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AN APPLICATION OF LYAPUNOV'S DIRECT METHOD TO THE STUDY OF OSCILLATIONS OF A DELAY DIFFERENTIAL EQUATION OF EVEN ORDER

RAYMOND D. TERRY

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Abstract

The direct method of Lyapunov is utilized to obtain a variety of criteria for the nonexistence of certain types of positive solutions of a delay differential equation of even order. Previous results of Terry (*Pacific J. Math.* **52** (1974), 269–282) are seen to be corollaries of the more general results of this paper.

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In this paper we consider the general delay differential equation of even order

$$D^{2n-i}[r(t)\,D^i y(t)]+y_\tau(t)\,f[t,y_\tau(t)]=0, \tag{1}$$

where $0<m\leqslant r(t)\leqslant M<\infty$, $0\leqslant \tau(t)\leqslant T<\infty$, $y_\tau(t)=y[t-\tau(t)]$ and $f(t,u)$ satisfies the following properties:

(F1) $f(t,u)$ is a continuous real valued function on $[0,\infty)\times R$;

(F2) for each fixed t in $[0,\infty)$, $f(t,u)<f(t,v)$ for $0<u<v$;

(F3) for each fixed t in $[0,\infty)$, $f(t,u)>0$ and $f(t,-u)=f(t,u)$ for $u\neq 0$.

We first let

$$y_j(t)=\begin{cases} D^j y(t), & j=0,\dots,i-1,\\ D^{j-i}[r(t)\,D^i y(t)], & j=i,\dots,2n-1.\end{cases}$$

Following Terry (1974), we say that a positive solution $y(t)$ of (1) is of *type* B_j on $[T_0,\infty)$ if for $t\geqslant T_0$ the $y_k(t)>0$ $(k=0,\dots,2j+1)$ and $(-1)^{k+1}y_k(t)>0$ $(k=2j+2,\dots,2n-1)$. It is of *type* B_j if there is a $T_0>0$ such that it is of type B_j on $[T_0,\infty)$. As in Terry (1974), it is evident that a positive solution of (1) is necessarily of type B_j for some $j=0,\dots,n-1$. Moreover, the following lemmas may be established.

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LEMMA 1. *Let $y(t)$ be a solution of* (1) *of type B_j on $[T_0, \infty)$, where either* (i) *i is even and $j \leq (i-2)/2$ or* (ii) *i is odd and $j \leq (i-3)/2$. Then for $t \geq T_1 = T_0 + T$*

$$(t-T_1)\, y_k(t) \leq (2j+2-k)\, y_{k-1}(t), \quad k = 1, \ldots, 2j+1.$$

LEMMA 2. *Let $y(t)$ be a solution of* (1) *of type B_j on $[T_0, \infty)$, where i is odd and $j = (i-1)/2$. Then for $t \geq T_1 = T_0 + T$*

$$(t-T_1)\, y_i(t) \leq M y_{i-1}(t)$$

and

$$(t-T_1)\, y_k(t) \leq [Mm^{-1} + (i-k)]\, y_{k-1}(t), \quad k = 1, \ldots, i-1.$$

LEMMA 3. *Let $y(t)$ be a solution of* (1) *of type B_j on $[T_0, \infty)$, where either* (i) *i is even and $j \geq i/2$ or* (ii) *i is odd and $j \geq (i+1)/2$. Then for $t \geq T_1$*

(a) $(t-T_1)\, y_k(t) \leq (2j+2-k)\, y_{k-1}(t)$ $(k = i+1, \ldots, 2j+1)$;

(b) $(t-T_1)\, y_i(t) \leq M(2j+2-i)\, y_{i-1}(t)$;

and

(c) $(t-T_1)\, y_k(t) \leq [M(2j+2-i)\, m^{-1} + (i-k)]\, y_{k-1}(t)$ $(k = 1, \ldots, i-1)$.

The proof of each of these three lemmas is elementary using only integration by parts and the definition of a B_j-solution. The case $i = n$ is considered in Terry (1974), where the three results reduce to Lemmas 2.1, 2.2 and 2.3 respectively.

LEMMA 4. *Let $y(t)$ be a solution of* (1) *of type B_j on $[T_0, \infty)$. Then there exist constants $k_l > 0$ and $t_l \geq T_1$ such that*

$$y_l[t-\tau(t)] \geq k_l y_l(t), \quad t \geq t, \quad l = 0, \ldots, 2j.$$

As in Terry (1973), each of the four lemmas may be extended to the case in which $\tau(t)$ satisfies either

(T1) $0 \leq \tau(t) \leq \mu t$, $0 \leq \mu < m/(m+M)$;

or

(T2) $0 \leq \tau(t) \leq \mu t^{\beta}$, $0 \leq \mu < \infty$ and $0 \leq \beta < 1$,

provided the number T_1 is reinterpreted as $\min\{t \geq T_0 \colon t - \tau(t) \geq T_0 \text{ for } t \geq T_1\}$.

In this paper we use Lyapunov's second method to obtain criteria for the nonexistence of B_j-solutions of equation (1). We assume $n \geq 2$ and consider the system

$$(2) \qquad \begin{cases} Dy_k(t) = y_{k+1}(t), & k = 0, \ldots, i-2; \\ Dy_{i-1}(t) = y_i(t)/r(t); & \\ Dy_k(t) = y_{k+1}(t), & k = i, \ldots, 2n-2; \\ Dy_{2n-1}(t) = -y_\tau(t)\, ft, y_\tau(t). & \end{cases}$$

By a solution of (2) we mean an ordered $2n$-tuple $\sigma(t) = (y_0(t), \ldots, y_{2n-1}(t))$ which satisfies (2). To simplify the discussion we shall let $R_a = [a, \infty)$, $a \geq 0$; $R^* = (0, \infty)$; $R_* = (-\infty, 0)$; $R^1 = R = (-\infty, \infty)$. As in Terry (1974), we shall abbreviate certain frequently occurring cartesian products:

$$R^{p*} = R^* \times \ldots \times R^*, \quad p \text{ times};$$
$$R_{p*} = R_* \times \ldots \times R_*, \quad p \text{ times};$$
$$R_*{}^* = R_* \times R^*; \quad R^*{}_* = R^* \times R_*;$$
$$R_a{}^{p*} = R_a \times R^{p*};$$
$$\Pi_j = R^{(2j+1)*} \times (R^*{}_*)^{n-1-j} \times R; \quad \Pi_j{}^* = R^{(2j+1)*} \times (R^*{}_*)^{n-1-j} \times R^*;$$
$$\Pi^j = R_{(2j+1)*} \times (R_*{}^*)^{n-1-j} \times R; \quad \Pi^j{}_* = R_{(2j+1)*} \times (R^*{}_*)^{n-1-j} \times R_*.$$

In the following a scalar function $v(t,\sigma(t))$ will be called a Lyapunov function for the system (2) if it is continuous in $(t,\sigma(t))$ in its domain of definition and is locally Lipschitzian in $\sigma(t)$. Following Yoshizawa (1970), we define

$$\dot{v}_{(1)}(t,\sigma(t)) = \limsup_{h\to 0+} \frac{v(t+h,\sigma(t+h)) - v(t,\sigma(t))}{h}. \tag{3}$$

THEOREM 1. *Suppose that there exist two continuous functions $V(t,\sigma(t))$ and $W(t,\sigma(t))$ which are defined on $R_T \times \Pi_j$ and $R_T \times \Pi^j$ respectively for some fixed T. Assume further that $V(t,\sigma(t))$ and $W(t,\sigma(t))$ satisfy:*

(i) *both $V(t,\sigma(t))$ and $W(t,\sigma(t))$ tend to infinity as $t\to\infty$ uniformly for $\sigma(t)\in\Pi_j$ or $\sigma(t)\in\Pi^j$ respectively;*

(ii) *$\dot{V}_{(1)}(t,\sigma(t))\leqslant 0$ for all sufficiently large t, where $\sigma(t)$ is a solution of* (2) *which for large t lies in the region Π_j; and*

(iii) *$\dot{W}_{(1)}(t,\sigma(t))\leqslant 0$ for all sufficiently large t, where $\sigma(t)$ is a solution of* (2) *which for large t lies in the region Π^j.*

Then (1) *has no solutions of type B_j. Moreover,* (1) *has no negative solutions $y(t)$ such that $-y(t)$ is of type B_j.*

PROOF. Let $y(t)$ be a solution of (1) of type B_j. Since $y(t)$ and $y_1(t)$ are positive for large t, there is a positive T_0 for which $\sigma(t)$ lies in Π_j for $t\geqslant T_0$. By (ii), for t sufficiently large, for example, for $t\geqslant T_1\geqslant T_0$, $V(t,\sigma(t)) < V(T_1,\sigma(T_1))$. On the other hand, condition (i) implies that there is a $T_2 > T_1$ for which $V(t,\sigma(t)) > V(T_1,\sigma(T_1))$ for $t\geqslant T_2$, which is a contradiction. By letting $y(t)$ be a negative solution of (1) and considering $W(t,\sigma(t))$, we obtain an analogous contradiction.

Let us assume for the moment that $f(t,u)$ satisfies only (F1).

THEOREM 2. *For $(t,\sigma(t))\in R_T{}^*\times\Pi_{n-1}$ assume that there exists a Lyapunov function $v(t,\sigma(t))$ satisfying:*

(i) *$y_{2n-1}(t)\, v(t,\sigma(t)) > 0$;*

(ii) *$\dot{v}_{(1)}(t,\sigma(t))\leqslant -\lambda(t)$, where $\lambda(t)$ is a continuous function defined on R_T such that*

$$\liminf_{t\to\infty} \int_T^t \lambda(s)\,ds \geqslant 0 \tag{4}$$

for $t\geqslant T\geqslant T^*$.

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Moreover, suppose that there exists a T_1 *and a function* $w(t, \sigma(t))$ *which for* $(t, \sigma(t))$ *in the region* $R_{T_1} \times R^{(2n-1)*} \times R_*$ *is a Lyapunov function satisfying:*

(iii) $y_{2n-1}(t) \leqslant w(t, \sigma(t)) \leqslant b(y_{2n-1}(t))$, *where* $b(u)$ *is a continuous function,* $b(0) = 0$ *and* $b(u) < 0$ *for* $u \neq 0$;

and

(iv) $\dot{w}_{(1)}(t, \sigma(t)) \leqslant -\rho(t)\, w(t, \sigma(t))$, *where* $\rho(t) \geqslant 0$ *is a continuous function such that*

$$\int^{\infty} \exp\left\{ -\int_T^t \rho(s)\, ds \right\} dt = +\infty. \tag{5}$$

If $\sigma(t)$ *is a solution of* (2) *which lies in the region* Π_{n-1} *for sufficiently large values of* t, *then* $y_{2n-1}(t) \geqslant 0$, that is, $\sigma(t) \in \Pi^*_{n-1}$.

PROOF. Suppose that there is a sequence $\{t_k\}$ for which $t_k \to \infty$ as $k \to \infty$ and $y_{2n-1}(t_k) < 0$. Assume that $t_k \geqslant T^*$ and that t_k is sufficiently large so that by (4),

$$\liminf_{t\to\infty} \int_{t_k}^t \lambda(s)\, ds \geqslant 0, \quad t \geqslant t_k,$$

and $y_0(t), \ldots, y_{2n-2}(t)$ are positive, where we assume that $n \geqslant 2$. For the case $n = 1$, see Yoshizawa (1970). Consider the function $v(t, \sigma(t))$ for $t \geqslant t_k$.

$$v(t, \sigma(t)) \leqslant v(t_k, \sigma(t_k)) + \int_{t_k}^t \dot{v}_{(1)}(s, \sigma(s))\, ds$$

$$\leqslant v(t_k, \sigma(t_k)) - \int_{t_k}^t \lambda(s)\, ds. \tag{6}$$

Since $y_{2n-1}(t_k) < 0$, $v(t_k, \sigma(t_k)) < 0$, there is a $T_1 \geqslant t_k$ for which

$$\int_{t_k}^t \lambda(s)\, ds \geqslant v(t_k, \sigma(t_k))/2,$$

which implies that for $t \geqslant T_1$

$$v(t, \sigma(t)) \leqslant v(t_k, \sigma(t_k))/2 < 0.$$

By (i), $y_{2n-1}(t) < 0$ for $t > T_1$. By (iii), there is a $T_2 > T_1$ and a Lyapunov function $w(t, \sigma(t))$ defined on $R_{T_2} \times R^{(2n-1)*} \times R_*$. For this $w(t, \sigma(t))$ we have by (iv)

$$y_{2n-1}(t) \leqslant w(t, \sigma(t)) \leqslant w(T_2, \sigma(T_2)) \exp\left[-\int_{T_2}^t \rho(s)\, ds \right],$$

where $T_2 > T_1$. By (iii),

$$y_{2n-1}(t) \leqslant b(y_{2n-1}(T_2)) \exp\left[-\int_{T_2}^t \rho(s)\, ds \right].$$

Substituting this into the above expression, we get

$$y_{2n-1}(u) = [y_{2n-2}(u)]' \leqslant b(y_{2n-1}(T_2)) \exp\left[-\int_{T_2}^u \rho(s) \right] ds.$$

Integrating from T_2 to t, we arrive at

$$y_{2n-2}(t) \leqslant y_{2n-2}(T_2) + b(y_{2n-1}(T_2)) \int_{T_2}^{t} \exp\left[-\int_{T_2}^{u} \rho(s)\,ds\right] dt.$$

Letting $t \to \infty$ and using (5), it follows that $y_{2n-2}(t) < 0$ for sufficiently large t, which is a contradiction.

By the same argument we can prove the following result.

THEOREM 2′. *For* $(t, \sigma(t)) \in R_{T^*} \times \Pi^{n-1}$ *assume that there exists a Lyapunov function* $v(t, \sigma(t))$ *satisfying:*

(i) $y_{2n-1}(t)\, v(t, \sigma(t)) > 0$;

and

(ii) $\dot{v}_{(1)}(t, \sigma(t)) < -\lambda(t)$, *where* $\lambda(t)$ *is a continuous function defined on* R_T *such that for* $t \geqslant T \geqslant T^*$

$$\liminf_{t\to\infty} \int_{T}^{t} \lambda(s)\,ds \geqslant 0.$$

Moreover, suppose that there exists a T_1 *and a function* $w(t, \sigma(t))$ *which for* $(t, \sigma(t))$ *in the region* $R_{T_1} \times R_{(2n-1)^*} \times R^*$ *is a Lyapunov function satisfying:*

(iii) $y_{2n-1}(t) \leqslant w(t, \sigma(t)) \leqslant b(y_{2n-1}(t))$, *where* $b(u)$ *is a continuous function,* $b(0) = 0$ *and* $b(u) < 0$ *for* $u \neq 0$;

and

(iv) $\dot{w}_{(1)}(t, \sigma(t)) \leqslant -\rho(t)\, w(t, \sigma(t))$, *where* $\rho(t) \geqslant 0$ *is a continuous function for which*

$$\int^{\infty} \exp\left[-\int_{T_1}^{t} \rho(s)\,ds\right] dt = +\infty.$$

If $\sigma(t)$ *is a solution of* (2) *which lies in the region* Π^{n-1} *for sufficiently large values of* t, *then* $y_{2n-1}(t) \leqslant 0$ *for large* t, *that is,* $\sigma(t) \in \Pi_*^{n-1}$.

REMARK 1. Since $0 < m \leqslant r(t) \leqslant M$, condition (5) is equivalent to

$$\int^{\infty} \frac{1}{r(t)} \left\{\exp\left[-\int_{T_1}^{t} \rho(s)\,ds\right]\right\} dt = +\infty. \tag{7}$$

To see this we merely note that

$$M \int^{t} \frac{1}{r(u)} \exp\left[-\int_{T}^{u} \rho(s)\,ds\right] du \geqslant \int^{t} \exp\left[-\int_{T}^{u} \rho(s)\,ds\right] du$$

$$\geqslant m \int^{t} \frac{1}{r(u)} \exp\left[-\int_{T}^{u} \rho(s)\,ds\right] du.$$

In the case $n = 1$, we have $v(t, \sigma(t)) = v(t, y, y')$ since $y_0 = y$ and $y_1 = ry'$. Condition (7) arises naturally in the proof of Theorem 2. For the details see Yoshizawa (1970).

REMARK 2. Suppose we let $\sigma(t)\equiv 0$ in each of the two theorems. Condition (5) is then trivially valid, and the alternative condition (7) reduces to

$$\int^{\infty} dt/r(t) = +\infty.$$

Thus, we may replace condition (iv) by $\dot{w}_{(1)}(t,\sigma(t))\leqslant 0$ and obtain two easy corollaries whose statements are left to the reader.

REMARK 3. Let $r(t)\equiv 1$ and $f[t,y_\tau(t)]$ be nonnegative. As already noted, solutions of type B_1 are solutions of type A_1 (see Terry, 1974). Theorem 2 asserts that a solution $y(t)$ for which $D^k y(t)>0$, $k=0,1,\ldots,2n-2$, must satisfy $D^{2n-1}y(t)>0$, that is, $y(t)$ must be a solution of type A_{n-1}, which is obvious from the lemma of Kiguradze (1962).

Theorem 3. *Suppose there are continuous functions* $a(t)$, $b(t)$, $\alpha(y_{2n-2})$ *and* $\beta(y_{2n-2})$ *satisfying:*

(a) *for large* T,

$$\liminf_{t\to\infty}\int_T^t a(s)\,ds\geqslant 0,\quad \liminf_{t\to\infty}\int_T^t b(s)\,ds\geqslant 0;$$

(b) *for* $u=y_{2n-2}(t)$, $u\alpha(u)>0$ *and* $D_u\alpha(u)\geqslant 0$, *where* $y_k(t)$, $k=0,\ldots,2n-2$, *are nonnegative for large* t;
for $u=y_{2n-2}(t)$, $u\beta(u)>0$ *and* $D_u\beta(u)\geqslant 0$, *where* $y_k(t)$, $k=0,\ldots,2n-2$, *are nonpositive for large* t;

(c) $a(t)\,\alpha[y_{2n-2}(t)]\leqslant f[t,y_\tau(t)]\,y_\tau(t)$ *for large* t, $y(t)\geqslant 0$;
$b(t)\,\beta[y_{2n-2}(t)]\geqslant f[t,y_\tau(t)]\,y_\tau(t)$ *for large* t, $y(t)\leqslant 0$.

If $\sigma(t)$ *is a solution of* (2) *which for large* t *lies in the region* Π_{n-1}, *then* $y_{2n-1}(t)\geqslant 0$ *for large* t. *If* $\sigma(t)$ *is a solution of* (2) *which for large* t *lies in the region* Π^{n-1}, *then* $y_{2n-1}(t)\leqslant 0$ *for large* t.

PROOF. Let $\lambda(t)=a(t)$, $\rho(t)=0$ and define $v(t,\sigma(t))$ and $w(t,\sigma(t))$ by

$$v(t,\sigma(t))=\frac{y_{2n-1}(t)}{\alpha[y_{2n-2}(t)]};\quad w(t,\sigma(t))=y_{2n-1}(t)+\alpha[y_{2n-2}(t)]\int_T^t a(s)\,ds.$$

Conditions (i), (ii) and (iii) of Theorem 2 hold. In particular,

(i) $y_{2n-1}(t)\,v(t,\sigma(t))=\dfrac{y_{2n-1}^2(t)}{\alpha[y_{2n-2}(t)]}>0$ since $y_{2n-2}(t)>0$;

(ii)
$$\begin{aligned}\dot{v}_{(1)}(t,\sigma(t)) &= \{\alpha Dy_{2n-1}(t)-y_{2n-1}^2(t)\,\alpha'(y_{2n-2}(t))\}/\alpha^2(y_{2n-2}(t))\\ &< \frac{Dy_{2n-1}(t)}{\alpha[y_{2n-2}(t)]} \qquad \text{(by condition (b))}\\ &= \frac{-f[t,y_\tau(t)]\,y_\tau(t)}{\alpha[y_{2n-2}(t)]} \qquad \text{(from (1))}\\ &\leqslant -a(t)=-\lambda(t) \qquad \text{(by condition (a)).}\end{aligned}$$

Moreover,

$$\liminf_{t\to\infty}\int_T^t \lambda(s)\,ds = \liminf_{t\to\infty}\int_T^t a(s)\,ds \geqslant 0$$

for large t by condition (a).

(iii) $y_{2n-1}(t) \leqslant w(t,\sigma(t)) \leqslant y_{2n-1}(t) + \alpha[y_{2n-2}(t)]\int_T^t a(s)\,ds,$
since $y_{2n-2}(t) \geqslant 0$. Also,

$$\begin{aligned}\alpha[y_{2n-2}(t)]\int_T^t a(s)\,ds &\leqslant \int_T^t \alpha[y_{2n-2}(s)]\,ds\\ &\leqslant \int_T^t f[s,y_\tau(s)]\,y_\tau(s)\,ds\\ &= y_{2n-1}(T) - y_{2n-1}(t).\end{aligned}$$

For, if we assume as in (iii) of Theorem 2 that $\sigma(t) \in R^{(2n-1)*} \times R_*$, $y_{2n-2}(t)$ is a positive decreasing function of t. Thus, for $s < t$,

$$y_{2n-2}(t) < y_{2n-2}(s) \quad \text{and} \quad \alpha[y_{2n-2}(t)] < \alpha[y_{2n-2}(s)]$$

since $D_u\alpha(u) > 0$. It follows that $w(t,\sigma(t)) \leqslant y_{2n-1}(T) < 0$. Thus, we may take $b(u)$ to be the continuous function for which $b(0) = 0$, $b(u) = y_{2n-1}(T)$ for $u \notin (-\varepsilon,\varepsilon)$ and $b(u)$ is defined linearly on $(-\varepsilon,\varepsilon)$.

Moreover, to prove the second assertion of the theorem, suppose we let $v(t,\sigma(t))$ and $w(t,\sigma(t))$ be defined by

$$v(t,\sigma(t)) = \frac{y_{2n-1}(t)}{\beta[y_{2n-2}(t)]}; \quad w(t,\sigma(t)) = -y_{2n-1}(t) - \beta[y_{2n-1}(t)]\int_T^t b(s)\,ds.$$

Routine computations, similar to those just performed, show that $v(t,\sigma(t))$ and $w(t,\sigma(t))$ satisfy the four conditions of Theorem 2′.

THEOREM 4. *Suppose that, in addition to the hypotheses of Theorem* 3,

$$\int^\infty a(s)\,ds = \int^\infty b(s)\,ds = +\infty.$$

Then (1) *has no solutions of type* B_{n-1}.

PROOF. Suppose we define

$$V(t,\sigma(t)) = \begin{cases} \dfrac{y_{2n-1}(t)}{\alpha[y_{2n-2}(t)]} + \displaystyle\int_0^t a(s)\,ds, & y \geqslant 0,\\[2ex] \displaystyle\int_0^t a(s)\,ds, & y < 0;\end{cases}$$

$$W(t,\sigma(t)) = \begin{cases} \dfrac{y_{2n-1}(t)}{\beta[y_{2n-2}(t)]} + \displaystyle\int_0^t b(s)\,ds, & y < 0,\\[2ex] \displaystyle\int^t b(s)\,ds, & y \geqslant 0.\end{cases}$$

Assume that $y(t)$ is a solution of (1) of type B_{n-1}. Then for large t, $y_k(t) \geqslant 0$ for $k = 0, \ldots, 2n-1$. It follows that

$$V(t, \sigma(t)) \geqslant \int_0^t a(s)\, ds \quad \text{and} \quad W(t, \sigma(t)) \geqslant \int_0^t b(s)\, ds.$$

Because of the additional requirement in the hypothesis of this theorem, both $V(t, \sigma(t))$ and $W(t, \sigma(t))$ tend to infinity as $t \to \infty$ uniformly. Next, referring to (ii) of the proof of Theorem 3

$$\dot{V}_{(1)}(t, \sigma(t)) = D \frac{y_{2n-1}(t)}{\alpha[y_{2n-2}(t)]} + a(t) \leqslant 0;$$

$$\dot{W}_{(1)}(t, \sigma(t)) = D \frac{y_{2n-1}(t)}{\beta[y_{2n-2}(t)]} + b(t) \leqslant 0.$$

Hence, $V(t, \sigma(t))$ and $W(t, \sigma(t))$ satisfy the three conditions of Theorem 1 and the proof is complete. As in Theorem 1, we may also conclude that there are no negative solutions $y(t)$ of (1) such that $-y(t)$ is of type B_{n-1}.

We observe that Theorem 4 is only one of a sequence of similar results. Let us now consider the more general formulation.

THEOREM 5. *Suppose that there are continuous functions* $a(t)$, $b(t)$, $\alpha(u)$ *and* $\beta(u)$ *satisfying:*

(a) $$\int^\infty a(s)\, ds = \int^\infty b(s)\, ds = +\infty;$$

(b) $u\alpha(u) > 0$, $D_u \alpha(u) \geqslant 0$, *where u and u' are nonnegative for large t;*
$u\beta(u) > 0$, $D_u \beta(u) \geqslant 0$, *where u and u' are nonpositive for large t;*

(c) $a(t)\,\alpha[y_{2j}(t)] \leqslant f[t, y_r(t)]\, y_r(t)$ and $b(t)\,\beta[y_{2j}(t)] \leqslant f[t, y_r(t)]\, y_r(t)$.

Then (1) *has no solutions of type* B_r $(r = j, \ldots, n-1)$.

PROOF. Let

$$V(t, \sigma(t)) = \begin{cases} \dfrac{y_{2n-1}(t)}{\alpha[y_{2j}(t)]} + \displaystyle\int_0^t a(s)\, ds, & y < 0; \\[2ex] \displaystyle\int_0^t a(s)\, ds, y = 0; \end{cases}$$

$$W(t, \sigma(t)) = \begin{cases} \dfrac{y_{2n-1}(t)}{\beta[y_{2j}(t)]} + \displaystyle\int_0^t b(s)\, ds, y > 0, \\[2ex] \displaystyle\int_0^t b(s)\, ds, & y \leqslant 0. \end{cases}$$

As in the proof of Theorem 4, $V(t, \sigma(t))$ and $W(t, \sigma(t))$ will satisfy the three conditions of Theorem 1. The details are omitted. Moreover, there are no negative solutions $y(t)$ of (1) such that $-y(t)$ is of type B_r $(r = j, \ldots, n-1)$.

COROLLARY 1. *Let* $p(t)>0$. *If* $\int^{\infty} t^{2j} p(t)\,dt=+\infty$, *then there are no solutions of*

$$D^n[r(t)\,D^n y(t)]+p(t)\,y_\tau(t)=0 \tag{8}$$

of type B_r $(r=j,\ldots,n-1)$.

PROOF. Let $y(t)$ be a solution of (8) of type B_r $(r=j,\ldots,n-1)$. Then

$$f[t,y_\tau(t)]\,y_\tau(t)=p(t)\,y_\tau(t)\geqslant \mu t^{2j}p(t)\,y_{2j}(t).$$

We let $\alpha(u)=\beta(u)=\mu u$ and $\lambda(t)=a(t)=b(t)=t^{2j}p(t)$. With the choices of $V(t,\sigma(t))$ and $W(t,\sigma(t))$ prescribed by Theorem 4, it follows that (8) has no solutions of type B_r.

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California Polytechnic State University
San Luis Obispo, California 93407
U.S.A.