# RADIUS OF UNIVALENCE AND STARLIKENESS OF A CLASS OF ANALYTIC FUNCTIONS

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Let $\mathscr{P}$ denote the family of functions

$$P(z) = 1 + 2a_1 z + a_2 z^2 + \cdots$$

regular in $E\{z: |z| < 1\}$ and with positive real part there. We propose to study, in this article, the subclass $\mathscr{P}_{2a_1}$ of $\mathscr{P}$ whose functions $P(z)$ have pre-assigned second coefficient $2a_1$. In what follows we may assume, without loss in generality, that $a_1$ is real and non-negative. This assumption will be made throughout. As is well known [2], $0 \leqq a_1 \leqq 1$. In Theorem 1 we derive a generalization of Zmorovic's theorem 1, [3]. The result so obtained is then utilized in Theorem 2 to determine the radius of univalence and starlikeness of the class of functions

$$\Gamma(z) = P(z) - 1 = 2a_1 z + a_2 z^2 + \cdots$$

where $P(z) \in \mathscr{P}_{2a_1}$.

THEOREM 1. *Let $P \in \mathscr{P}_{2a_1}$. Then we have on $|z| = r$*

$$\left| zP' - \frac{P^2 - 1}{2} \right| \leqq \frac{\rho^2 - \rho_0^2}{2} \tag{1.1}$$

where

$$\rho = \frac{2r}{1 - r^2}, \quad a = \frac{1 + r^2}{1 - r^2}, \quad |P - a| = \rho_0 \leqq \rho$$

*This estimate is sharp.*

PROOF. We shall first prove (1.1) for the general class $\mathscr{P}$. The proof of the theorem will then be completed by showing the existence of a function belonging to $\mathscr{P}_{2a_1}$ for which equality holds in (1.1). We first observe that if $\phi(z)$ is regular and bounded in $E$, (by the term 'bounded' we shall always mean 'bounded by one') with $\phi(0) = 0$, then the function

$$\psi(z) = \frac{\phi(z)}{z} \tag{1.2}$$

is likewise bounded in $E$. Differentiating (1.2) we obtain

$$\psi'(z) = \frac{\phi'(z)}{z} - \frac{\phi(z)}{z^2} \tag{1.3}$$

Therefore, [2],

$$\left| \frac{\phi'(z)}{z} - \frac{\phi(z)}{z^2} \right| = |\psi'(z)| \leqq \frac{1 - |\psi(z)|^2}{1 - r^2} \tag{1.4}$$

Substituting the value of $\psi(z)$ from (1.2) and simplifying, (1.4) yields

$$|z\phi'(z) - \phi(z)| \leqq \frac{r^2 - |\phi(z)|^2}{1 - r^2} \tag{1.5}$$

Now

$$\phi(z) = \frac{P(z) - 1}{P(z) + 1} \tag{1.6}$$

from which we obtain after differentiation

$$z\phi'(z) = \frac{2zP'(z)}{(P(z) + 1)^2} \tag{1.7}$$

Substituting (1.6) and (1.7) in (1.5) we get

$$\left| zP' - \frac{P^2 - 1}{2} \right| \leqq \frac{r^2(|P + 1|)^2 - |P - 1|^2}{2(1 - r^2)} = \frac{\left(\frac{2r}{1 - r^2}\right)^2 - \left| P - \frac{1 + r^2}{1 - r^2} \right|^2}{2} \tag{1.8}$$

This gives (1.1) for $P \in \mathscr{P}$. For the class $\mathscr{P}_{2a_1}$ it is readily verified that the function

$$P_0(z) = \frac{1 + 2a_1 z + z^2}{1 - z^2} \tag{1.9}$$

belonging to $\mathscr{P}_{2a_1}$ yields equality in (1.1). Thus the estimate (1.1) holds for $P \in \mathscr{P}_{2a_1}$ and the proof of the theorem is complete.

It may be remarked that Zmorovic [3] proved the inequality (1.1) under the condition that the function $P(z) \in \mathscr{P}$ has the form

$$P(z) = \lambda_1 \frac{1 + z_1^m}{1 - z_1^m} \cdot + \lambda_2 \frac{1 + z_2^m}{1 - z_2^m},$$

where $z_1$ and $z_2$ are arbitrary points on $|z| = r$, $m$ is a positive integer, $\lambda_1 \geqq 0$, $\lambda_2 \geqq 0$, $\lambda_1 + \lambda_2 = 1$. We have proved (1.1) without this assumption.

THEOREM 2. *Let $P \in \mathscr{P}_{2a_1}$. Then the radius of univalence and starlikeness, $r_0$ of the class of functions*

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$$\Gamma(z) = P(z) - 1 = 2a_1 z + a_2 z^2 + \cdots \tag{2.1}$$

*is given by*

$$r_0 = \frac{a_1}{1 + \sqrt{1 - a_1^2}} \tag{2.2}$$

In order to prove the theorem we need the following

LEMMA 1. *Let $P \in \mathscr{P}_{2a_1}$. Then on $|z| = r < a_1$, we have*

$$|P - 1 - x| \geqq y \tag{2.3}$$

where $x = \dfrac{2r^2(a_1 - r)^2}{(1 - r^2)(1 - 2a_1 r + r^2)}$ and $y = \dfrac{2r(a_1 - r)(1 - a_1 r)}{(1 - r^2)(1 - 2a_1 r + r^2)}$

*This result is sharp in the sense that $P - 1$ take the value*

$$A = \frac{-2r(a_1 - r)}{1 - r^2} \text{ and } B = \frac{2r(a_1 - r)}{1 - 2a_1 r + r^2}$$

*which are the extremities of the diameter of the circle ALB whose centre is at x and radius is y.*

PROOF. For $P \in \mathscr{P}_{2a_1}$, we may write

$$\frac{P - 1}{P + 1} = \phi(z) \tag{2.4}$$

where $\phi(z)$ is regular and bounded in $E$, $\phi(0) = 0$, $\phi'(0) = a_1$, from which it follows that, [2]

$$\left|\frac{P - 1}{P + 1}\right| = |\phi(z)| \geqq \frac{r(a_1 - r)}{1 - a_1 r} \tag{2.5}$$

and this yields

$$\left|P - \frac{1 - 2a_1 r + 2a_1^2 r^2 - 2a_1 r^3 + r^4}{(1 - r^2)(1 - 2a_1 r + r^2)}\right| \geqq y \tag{2.6}$$

which is equivalent to (2.3). The values $A$ and $B$ are respectively taken by the functions

$$\frac{1 + 2a_1 z + z^2}{1 - z^2} \qquad \text{at } z = -r$$

and

$$\frac{1 - z^2}{1 - 2a_1 z + z^2} \qquad \text{at } z = +r.$$

This completes the proof of the lemma.

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LEMMA 2. *Let $P \in \mathscr{P}_{2a_1}$. Then on $|z| = r$, $0 < r < 1$ we have*

$$|P - 1 - X| \leqq Y \tag{2.7}$$

*where* $$X = \frac{2r^2(a_1 + r)^2}{(1 + 2a_1 r + r^2)(1 - r^2)} \text{ and } Y = \frac{2r(1 + a_1 r)(a_1 + r)}{(1 + 2a_1 r + r^2)(1 - r^2)}$$

*This result is sharp in the sense that $P - 1$ take the values*

$$A' = \frac{-2r(a_1 + r)}{1 + 2a_1 r + r^2} \text{ and } B' = \frac{2r(a_1 + r)}{1 - r^2}$$

*which are the extremities of the diameter of the circle $A'L'B'$ whose centre is at $X$ and radius is $Y$.*

PROOF. The proof is similar to that of Lemma 1 and follows from the inequality, [2],

$$\left|\frac{P-1}{P+1}\right| = |\phi(z)| \leqq \frac{r(a_1 + r)}{1 + a_1 r}$$

where $\phi(z)$ is regular and bounded in $E$, $\phi(0) = 0$, $\phi'(0) = a_1$. The functions of Lemma 1 are extremal in this case as well.

PROOF OF THEOREM 2. The function $\Gamma(z) = P(z) - 1$ is regular in $E$ and from Lemma 1 we see that

$$\left|\frac{P(z) - 1}{z}\right| \geqq \frac{2(a_1 - r)}{1 - r^2}$$

so that $\Gamma(z)$ has no zeros in $|z| < a_1$ except a simple zero at the origin. A necessary and sufficient condition that $\Gamma(z)$ be starlike in $|z| < r_0$ is that

$$\text{Re}\,\frac{z\Gamma'(z)}{\Gamma(z)} = \text{Re}\,\frac{zP'(z)}{P(z)-1} > 0 \tag{2.8}$$

in $|z| < r_0$. Since $\text{Re}\,\dfrac{zP'(z)}{P(z)-1}$ is harmonic in $|z| < a_1$, it is sufficient to obtain the radius of the largest circle on which this is non-negative.

Making use of (1.1) we get

$$\text{Re}\,\frac{zP'}{P-1} \geqq \text{Re}\,\frac{P+1}{2} - \frac{\rho^2 - \rho_0^2}{2|P-1|} \tag{2.9}$$

We now have the following extremal problem: Given $|z| = r$, to find the minimum of the right side of (2.9) as $P$ runs over the class $\mathscr{P}_{2a_1}$. From Lemma 1 and 2 we see that we need to find this minimum for $P - 1$ lying in the region enclosed by the circles $A'L'B'$ and $ALB$.

Putting $P-a=\xi+i\eta$ and denoting the right side of (2.9) by $\psi_\rho(\xi,\eta)$ we obtain

$$\psi_\rho(\xi,\eta)=\frac{1}{2}\left[a+\xi+1-\frac{\rho^2-(\xi^2+\eta^2)}{R}\right] \tag{2.10}$$

where $R=|P-1|$.

We now divide the range of $\xi=\operatorname{Re}(P-a)$ into two parts:

$$\xi\leqq\frac{-2a_1r}{1-r^2}\ \text{and}\ \xi\geqq\frac{2r(a_1-2r+a_1r^2)}{(1-2a_1r+r^2)(1-r^2)} \tag{A}$$

and

$$-\frac{2a_1r}{1-r^2}\leqq\xi\leqq\frac{2r(a_1-2r+a_1r^2)}{(1-2a_1r+r^2)(1-r^2)} \tag{B}$$

We shall show that in case (A) the minimum of $\psi_\rho(\xi,\eta)$ inside the circle $\xi^2+\eta^2=\rho_0^2\leqq\rho^2$ is attained on the diameter $\eta=0$. We differentiate (2.10) with respect to $\eta$ and obtain

$$\frac{\partial\psi}{\partial\eta}=\frac{1}{2}\left[\frac{\rho^2-(\xi^2+\eta^2)}{R^2}\cdot\frac{\eta}{R}+\frac{2\eta}{R}\right]=\frac{\eta}{2R^3}[\rho^2-(\xi^2+\eta^2)+2R^2] \tag{2.11}$$

The expression within the brackets is positive. Hence for each fixed $\xi$, the minimum is attained at $\eta=0$. Therefore, inside the circle $\xi^2+\eta^2=\rho_0^2$ (subject to (A)) the minimum occurs on the diameter $\eta=0$.

Putting $\eta=0$ in (2.10) we have the following problem: To find the minimum of

$$l(\xi)=\frac{1}{2}\left[a+\xi+1-\frac{\rho^2-\xi^2}{|a+\xi-1|}\right] \tag{2.12}$$

If $\xi\leqq-2a_1r/(1-r^2)$, then $a+\xi-1$ is negative and so

$$l(\xi)=\frac{1}{2}\left[a+\xi+1+\frac{\rho^2-\xi^2}{a+\xi-1}\right]=\frac{2r^2}{(1-r^2)(a+\xi-1)}+\frac{1+r^2}{1-r^2} \tag{2.13}$$

from which we see that the minimum of $l(\xi)(=l_1(\xi))l$ is given by the smallest numerical value of $a+\xi-1$. Substituting $\xi=-2a_1r/(1-r^2)$ in (2.13) we obtain

$$l(\xi)\geqq l_1(\xi)=\frac{a_1-2r+a_1r^2}{(a_1-r)(1-r^2)} \tag{2.14}$$

If

$$\xi\geqq\frac{2r(a_1-2r+a_1r^2)}{(1-2a_1r+r^2)(1-r^2)},$$

then $a+\xi-1$ is positive. In this case

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$$(2.15)\quad \begin{aligned} l(\xi) &= \frac{1}{2}\left[a+\xi+1-\frac{\rho^2-\xi^2}{a+\xi-1}\right] \\ &= a+\xi-1-\frac{2r^2}{(1-r^2)(a+\xi-1)}+\frac{1-3r^2}{1-r^2} \end{aligned}$$

Since $a+\xi-1>0$, the minimum of $l(\xi)\,(=l_2(\xi))$ occurs for the smallest numerical value of $a+\xi-1$. Putting

$$\xi = \frac{2r(a_1-2r+a_1r^2)}{(1-2a_1r+r^2)(1-r^2)}$$

in (2.15) we obtain

$$(2.16)\quad l(\xi) \geqq l_2(\xi) = \frac{2r(a_1-r)}{1-2a_1r+r^2}-\frac{2r^2(1-2a_1r+r^2)}{(1-r^2)\cdot 2r(a_1-r)}+\frac{1-3r^2}{1-r^2}$$

From (2.14) and (2.16) we see that

$$(2.17)\quad l_2(\xi) \geqq l_1(\xi)$$

if

$$a_1-2r+a_1r^2 \geqq 0, \text{ that is, if } r \leqq a_1/(1+\sqrt{1-a_1^2})$$

In case (B) let us assume that every value on the circumference of the circle $ALB$ is taken by some $P-1$, $P\in\mathscr{P}_{2a_1}$, for some $z$, $|z|=r<a_1$. We see then from (2.11) that for each fixed $\xi$, the minimum of the right side of (2.9) occurs for points on the circumference of the circle $ALB$. Therefore, for the admissible range of $\xi$, the minimum occurs on the circumference of the above circle. Also, from (2.3) we see that any point on the circumference of the circle can be written as

$$P-1 = x+ye^{i\theta} \qquad 0 \leqq \theta < 2\pi$$

so that our problem reduces to minimizing the expression:

$$(2.18)\quad \begin{aligned} \psi &= 1+\frac{1}{2}\left[\operatorname{Re}(P-1)-\frac{\rho^2-|P-a|^2}{|P-1|}\right] \\ &= 1+\frac{1}{2}\left[x+y\cos\theta-\left(\frac{4r^2}{(1-r^2)\sqrt{2x}}-\sqrt{2x}\right)(1+x+y\cos\theta)^{\frac{1}{2}}\right] \end{aligned}$$

where we have made use of the fact that

$$(2.19)\quad x^2-y^2+2x \equiv 0$$

Differentiating (2.18) with respect to $\theta$ we obtain

$$(2.20)\quad \frac{\partial\psi}{\partial\theta} = -\frac{1}{2}y\sin\theta\left[1-\frac{1}{2}\left(\frac{4r^2}{(1-r^2)\sqrt{2x}}-\sqrt{2x}\right)(1+x+y\cos\theta)^{-\frac{1}{2}}\right]$$

We propose to show that the expression within the brackets retains a positive sign at least when $|z| = r < a_1/(1+\sqrt{1-a_1^2})$. It will then follow that the minimum of $\psi$ can occur only when $\theta = 0$ or $\theta = \pi$, that is, at $A$ or $B$. In other words, the minimum of $\psi$ in case (A) and case (B) is the same if $|z| = r < a_1/(1+\sqrt{1-a^2})$.

To show that the expression within the brackets retains a positive sign, let us put

$$\Phi = 1 - \frac{1}{2}\left(\frac{4r^2}{(1-r^2)\sqrt{2x}} - \sqrt{2x}\right)\cdot(1+x+y\cos\theta)^{-\frac{1}{2}}$$

then

$$\frac{\partial\Phi}{\partial\theta} = -\frac{1}{4}y\sin\theta\left(\frac{4r^2}{(1-r^2)\sqrt{2x}} - \sqrt{2x}\right)\cdot(1+x+y\cos\theta)^{-\frac{3}{2}}$$

$$= xy\sin\theta\left(x - \frac{2r^2}{1-r^2}\right)\cdot(x^2+y^2+2xy\cos\theta)^{-\frac{3}{2}}$$

Since

$$x - \frac{2r^2}{1-r^2} = -\frac{2r^2}{(1-r^2)}\frac{(1-a_1^2)}{(1-2a_1r+r^2)},\quad x^2+y^2+2xy\cos\theta > 0,$$

the extrema of $\Phi$ occurs for $\theta = 0, \pi$ (if $a_1 = 1$, $\Phi \equiv 1$). For $\theta = 0$,

$$\Phi = \Phi_0 = \frac{a_1 - 2r - a_1r^2 + a_1^2r + r^3}{(a_1-r)(1-r^2)}$$

For $\theta = \pi$

$$\Phi = \Phi_\pi = \frac{a_1 - 3a_1^2r + 3a_1r^2 - r^3}{(a_1-r)(1-2a_1r+r^2)}$$

and we will show that when $r < a_1/(1+\sqrt{1-a_1^2})$ both $\Phi_0$ and $\Phi_\pi$ are positive. The numerator of

$$\begin{aligned}\Phi_0 &= a_1 - 2r + a_1r^2 - 2a_1r^2 + a_1^2r + r^3\\ &\geqq a_1 - 2r + a_1r^2 - 2a_1r^2 + a_1^2r + a_1^2r^3\\ &= (a_1 - 2r + a_1r^2)(1-a_1r) > 0 \quad \text{if } r < a_1/(1+\sqrt{1-a_1^2})\end{aligned}$$

If $a_1 = 1$, it is easy to see that $\Phi_\pi = 1$. Otherwise the numerator of $\Phi_\pi$ is a monotonic decreasing function of $r$. Putting $r = a_1$, the numerator becomes $a_1 - a_1^3 > 0$. Therefore if $r < a_1/(1+\sqrt{1-a_1^2}) < a_1$, $\phi_\pi$ is also positive.

Finally, if $P-1$ omits a larger set of values than the interior of the circle $ALB$, this omitted set of values will include the interior of the above circle but not the points $A$ and $B$ and so the minimum/max will again occur at $A$ or $B$.

Summing up, we have proved that for $|z| = r < a_1/(1+\sqrt{1-a^1})$

$$\operatorname*{Re}_{P\in\mathscr{P}_{2_1}} \frac{zP'}{P-1} \geqq \frac{a_1 - 2r + a_1r^2}{(a_1-r)(1-r^2)} \tag{2.21}$$

Also the right side of this inequality is non-negative for

$$|z| \leqq r_0 = a_1/(1+\sqrt{1-a_1^2}).$$

Therefore $\Gamma(z)$ is starlike in $|z| < r_0$. That $\Gamma(z)$ may not be starlike in a larger circle may be shown by considering the function

$$P_0(z) = \frac{1+2a_1z+z^2}{1-z^2} \in \mathscr{P}_{2a_1}$$

for which $\operatorname{Re} zP'/(P-1)$ vanishes on $|z| = r_0$. Thus the estimate (2.2) for the radius of starlikeness is correct. Since the derivative of $P_0(z)$ vanishes for $|z| = r_0 = a_1/(1+\sqrt{1-a_1^2})$, we see that $r_0$ is also the radius of univalence of the class $\Gamma(z) = P(z) - 1$, $P \in \mathscr{P}_{2a_1}$. This completes the proof of the theorem.

It may be pointed out that the radius of univalence of $\Gamma(z)$ follows immediately from a result of Landau [1] who showed that a function $\phi(z) = a_1z + \cdots$ which is regular and bounded in $E$ is univalent in the disc

$$(2.22) \qquad |z| < \frac{|a_1|}{1+\sqrt{1-|a_1|^2}}$$

Since we may write $\Gamma(z) = P(z) - 1 = 2\phi/(1-\phi)$ where $P \in \mathscr{P}_{2a_1}$, $\phi$ is regular and bounded in $E$ and $\phi(0) = 0$, $\phi'(0) = a_1$, the univalence of $\Gamma(z)$ in the disc (2.22) follows from the univalence of $\phi$ in the same disc. Of course, insofar as starlikeness is concerned the situation for $\Gamma(z)$ and $\phi(z)$ would be quite different because of the intervention of the linear transformation.

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## References

[1] E. Landau, 'Der Picard-Schottkysche Satz un die Blochsche Konstante', Sitzungsb. *Akad. d. Wiss. Berlin, Phys. Math. Klasse* (1926), 467–474.

[2] Z. Nehari, *Conformal Mapping* (McGraw-Hill, 1952).

[3] V. A. Zmorovic, 'On the radius of convexity of starlike functions of order $\alpha$ regular in $|z| < 1$ and in $0 < |z| < 1$' (Russian), *Mat. Sbornik* (N. S.) 68 (110) (1965), 518–526.

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