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CONVOLUTIONS WITH UNBOUNDED UNITY

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ABSTRACT. On the set F of complex-valued arithmetic functions we construct an infinite family of convolutions, that is, binary operations ψ of the form

$$(f\psi g)(n) = \sum_{\substack{f,g\in F\\ \psi(x,y)=n}} f(x)g(y)$$

so that $(F, +, \psi)$ is a commutative ring, for which the unity is unbounded. Here + denotes pointwise addition.

1. **Introduction.** In all the well known arithmetical convolutions on the set of arithmetical functions that exist in the literature (see [4]) such as, the Dirichlet, unitary (cf. [1]) and more generally Narkiewicz regular convolution [3], the unity is a bounded function, namely $[1/n]$ = the integral part of $1/n$, $n = 1, 2, \ldots$. That there can exist convolutions whose unity may be unbounded does not seem to have been noticed so far. We here construct an infinite family of arithmetical convolutions, all having an unbounded unity.

2. **Preliminaries.** Let $\mathbb{Z}^+$ denote the set of positive integers and F denote the set of arithmetic functions i.e., complex valued functions whose domain is $\mathbb{Z}^+$.

Let T be a non-empty subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$ and $\psi : T \longrightarrow \mathbb{Z}^+$ be a mapping satisfying the following conditions:

(2.1) For each $n \in \mathbb{Z}^+$, $\psi(x, y) = n$ has a finite number of solutions.

(2.2) If $(x, y) \in T$, then $(y, x) \in T$ and $\psi(x, y) = \psi(y, x)$.

(2.3) The statements "$(y, z) \in T$ and $\big(x, \psi(y, z)\big) \in T$" and "$(x, y) \in T$ and $\big(\psi(x, y), z\big) \in T$" are equivalent; whenever one of these statements holds, we have $\psi\big(x, \psi(y, z)\big) = \psi\big(\psi(x, y), z\big)$.

If we define the binary operation ψ on F by

$$(2.4) \qquad (f\,\psi\, g)(n) = \sum_{\psi(x,y)=n} f(x)g(y),$$

for each $n \in \mathbb{Z}^+$ and $f, g \in F$, then using the conditions (2.1), (2.2) and (2.3), it is not difficult to see that $(F, +, \psi)$ is a commutative ring (cf. [2]), where '+' as usual denotes the pointwise addition.

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3. **An infinite family of convolutions.** For each positive integer r, let $T_r \subseteq \mathbb{Z}^+$ be defined by

$$T_r = \{ i_r^{(1)}, i_r^{(2)}, \dots, i_r^{(r+1)}, i_r^{(r+2)} \}, \tag{3.1}$$

with

$$i_r^{(1)} < i_r^{(2)} < \cdots < i_r^{(r+2)}$$

and

$$i_r^{(r+2)} < i_{r+1}^{(1)}.$$

Clearly $T_r \cap T_s = \phi$ if $r \neq s$. Let

$$L = \bigcup_{r=1}^{\infty} T_r, \tag{3.2}$$

and

$$T = L_1 \cup L_2, \tag{3.3}$$

where

$$L_1 = \{ (k,k) : k \in \mathbb{Z}^+, k \notin L \}, \tag{3.4}$$

and

$$L_2 = \bigcup_{r=1}^{\infty} (T_r \times T_r). \tag{3.5}$$

We may note that $L_1 \cap L_2 = \emptyset$. We define $\psi : T \longrightarrow \mathbb{Z}^+$ as follows: on L_1, we define $\psi(k,k) = k$. Let $(x,y) \in L_2$ so that $x, y \in T_r$ for some positive integer r. From the definition of T_r given in (3.1) we may assume that $x = i_r^{(m)}$ and $y = i_r^{(n)}$, where $1 \leq m$, $n \leq r+2$. We now define

$$\psi(x,y) = \begin{cases} i_r^{(m)}, & \text{if } m = n \\ i_r^{(r+2)}, & \text{if } m \neq n. \end{cases}$$

It is easily seen that $\psi(x,y) \geq \max\{x,y\}$ for all $(x,y) \in T$ and hence ψ satisfies the condition (2.1). Also, it is clear that ψ satisfies (2.2). We will now verify that ψ satisfies (2.3). Let $(y,z) \in T$ and $\big(x, \psi(y,z)\big) \in T$. We recall that $T = L_1 \cup L_2$ and $L_1 \cap L_2 = \emptyset$. If $(y,z) \in L_1$, then $y = z$ and $\psi(y,z) = y$. Also, $y \notin L$. Now $(x,y) = \big(x, \psi(y,z)\big) \in T$ and since $y \notin L$, we must have $(x,y) \in L_1$. Hence $x = y$. Thus $x = y = z$ and $x \notin L$. Therefore, $(x,y) \in L_1 \subseteq T$ and $\big(\psi(x,y), z\big) = (y,z) \in L_1 \subseteq T$. If $(y,z) \in L_2$, then $y, z \in T_r$ for some r and so from the definition of ψ, $\psi(y,z) \in T_r$. Since $\big(x, \psi(y,z)\big) \in T$ and $\psi(y,z) \in T_r \subseteq L$, it follows that $\big(x, \psi(y,z)\big) \in L_2$ and $x \in T_r$. Thus we have that $x, y, z \in T_r$ and therefore $(x,y) \in L_2 \subseteq T$ and $\big(\psi(x,y), z\big) \in L_2 \subseteq T$. In any case we proved that $(y,z) \in T$ and $\big(x, \psi(y,z)\big) \in T$ imply that $(x,y) \in T$ and $\big(\psi(x,y), z\big) \in T$. The converse can be proved in a similar way.

If $(y,z) \in T$ and $\left(x, \psi(y,z)\right) \in T$, we now verify that $\psi\left(x, \psi(y,z)\right) = \psi\left(\psi(x,y), z\right)$.

If $(y,z) \in L_1$, then the elements $\left(x, \psi(y,z)\right), (x,y)$ and $\left(\psi(x,y), z\right)$ are all in L_1 and hence trivially $\psi\left(x, \psi(y,z)\right) = \psi\left(\psi(x,y), z\right)$.

If $(y,z) \in L_2$, then the elements $x, y, z, \psi(x,y)$ and $\psi(y,z)$ are all in T_r for some $r \in \mathbb{Z}^+$. We may assume that $x = i_r^{(a)}$, $y = i_r^{(b)}$ and $z = i_r^{(c)}$, where a, b and c are positive integers with $1 \le a, b, c \le r+2$. From the definition of ψ we have,

$$\psi\left(x, \psi(y,z)\right) = \begin{cases} i_r^{(a)}, & \text{if } a = b = c \\ i_r^{(r+2)}, & \text{if } a \ne b \text{ and } b = c \\ i_r^{(r+2)}, & \text{if } b \ne c, \end{cases} \tag{3.6}$$

and

$$\psi\left(\psi(x,y), z\right) = \begin{cases} i_r^{(a)}, & \text{if } a = b = c \\ i_r^{(r+2)}, & \text{if } a = b \text{ and } a \ne c \\ i_r^{(r+2)}, & \text{if } a \ne b. \end{cases} \tag{3.7}$$

From (3.6) and (3.7) it is clear that $\psi\left(x, \psi(y,z)\right) = \psi\left(\psi(x,y), z\right)$. Thus we have verified that ψ satisfies the condition (2.3).

4. **The Unity.** Let $g \in F$ be defined by

$$g(k) = \begin{cases} 1, \text{ if } k \notin L \text{ or } k = i_r^{(m)} & \text{for some positive integers } r \text{ and } m \text{ with } 1 \le m \le r+1 \\ -r, \text{ if } k = i_r^{(r+2)} & \text{for some } r \in \mathbb{Z}^+, \end{cases}$$

where $k \in \mathbb{Z}^+$.

We now prove:

THEOREM 4.1. *The element g is the unity of the commutative ring $(F, +, \psi)$.*

PROOF. We fix $k \in \mathbb{Z}^+$. Let $f \in F$. From (2.4) we have

$$(f \psi g)(k) = \sum_{\substack{\psi(x,y)=k \\ (x,y)\in T}} f(x)g(y) = \sum_{\substack{\psi(x,y)=k \\ (x,y)\in L_1}} f(x)g(y) + \sum_{\substack{\psi(x,y)=k \\ (x,y)\in L_2}} f(x)g(y)$$
$$= \sum\nolimits_1 + \sum\nolimits_2,$$

say. Let $k \notin L$. Then the only pair (x,y) in L_1 satisfying $\psi(x,y) = k$ is $(x,y) = (k,k)$. Hence we have

$$\sum\nolimits_1 = f(k)g(k) = f(k),$$

since $g(k) = 1$ for $k \notin L$. Also, there is no pair $(x,y) \in L_2$ such that $\psi(x,y) = k$. For, if such a pair $(x,y) \in L_2$ existed, then x and y are in T_r for some $r \in \mathbb{Z}^+$ so that $k = \psi(x,y) \in T_r \subseteq L$. This is a contradiction since $k \notin L$. Therefore $\sum_2$ is an empty sum. Hence for $k \notin L$, we have $(f \psi g)(k) = f(k)$.

Suppose $k \in L$. Hence $k \in T_r$ for some positive integer r. We can assume that $k = i_r^{(m)}$ where $1 \le m \le r+2$. First we assume that $1 \le m \le r+1$. In this case it is clear that

$\sum_1$ is an empty sum. We consider the sum $\sum_2$. Since $k = i_r^{(m)} \in T_r$, $(x, y) \in L_2$ and $\psi(x, y) = k$ imply that $x \in T_r$ and $y \in T_r$. Since $1 \leq m \leq r + 1$, from the definition of ψ it follows that the only solution to $\psi(x, y) = k$ is (k, k). Hence we have

$$\sum\nolimits_2 = f(k)g(k) = f(k),$$

since $g(k) = g\big(i_r^{(m)}\big) = 1$ for $1 \leq m \leq r + 1$. Thus in this case also we have $(f \,\psi\, g)(k) = f(k)$.

Finally, let $k = i_r^{(r+2)}$. Clearly $\sum_1$ is empty. We have

$$\begin{aligned}
\sum\nolimits_2 = \sum_{\substack{\psi(x,y)=k \\ (x,y)\in L_2}} f(x)g(y) &= \sum_{\substack{x\in T_r \\ y\in T_r \\ \psi(x,y)=k}} f(x)g(y) \\
&= \sum_{x\in T_r} f(x) \sum_{\substack{y\in T_r \\ \psi(x,y)=k}} g(y) \\
&= f(k)\sum_{m=1}^{r+2} g\big(i_r^{(m)}\big) + \sum_{\substack{x\in T_r \\ x\neq k}} f(x) \sum_{\substack{\psi(x,y)=k \\ y\in T_r}} g(y) \\
&= f(k)\sum_{m=1}^{r+2} g\big(i_r^{(m)}\big),
\end{aligned}$$

since the double sum on the right can be shown to be zero, as follows with $k = i_r^{(r+2)}$.

$$\begin{aligned}
\sum_{\substack{x\in T_r \\ x\neq k}} f(x) \sum_{\substack{\psi(x,y)=k \\ y\in T_r}} g(y) &= \sum_{\substack{x\in T_r \\ x\neq k}} f(x)\Big[\sum_{\substack{y\neq x \\ y\in T_r}} g(y)\Big] \\
&= \sum_{\substack{x\in T_r \\ x\neq k}} f(x)\Big[\Big(\sum_{\substack{y\neq x \\ y=i_r^{(n)},1\leq n\leq r+1}} 1\Big) - r\Big] \\
&= \sum_{\substack{x\in T_r \\ x\neq k}} f(x)[r - r] = 0.
\end{aligned}$$

Thus we have

$$\sum\nolimits_2 = f(k)\sum_{m=1}^{r+2} g\big(i_r^{(m)}\big) = f(k)\{(r+1) - r\} = f(k),$$

since $g\big(i_r^{(m)}\big) = 1$ for $1 \leq m \leq r + 1$ and $g\big(i_r^{(r+2)}\big) = -r$. Thus in any case we proved that $(f \,\psi\, g)(k) = f(k)$, for each $k \in \mathbb{Z}^+$. Hence g is the unity of $(F, +, \psi)$. Clearly g is unbounded.

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