Canad. Math. Bull. Vol. **13** (4), 1970

ON A SOLUTION OF THE HAMMERSTEIN EQUATION WITH SINGULAR NORMAL KERNELS

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We consider here the equation

$$\varphi(x)+\int_0^1 K(x, y)f(y, \varphi(y))\,dy = 0. \tag{1}$$

This equation was first studied by Hammerstein [4] under the assumption that the linear operator

$$Af = \int_0^1 K(x, y)f(y)\,dy$$

is selfadjoint and completely continuous. V. Nemytsky [5] and M. Golomb [3] dropped the assumption that A be selfadjoint and positive. M. Vainberg [6] considered (among other cases) the case in which A is a bounded operator generated by a Carleman kernel. The kernels considered in this work do not necessarily generate bounded, completely continuous or selfadjoint, operators. Although our theorem may be established with the aid of Schauder's second theorem our purpose here is to prove it directly by the classical method. For convenience we will use the familiar notation

$$A = Kf = \int_0^1 K(x, y)f(y)\,dy \quad \text{and} \quad \|\varphi\| = \left\{\int_0^1 \varphi^2(x)\,dx\right\}^{1/2}$$

when there is no likelihood of confusion.

THEOREM. *Let $K(x, y)$ be a singular normal kernel and $f(t, u)$ be such that*

(i) $f(t, u) = -u-h(t, u)$

where

(ii) $|h(t, u')-h(t, u)| \leq uK^{-1}(t)|u'-u|$,

(iii) $0 < u < \dfrac{1}{1+\delta^{-1}}$

(iv) $\gamma(x) = \int_0^1 K(x, t)h(t, 0)\,dt \subset L_2$

(*for definition of singular normal, and* δ *see* [2]).
Then

$$\varphi(x)+\int_0^1 K(x, t)f(t, \varphi(t))\,dt = 0$$

has a solution $\varphi(x)$ in L_2.

Received by the editors December 5, 1969.

Proof. Consider the associated inhomogeneous linear integral equation

(2) $$\varphi(x) = \int_0^1 K(x, t)\varphi(t)\,dt + f(x)$$

where $f(x)$, possibly complex valued is in L_2.

It is shown in [2] that equation (2) admits a solution $\varphi(x)$ in L_2 such that

(3) $$\|\varphi_m\| \le (1+\delta^{-1})\|f\|$$

(φ_m being solution of the inhomogeneous equation with the approximating kernel K_m involved in definition of singular normal kernel)

(4) $$\|\varphi\| \le (1+\delta^{-1})\|f\|.$$

Put

(5) $$h(t, u) = h(t, 0) + g(t, u).$$

Clearly

$$g(t, 0) = 0$$

(6) $$|g(t, u') - g(t, u)| \le \mu\, |K^{-1}(t)|\, |u'-u| : K^2(x) = \int_0^1 K^2(x, t)\,dt$$

and we now write (1) in form

(7) $$\varphi(x) - \int_0^1 K(x, t)\varphi(t)\,dt = \gamma(x) + \int_0^1 K(x, t)g(t, \varphi(t))\,dt.$$

Define $\{\varphi_n\}$, as follows

$$\varphi_0 = 0$$

(8) $$\varphi_\nu - K\varphi_\nu = \gamma(x) + \int_0^1 K(x, t)g(t, \varphi_{\nu-1}(t))\,dt, \quad \nu \ge 1.$$

Let

$$r_\nu(x) = \varphi_\nu - \varphi_{\nu-1}.$$

By (7)

(9) $$r_\nu - Kr_\nu = \int_0^1 K(x, t)\{g(t, \varphi_{\nu-1}(t)) - g(t, \varphi_{\nu-2}(t))\}\,dt = W_{\nu-1}, \quad \nu \ge 2$$

(10) $$r_1 - Kr_1 = \gamma(x) + \int_0^1 K(x, t)g(t, 0)\,dt = \gamma(x) = W_0.$$

From (4)

(11) $$\|r_\nu\| \le (1+\delta^{-1})\|W_{\nu-1}\| \quad (\nu = 1, 2, \ldots).$$

In particular

$$\|r_1\| \le (1+\delta^{-1})\|\gamma\|.$$

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From (9) and (6)

$$|W_{\nu-1}| < \mu \int_0^1 |K(x,t)|\, K^{-1}(t)\, |\varphi_{\nu-1}(t)-\varphi_{\nu-2}(t)|\, dt, \quad \nu \geq 2 \tag{12}$$

$$W_{\nu-1}^2 \leq \mu^2 \int_0^1 (K(x,t)K^{-1}(t))^2\, dt \|r_{\nu-1}\|^2 \tag{13}$$

and

$$\|W_\nu\| \leq \mu\, \|r_\nu\|. \tag{14}$$

In particular, from (11)

$$\|W_1\| \leq (1+\delta^{-1})\mu\|\gamma\|.$$

With induction in view, suppose for some $\nu \geq 2$,

$$\|r_i\| \leq (1+\delta^{-1})^i \mu^{i-1}\|\gamma\| = \delta_i \tag{15}$$

$$\|W_i\| \leq (1+\delta^{-1})^i \mu^i \|\gamma\| \quad \text{for } i = 1, 2, \ldots, \nu-1. \tag{16}$$

From (11) and (16) for $i=\nu-1$

$$\|r_\nu\| \leq (1+\delta^{-1})^\nu \mu^{\nu-1}\|\gamma\|. \tag{17}$$

With the aid of (14) we have

$$\|W_\nu\| \leq (1+\delta^{-1})^\nu \mu^\nu (\|\gamma\|) \tag{18}$$

proving the induction for (15), (16) since, from (11 and 14),

$$\|r_1\| \leq (1+\delta^{-1})\|\gamma\| \quad \text{and} \quad \|W_1\| \leq (1+\delta^{-1})\mu\|\gamma\| \quad \text{for } \nu = 1.$$

From (15) and (iii) the series

$$S = \sum_i \delta_i < \infty. \tag{19}$$

This implies that

$$\{\varphi_\nu\} \text{ converges weakly to } \varphi(x) \text{ and} \quad \|\varphi_\nu\| < S, \quad \|\varphi\| < S. \tag{20}$$

Consequently, with aid of (6),

$$\|K(t)g(t, \varphi_{\nu-1}(t))\| \leq \mu\|\varphi_{\nu-1}\| \leq \mu S;$$

and there exists a function W in L_2 such that

$$K(t)g(t, \varphi_{\nu-1}(t)) \text{ converge weakly to } W(t), \text{ and} \quad \|W(t)\| \leq uS. \tag{21}$$

We now write (7) in the form

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(22) $$\varphi_{\nu_i}-K\varphi_{\nu_i} = \gamma(x)+\int_0^1 K(x,t)K^{-1}(t)K(t)g(t,\varphi_{\nu-1}(t))\,dt = P_i(x).$$

Since $K(x,t)K^{-1}(t)$ is in $L_2(t)$, from (21) it follows that

(23) $$\lim_i p_i(x) = p(x) = \gamma(x)+\int_0^1 K(x,t)K^{-1}(t)W(t)\,dt.$$

Since $\int K^2(x,t)\,dt<\infty$, from (20) it follows that

$$\lim_i K\varphi_{\nu_i} = K\varphi.$$

From (22)

(24) $$\lim_i \varphi_{\nu_i} = K\varphi+p(x) = \varphi^*$$

in ordinary sense, so from (21) we conclude that

(25) $$\varphi^* = \varphi \quad \text{a.e.}$$

Since h is continuous in u we have

$$\lim_i K(t)g(t,\varphi_{\nu_i}(t)) = K(t)g(t,\varphi(t))$$

and

(26) $$\lim_i \int_0^1 K(x,t)g(t,\varphi_{\nu_i}(t))\,dt = \lim_i \int_0^1 K(x,t)K^{-1}(t)K(t)g(t,\varphi_{\nu_i}(t))\,dt$$
$$= \int_0^1 K(x,t)g(t,\varphi(t))\,dt.$$

Putting $\nu=\nu_i$ in (8) we see that

$$\varphi_{\nu_i}-K\varphi_{\nu_i}-\gamma(x)-\int_0^1 K(x,t)g(t,\varphi_{\nu_i}(t))\,dt$$
$$=\int K(x,t)\{g(t,\varphi_{\nu_i-1}(t))-g(t,\varphi_{\nu_i}(t))\}\,dt = -W_\nu \quad \text{(cf. 9)}.$$

From (13), (15), and (19) it follows that

$$\lim_i W_{\nu_i}^2 = 0$$

and we have

$$\varphi-K\varphi-\gamma(x)-\int_0^1 K(x,t)g(t,\varphi(t))\,dt = 0.$$

In view of (iv) and (5) we conclude that $\varphi(x)$ is a solution of (1).

EXAMPLE 1. To show explicitly that our singular kernel does not necessarily generate the selfadjoint operators we consider the following.

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Let $\varphi_K(x)$ be Haar functions, let

$$K(x, y) = \sum_{p=0}^{\infty} a_p\varphi_p(x)\varphi_p(y).$$

Then $A = Kf$ is selfadjoint only if $\sum_\nu [2^\nu/(1+a_\nu^2)]$ is divergent (see [1, pp. 62–66]). If $\sum_\nu [2^\nu/(1+a_\nu^2)]$ is convergent as is the case when $a_\nu = \nu.2^{\nu/2}$, i.e.

$$K(x, t) = \sum_\nu \nu.2^{\nu/2}\varphi_\nu(x)\varphi_\nu(y),$$

then $K(x, y)$ is of Carleman class 2 and generates a nonselfadjoint operator (see [7, p. 422]). In either case the kernel is singular normal.

EXAMPLE 2. Another construction of a singular normal kernel is as follows. Let $\varphi_{m\nu}$ be real functions in $L_2[0, 1]$ such that

$$\int_0^1 \varphi_{m\nu}(x)\varphi_{mj}(x)\,dx = 0, \quad \nu \neq j \tag{1}$$

$$\int_0^1 \varphi_{m\nu}^2(x)\,dx = 1, \quad m, \nu = 1, 2, \ldots \tag{2}$$

$$\int_0^1 \varphi_{m\nu}(x)\varphi_{pj}(x)\,dx = 0, \quad m \neq p; \quad i = 1, 2, \ldots. \tag{3}$$

Let $\lambda_{m,\nu}$ be real numbers such that the series

$$S_m = \sum_\nu \frac{1}{\lambda_{m,\nu}^2}; \quad \sum_m \sum_\nu \frac{1}{\lambda_{m,\nu}^2}\varphi_{m\nu}^2(x); \quad \sum_\nu \frac{\varphi_{m\nu}^2(x)}{|\lambda_{m,\nu}|} \tag{4}$$

all converge for almost all x on $[0, 1]$ while

$$S = \sum_{m=1}^{\infty} S_m \tag{5}$$

diverges. Consider

$$g_m(x, y) = \sum_\nu \frac{1}{\lambda_{m,\nu}}\varphi_{m,\nu}(x)\varphi_{m\nu}(y). \tag{6}$$

Clearly $\int_0^1\int_0^1 g_m(x, y)^2 dx\,dy = S_m (m=1, 2, \ldots)$ exists. The $g_m(x, y) (m=1, 2, \ldots)$ are regular kernels.

Define

$$K_n(x, y) = g_1(x, y) + \cdots + g_n(x, y), \quad n = 1, 2, \ldots.$$

Now

$$\int_0^1\int_0^1 K_n^2(x, y)\,dx\,dy = S_1 + \cdots + S_n < \infty.$$

From (1) and (2)

$$\int_0^1 g_m^2(x, y)\,dy = \sum_\nu \frac{1}{\lambda_{m,\nu}^2}\varphi_{m\nu}^2$$

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and

$$\int_0^1 K_n^2(x, y)\, dy = \sum_{m=1}^{n} \int_0^1 g_m^2(x, y)\, dy = \sum_{m=1}^{n} \sum_{\nu=1}^{\infty} \frac{1}{\lambda_{m\nu}^2} \varphi_{m\nu}^2(x).$$

Now

$$K(x, y) = \sum_{m=1}^{\infty} g_m(x, y) = \lim_n K_n(x, y).$$

The $\lambda_{m\nu}(m=1,\ldots, n, \nu=1, 2,\ldots)$ are characteristic values of the approximating kernels K_n (see definition of singular normal kernels). Actually, these $K_n(n=1, 2, \ldots)$ are symmetric and therefore are also normal. Since $\int_0^1 \int_0^1 K^2(x, y)\, dx\, dy$ does not necessarily exist, the kernel K is not necessarily completely continuous or compact.

THEOREM 2. *Let $K(x, t)$ be a singular normal kernel such that*

(i) $\hat{K}(x, y)=\int_0^1 K(x, t)\overline{K(y, t)}\, dt$ *belong to* L_2 *in* x *(also in* y*)*
(ii) $h(t, y, u)=\overline{K(y, t)u}-H(t, y, u)$ *where*
(iii) $|H(t, y, u')-H(t, y, u)| \le \mu\hat{K}^{-1}(t)|u'-u|$
(iv) $\gamma(x)=\int_0^1 \{\int_0^1 K(x, t)H(t, y, 0)\, dt\, dy$ *in* L_2
(v) $\rho(x)=\int_0^1 |K^2(x, t)|\, \hat{K}^{-2}(t)\, dt$ $\qquad \rho=\int_0^1 \rho(x)\, dx$ *exists, and*
(vi) $4\mu^2\rho<1$.

Then the Urysohn equation

$$\varphi(x)+\int_0^1 \left\{\int_0^1 K(x, t)h(t, y, \varphi(y))\, dt\right\} dy = 0 \tag{1}$$

has a solution in L_2.

Proof. Equation (1) may be put in the form

$$\varphi(x)+\gamma(x)+\int_0^1 \left\{\int_0^1 K(x, t)H(t, y, \varphi(y))dt\right\} dy = 0. \tag{2}$$

Define $\{\varphi_\nu\}$ recursively as follows.

$$\varphi_0 = 0$$

$$\varphi_\nu = -\gamma(x)-\int_0^1 \left\{\int_0^1 K(x, t)H(t, y, \varphi(y))\, dt\right\} dy, \quad v \ge 1.$$

The same technique as in Theorem 1 shows that $\{\varphi_\nu\}$ is uniformly bounded thus weakly compact so that $\{\varphi_\nu\}$ converges weakly to some function φ in L_2 and

$$\lim_{\nu\to\infty} \varphi_\nu = -\lim_{\nu\to\infty} \int_0^1 K(x, y)\varphi_\nu(y)\, dy+\gamma(x)+\lim_{\nu\to\infty} \int_0^1 \left\{\int_0^1 K(x, t)G(t, y, \varphi(y))\, dt\right\} dy$$

$$= \varphi' \quad \text{(convergence in ordinary sense).}$$

Thus $\varphi=\varphi'$ almost everywhere.

With the aid of the continuity of $G(t, y, u)$ in u and (v) we have

$$\lim_{\nu\to\infty}\int_0^1\left\{\int_0^1 K(x,t)G(t,y,\varphi_{\nu-1}(y))\,dt\right\}dy=\int_0^1\left\{\int_0^1 K(x,t)G(t,y,\varphi(y))\,dt\right\}dy.$$

In view of (2) and above our result follows.

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