I_t$. As before, left multiplication by x has an adjoint w.r.t. φ_t equal to left multiplication by x^*.

Let $I'(A) = \bigcap I_t$, taken over all skew-hermitian functionals of A. Suppose J is trivial and $\mathrm{char}(K) \neq 2$; skew-hermitian functionals are those which vanish on elements of the form $a + a^*$. If $x \in A$, and $ax \neq 0$ we know from Proposition 1 that $(ax)^* = -ax \neq ax$ so that $t(ax) \neq 0$ for some skew-hermitian functional t; hence $x \notin I_t$. In other words,

$$A.(I(A) \cap I'(A)) = 0 \tag{4}$$

is true in every case other than when J is trivial and $\mathrm{char}(K) = 2$.

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PROPOSITION 2. *Suppose A is a $*$-algebra over K satisfying* (1). *There exists an hermitian space (V, φ) over K and an injective $*$-algebra homomorphism $\lambda: A \rightarrow B(V, \varphi)$. If A is finite-dimensional, V may be chosen to be finite-dimensional, except possibly in the case when J is trivial and* $\operatorname{char}(K) = 2$.

Proof. Suppose that either J is nontrivial or $\operatorname{char}(K) \neq 2$. We conclude from (1) and (4) that $I(A) \cap I'(A) = 0$. Therefore there exists a family $\{s_i\}$ of hermitian or skew-hermitian functionals of A for which $\bigcap I_{s_i} = 0$. If A is finite-dimensional, we can choose a finite family with this property. Let V_i be either A/I_{s_i} if s_i is hermitian or $A/I_{s_i} \oplus A/I_{s_i}$ if s_i is skew-hermitian and put $V = \bigoplus V_i, \varphi = \bigoplus \varphi_{s_i}$. If A is finite-dimensional, so is V. For $a \in A$, define $\lambda(a)$ to be left multiplication by a; it follows from the preceding discussion that $\lambda(a)^*$ exists and equals $\lambda(a^*)$. If $\lambda(a) = 0$, we have $aA \subset \bigcap I_{s_i} = 0$ so that $Aa^* = 0$. By (1), $a^* = 0$ and hence $a = 0$.

Suppose now that J is trivial and $\operatorname{char}(K) = 2$. The rational function field $K(X)$ possesses the involution $J'(f(X)) = f(1/X)$. The algebra $A' = A \otimes_K K(X)$ with the involution $(a \otimes f)^* = a^* \otimes J'(f)$ is a $*$-algebra over $K(X)$. Since J' is nontrivial, the first part of the proof shows the existence of an imbedding $\lambda': A' \rightarrow B(V', \varphi')$ for some hermitian space (V', φ') over $K(X)$. Choose a nonzero hermitian functional σ of $K(X)$, regarded as a $*$-algebra over K. Let V be V' regarded as a vector space over K and φ the left hermitian form $\varphi(x, y) = \sigma(\varphi'(x, y))$ on V. For a fixed $x \neq 0$, $\varphi'(x, y)$ assumes every value in $K(X)$ so that $\sigma(\varphi'(x, y)) \neq 0$ for some y; i.e., φ is nondegenerate. Clearly $B(V', \varphi') \subset B(V, \varphi)$ and $*$ means the same in both algebras. Combining this inclusion with the canonical injection $A \rightarrow A'$, we obtain the desired homomorphism $\lambda: A \rightarrow B(V, \varphi)$.

It seems reasonable to conjecture that the second assertion of Proposition 2 holds without exception. This is true, for example, if K has a finite algebraic extension K' which has a nontrivial involution leaving K fixed.

2. Positive algebras. In this section we shall assume that F, the fixed field of J, is formally real and that K is either F or $F(\sqrt{(-\xi)})$, where ξ is a sum of squares in F; in the latter case, $J(\sqrt{(-\xi)}) = -\sqrt{(-\xi)}$. Let Ω be a fixed algebraic closure of K, $\{R_\lambda\}_{\lambda \in \Lambda}$ the set of real closures of F in Ω, and J_λ the involution of Ω which leaves R_λ fixed and sends $\sqrt{(-1)}$ to $-\sqrt{(-1)}$. The assumption on ξ implies that J_λ is always an extension of J. We shall denote by F^+ the set of elements in F which are sums of squares. If $\alpha \in K$, it is clear that $\alpha^J\alpha \in F^+$.

A left hermitian form φ on a vector space V over K is called positive if $\varphi(v, v) \in F^+$ for all $v \in V$. Starting from the fact that $\varphi(v + \alpha.u, v + \alpha.u) \in F^+$ for all $\alpha \in K$, the usual argument for the Cauchy-Schwarz inequality proves

PROPOSITION 3. *If φ is positive, then*:
(a) $\varphi(v, v)\, \varphi(u, u) - \varphi(v, u)^J \varphi(v, u) \in F^+$;
(b) *φ is nondegenerate if and only if $\varphi(v, v) = 0$ implies $v = 0$.*

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For each $\lambda \in \lambda$, let $V_\lambda = V \otimes_K \Omega$, regarded as a vector space over Ω, and φ_λ the left hermitian form (*w.r.t.* J_λ) on V_λ defined by

$$\varphi_\lambda(v \otimes \alpha, u \otimes \beta) = J_\lambda(\alpha)\varphi(v, u)\beta.$$

PROPOSITION 4. *If φ is positive, so is φ_λ.*

Proof. Let

$$w = \sum_{i=1}^{n} v_i \otimes \alpha_i$$

be an element of V_λ. We may assume $\varphi(v_1, v_1) \neq 0$ since, otherwise, $\varphi(v_1, u) = 0$ for all u by Proposition 3(a) and the element $v_1 \otimes \alpha_1$ makes no contribution to the value of $\varphi_\lambda(w, w)$. One can then write

$$w = v_1 \otimes \beta_1 + \sum_{i>1} v_i' \otimes \alpha_i,$$

where

$$v_i' = v_i - (\varphi(v_1, v_i)/\varphi(v_1, v_1))v_1$$

and

$$\beta_1 = \sum_{i=1}^{n} (\varphi(v_1, v_i)/\varphi(v_1, v_1))\alpha_i.$$

Induction on n shows that $\varphi_\lambda(w, w) \in R_\lambda^+$ since this is clearly true for $n = 1$.

We call a $*$-algebra A positive if it can be imbedded as a $*$-subalgebra of $B(v, \varphi)$ for some positive hermitian space (V, φ). Our aim is to find intrinsic conditions for positivity.

A functional s of A is called positive if it is hermitian and such that $s(a^*a) \in F^+$ for all $a \in A$. Let $I^+(A) = \bigcap I_s$, taken over all positive functionals of A. Applying Proposition 3(a) to the positive left hermitian form $(a, b) \mapsto s(a^*b)$ on A, we conclude that

$$s(a^*a)s(b^*b) - s(a^*b)^J s(a^*b) \in F^+. \tag{5}$$

Therefore in this case

$$I_s = \{b \in A \mid s(b^*b) = 0\}. \tag{6}$$

PROPOSITION 5. *$I^+(A)$ is an ideal of A. If A has a unit element, $I^+(A)$ is closed under $*$.*

Proof. Being an intersection of left ideals, $I^+(A)$ is clearly a left ideal. Suppose $x \in I^+(A)$; for every positive functional s of A and every $a \in A$, the functional $s'(b) = s(a^*ba)$ is also positive so that $s(a^*x^*xa) = s((xa)^*(xa)) = 0$. In view of (6), we must have $xa \in I^+(A)$; i.e., $I^+(A)$ is also a right ideal.

In particular, $s(xx^*xx^*) = 0$ for each positive functional s. If A has a unit element then, using (5) with $a = 1$ and $b = xx^*$, we conclude that $s(xx^*) = 0$ so that $x^* \in I^+(A)$ by (6).

PROPOSITION 6. *A $*$-algebra A satisfying* (1) *is positive if and only if* $I^+(A) = 0$.

Proof. Suppose $I^+(A) = 0$; since a positive functional s yields a positive form φ_s and a direct sum of positive forms is still positive, the same method as used in the proof of Proposition 2 shows that A is positive. (Furthermore, if A is finite-dimensional, the space V can also be chosen finite-dimensional.)

Conversely, suppose A is a $*$-subalgebra of $B(V, \varphi)$ for some positive hermitian space (V, φ). For each $v \in V$, $s_v(a) = \varphi(v, a(v))$ is a positive hermitian functional of A. If $x \in I^+(A)$,

$$s_v(x^*x) = \varphi(v, x^*x(v)) = \varphi(x(v), x(v)) = 0$$

for all $v \in V$ so that $x(v) = 0$; i.e., $x = 0$.

We now turn to an altogether different condition for positivity. Call a $*$-algebra A anisotropic if it satisfies

$$a^*a = 0 \text{ implies } a = 0 \tag{7}$$

and totally anisotropic if the algebra $A_\lambda = A \otimes_K \Omega$, with the involution $(a \otimes \alpha)^* = a^* \otimes J_\lambda(\alpha)$, is anisotropic for all $\lambda \in \Lambda$. Either property is obviously preserved in passing to a $*$-subalgebra.

PROPOSITION 7. *A positive $*$-algebra A is totally anisotropic.*

Proof. In view of the preceding remark, it suffices to verify that $B(V, \varphi)$ is totally anisotropic if (V, φ) is a positive hermitian space over K. On the other hand, we have an injective $*$-algebra homomorphism

$$\pi: B(V, \varphi) \otimes_K \Omega \to B(V_\lambda, \varphi_\lambda),$$

given by $\pi(f \otimes \alpha)(v \otimes \beta) = f(v) \otimes \alpha\beta$, so that again it suffices to prove that $B(V_\lambda, \varphi_\lambda)$ is anisotropic. Suppose $a^*a = 0$ holds in $B(V_\lambda, \varphi_\lambda)$; then

$$\varphi_\lambda(a^*a(v), v) = \varphi_\lambda(a(v), a(v)) = 0.$$

Since φ_λ is positive by Proposition 4, $a(v) = 0$ for all $v \in V$; i.e., $a = 0$.

As a partial converse, we have

PROPOSITION 8. *A finite-dimensional totally anisotropic $*$-algebra A is positive.*

Proof. Starting from (7), a well-known argument [**3**] shows that A has no nil ideals—in our context, this means that A must be semi-simple. Furthermore, if B is a minimal ideal of A, so is B^* and thus either $B^* = B$ or $B^*B = 0$; but the latter possibility is again excluded by (7). Since a product of positive algebras is easily seen to be positive, it is sufficient to prove the assertion in the case when A is simple.

Let tr: $A \to K$ be the reduced trace. If $a \in A$, we may compute $\mathrm{tr}(a^*a)$ in the extended algebra A_λ. Suppose $A_\lambda \cong \mathrm{End}_\Omega(V)$ for some finite-dimensional vector space V over Ω. It is well-known [**1**] that the involution induced by

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$*$ on $\mathrm{End}_\Omega(V)$ must be the adjoint involution corresponding to a nondegenerate left hermitian or skew-hermitian (w.r.t. J_λ) form ψ on V. We claim that ψ is hermitian and that either ψ or $-\psi$ is positive. If not, there would exist a nonzero $w \in V$ such that $\psi(w, w) = 0$. Choose a nonzero $f \in \mathrm{End}_\Omega(V)$ whose image is contained in $\Omega.w$. Then

$$\psi(v, f^*f(u)) = \psi(f(v), f(u)) = 0$$

for all $v, u \in V$ so that $f^*f = 0$, which contradicts (7) since A_λ is assumed to be anisotropic.

Since both ψ and $-\psi$ induce the same involution on $\mathrm{End}_\Omega(V)$, we may assume that ψ is positive. A standard argument [**4**] now shows that $\mathrm{tr}(f^*f) \in R_\lambda^+$. Since this holds for all $\lambda \in \Lambda$, we conclude that $\mathrm{tr}(a^*a) \in F^+$. Furthermore, $\mathrm{tr}(a^*a) = 0$ implies $a = 0$ since this is true in A_λ. In other words, $\mathrm{tr}: A \to K$ is a positive functional—it is obviously hermitian—such that $I_{\mathrm{tr}} = 0$; therefore, $I^+(A) = 0$ and A is positive by Proposition 6.

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