A NON-EXISTENCE THEOREM FOR (v, k, λ)-GRAPHS

W. D. WALLIS

(Received 25 June 1969)

Communicated by E. S. Barnes

A (v, k, λ)-graph is defined in [3] as a graph on v points, each of valency k, and such that for any two points P and Q there are exactly λ points which are joined to both. In other words, if S_i is the set of points joined to P_i, then

$$S_i \text{ has } k \text{ elements for any } i$$
$$S_i \cap S_j \text{ has } \lambda \text{ elements if } i \neq j.$$

The sets S_i are the blocks of a (v, k, λ)-configuration, so a necessary condition on v, k, and λ that a graph should exist is that a (v, k, λ)- configuration should exist. Another necessary condition, reported by Bose (see [1]) and others, is that there should be an integer m satisfying

$$\begin{aligned} &m^2 = k-\lambda \\ (1)\qquad &m|\lambda \\ &km^{-1} \text{ and } v-1 \text{ have equal parity.} \end{aligned}$$

We shall prove that these conditions are not sufficient.

Suppose there is a (v, k, λ)-graph with points P_i and with S_i as defined above. Then

$$P_i \notin S_i \text{ for any } i,$$
$$P_i \in S_j \Leftrightarrow P_j \in S_i;$$

and the corresponding (v, k, λ) configuration with varieties P_i and blocks S_j also has this property. Now consider the dual of this configuration [2, p. 250], which is a $(v, v-k, v-2k+\lambda)$-configuration with varieties P_i and blocks T_j, defined by

$$P_i \in T_j \Leftrightarrow P_i \notin S_j$$

This will have the property

$$\begin{aligned} &P_i \in T_i \text{ for any } i, \\ (2)\qquad &P_i \in T_j \Leftrightarrow P_j \in T_i, \end{aligned}$$

if a configuration exists then its dual exists, so we have proven

THEOREM 1. *If there is a* (v, k, λ)*-graph, then there is a* $(v, v-k, v-2k+\lambda)$*-configuration with the property* (2).

THEOREM 2. *There can be no* (v, j, μ)*-configuration with property* (2) *and with* $\mu = 1$.

PROOF. Suppose such a configuration existed. By definition $j > \mu$ (the trivial case of a (1, 1, 1)-configuration is normally excluded by definition), so T_1 has at least two members. P_1 is one; call the other P_i. Then by (2)

$$P_i \in T_i$$

$$P_1 \in T_i \text{ since } P_i \in T_1. \tag{3}$$

Therefore

$$\{P_1, P_i\} \subset T_i \cap T_1$$

so $T_i \cap T_1$ has at least two members, in contradiction of $\mu = 1$.

Theorems 1 and 2 together tell us that there can be no (v, k, λ)- graph with

$$v-2k+\lambda = 1.$$

This means that there is no graph with parameters

$$(r^2+r+1, r^2, r^2-r).$$

This is the dual of the triad $(r^2+r+1, r+1, 1)$, which corresponds to a projective plane of order r; such a configuration always exists when r is a prime power [2, p. 175], so (taking the dual gain) there is always an (r^2+r+1, r^2, r^2-r)-configuration when r is a prime power. In particular take $r = 2^{2n}$ and $m = 2^n$. Then these parameters satisfy (1).

COROLLARY. *If n is any natural number, there is always a* $(2^{4n}+2^{2n}+1, 2^{4n}, 2^{4n}-2^{2n})$*-configuration, and these parameters always satisfy* (1), *but there is no* (v, k, λ)*-graph with this* v, k *and* λ. *Thus the conditions stated are not suffcient; there are infinitely many counter-examples.*

Since $v-2k+\lambda$ is a non-negative integer, and cannot equal 1, there are two cases: either it is 0 (in which case we must have the complete graph of order v), or

$$v-2k+\lambda \geqq 2,$$

which is equivalent to

$$(k-\lambda)^2 \geqq k+\lambda.$$

Thus we can add to (1) the following further necessary condition:
(4) *either the graph is complete or*

$$m^4 \geqq k+\lambda.$$

https://doi.org/10.1017/S1446788700006820 Published online by Cambridge University Press

References

[1] R. W. Ahrens and C. Szekeres, 'On a combinatorial generalization of twenty-seven lines associated with a cubic surface', *J. Australian Math. Soc.* 10 (1969), 465—492.
[2] Marshall Hall Jr., *Combinatorial Theory* (Blaisdell, 1967).
[3] W. D. Wallis, 'Certain graphs arising from Hadamard matrices', *Bull. Australian Math. Soc.* 1 (1969), 325—331.

Department of Mathematics
La Trobe University
Bundoora, Victoria 3083