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COMPARISON THEOREMS FOR LINEAR ELLIPTIC EQUATIONS

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ABSTRACT. Two comparison theorems, one of pointwise type and one of integral type, will be obtained for linear elliptic equations of order $2m$ on an exterior domain in R^n.

1. **Introduction.** Using Gårding's inequality, we will obtain comparison theorems for the linear elliptic differential equations

$$(1) \qquad Lu := \sum_{|\alpha|,|\beta|=0}^{m} (-1)^{|\alpha|} D^{\alpha}[A_{\alpha\beta}(x)\, D^{\beta} u] = 0 \quad (x \in \Omega \subseteq R^n)$$

and

$$(2) \qquad \ell u := \sum_{\alpha|,|\beta|=0}^{m} (-1)^{|\alpha|} D^{\alpha}[a_{\alpha\beta}(x)\, D^{\beta} u] = 0.$$

Here, Ω is an unbounded open set, the coefficient functions are sufficiently smooth, and we make use of the multi-index notation employed in [1]. Our results generalize two known comparison theorems:

(i) work of the author [5] in which $\ell u = (-1)^m \Delta^m u + h(x)u$;

(ii) work of Butler and Erbe [2] on the ordinary differential equations

$$(3) \qquad L_N v + pv = 0$$

and

$$(4) \qquad L_N v + qv = 0,$$

where L_N is a linear, disconjugate, ordinary differential operator of order N.

We remind the reader that an N-th order, ordinary, linear differential operator L_N is said to be *disconjugate* on an interval J iff the equation $L_N y = 0$ has no nontrivial solution with N zeros, counting multiplicities, on J.

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REMARK 1.1. According to the Pólya-Levin disconjugacy criterion [9,7], an N-th order, ordinary, linear differential operator L_N with real-valued, locally integrable coefficients is disconjugate on a nondegenerate interval J if, and only if, there exist sufficiently smooth, nonvanishing functions $\rho_0, \rho_1, \ldots, \rho_N$ such that, in the interior of J, we have

$$L_N v = \rho_N \frac{d}{dt}\Big[\rho_{N-1}\frac{d}{dt}\Big[\cdots \rho_1 \frac{d}{dt}[\rho_0 v]\cdots\Big]\Big]. \tag{5}$$

2. **Definitions and preliminary results.** Let G be a nonempty, open (possibly unbounded) subset of Ω. If k is a nonnegative integer, we define the seminorm $|\cdot|_{k,G}$, the weighted seminorm $|\cdot|_{k,G,w}$ and the norm $\|\cdot\|_{k,G}$ as follows:

$$|u|_{k,G} = \Big[\sum_{|\alpha|=k}\int_G |D^\alpha u|^2\,dx\Big]^{1/2}, \tag{6}$$

$$|u|_{k,G,w} = \Big[\sum_{|\alpha|=k}\int_G (k!/\alpha!)|D^\alpha u|^2\,dx\Big]^{1/2}, \tag{7}$$

$$\|u\|_{k,G} = \Big[\sum_{j=0}^{k}|u|_{j,G}^2\Big]^{1/2}. \tag{8}$$

The definition of $|u|_{k,G,w}$ is motivated by the following formula, which is valid for any real-valued ϕ in $C_0^\infty(G)$:

$$(-1)^k\int_G \phi\Delta^k\phi\,dx = \sum_{|\alpha|=k}\int_G (k!/\alpha!)|D^\alpha\phi|^2\,dx. \tag{9}$$

To compare the seminorms $|\cdot|_{m,G}$ and $|\cdot|_{m,G,w}$, we let

$$c_0 = \max\{m!/\alpha! : |\alpha| = m\}. \tag{10}$$

Then it is easily seen that

$$|u|_{m,G} \le |u|_{m,G,w} \le c_0^{1/2}|u|_{m,G}. \tag{11}$$

In (6) and (8), when there is no danger of confusion, we will omit the subscript G.

Let the Sobolev spaces $H_k(G)$ and $H_k^0(G)$ be defined as in [4]. If G is bounded, and if there exists a nontrivial function u in $H_m^0(G)\cap C^{2m}(G)$ such that (1) holds, then G is called a *nodal domain* for L or a nodal domain for (1). If for every positive number r the region $\{x\in\Omega : |x| > r\}$ contains a nodal domain for L, then (1) is said to be *nodally oscillatory* in Ω.

Using integration by parts, we can easily show that if G is any nonempty, open (possibly unbounded) subset of Ω, then for every real-valued ϕ in $C_0^\infty(G)$ we have:

$$\begin{aligned}\int_G \phi L\phi\,dx &= \sum_{|\alpha|=|\beta|=m}\int_G A_{\alpha\beta}(x)\,D^\alpha\phi D^\beta\phi\,dx + \int_G \phi^2 A_{0,0}(x)\,dx\\ &\quad + \sum_{|\alpha|+|\beta|=1}^{2m-1}\int_G A_{\alpha\beta}D^\alpha\phi D^\beta\phi\,dx\\ &:= f_G[L;\phi] + \int_G \phi^2 A_{0,0}(x)\,dx\end{aligned} \tag{12}$$

and

$$\int_G \phi \ell \phi \, dx = \sum_{|\alpha|=|\beta|=m} \int_G a_{\alpha\beta}(x)\, D^\alpha \phi\, D^\beta \phi \, dx + \int_G \phi^2 a_{0,0}(x)\, dx$$

(13)
$$+ \sum_{|\alpha|+|\beta|=1}^{2m-1} \int_G a_{\alpha\beta}\, D^\alpha \phi\, D^\beta \phi \, dx$$

$$:= f_G[\ell; \phi] + \int_G \phi^2 a_{0,0}(x)\, dx.$$

Define the set $K(L;\Omega)$ as follows:

(14)
$$K(L;\Omega) = \Big\{ \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\xi^{\alpha+\beta} : x \in \Omega, \xi \in R^n, |\xi| = 1 \Big\};$$

and define $K(\ell;\Omega)$ by replacing $A_{\alpha,\beta}$ in (14) with $a_{\alpha\beta}$.

We will suppose that the differential operators L and ℓ are uniformly strongly elliptic in the following sense: there exist constants E_0, e_0, and e_1 such that

(15)
$$0 < E_0 := \inf K(L;\Omega)$$

and

(16)
$$0 < e_0 := \inf K(\ell;\Omega) \leq \sup K(\ell;\Omega) := e_1.$$

Modifying the well-known proof of the global version of Gårding's inequality [1, Theorem 7.6], we can establish the following two results.

LEMMA 2.1. *Suppose that the principal coefficient-functions $A_{\alpha,\beta}$ ($|\alpha| = |\beta| = m$) are uniformly continuous on Ω, and that the intermediate coefficient-functions $A_{\alpha\beta}$ ($1 \leq |\alpha| + |\beta| \leq 2m-1$) are bounded and measurable on Ω. Let G be any nonempty, open subset of Ω and let $f_G[L;\phi]$ be as in (12). Then there exist constants $c_1 \in (0,\infty)$ and $c_2 \in [0,\infty)$ such that, for every real-valued ϕ in $C_0^\infty(G)$, we have:*

(17)
$$f_G[L;\phi] \geq c_1 E_0 ||\phi||^2_{m,G} - c_2 |\phi|^2_{0,G}.$$

The constant c_1 may be expressed explicitly in terms of the integers m and n; the constant c_2 may be expressed explicitly in terms of the following quantities: $\sup\{|A_{\alpha\beta}(x)| : x \in \Omega;\ 1 \leq |\alpha| + |\beta| \leq 2m-1\}$, m, n, E_0, and the modulus of continuity for the principal coefficients.

LEMMA 2.2. *Suppose that the regularity hypotheses in Lemma 2.1 hold, with $A_{\alpha\beta}$ replaced by $a_{\alpha\beta}$. Let G be any nonempty, open subset of Ω, and let $f_G[\ell;\phi]$ be as in (13). Then there exist positive constants c_5 and c_6 such that, for every real-valued ϕ in $C_0^\infty(G)$, we have:*

(18)
$$f_G[\ell;\phi] \leq c_5 |\phi|^2_{m,G} + c_6 |\phi|^2_{0,G}.$$

The constants c_5 and c_6 may be expressed explicitly in terms of the following quantities:

$m, n, e_1, \sup\{|a_{\alpha,\beta}(x)| : x \in \Omega;\ 1 \leq |\alpha| + |\beta| \leq 2m-1\}$ and the modulus of continuity for the principal coefficients.

3. **The main results.** Our first comparison theorem is a generalization of the scalar case of [5, Theorem 3.1]. Note that in [5] we considered the case where L is vector-valued, and we compared the differential operator L with the differential operator $(-1)^m\Delta^m + h$, where $h\colon \Omega \longrightarrow R^{N\times N}$ is a continuous matrix-valued function, Δ denotes the Laplace operator, and $\Delta^m := \Delta(\Delta^{m-1})$ whenever $m \geq 2$.

THEOREM 3.1. *Suppose that*

$$0 < c_5 \leq c_4 := c_0^{-1} c_1 E_0 \tag{19}$$

and that for all $x \in \Omega$ we have

$$A_{0,0}(x) - a_{0,0}(x) \geq c_2 + c_6. \tag{20}$$

If (1) is nodally oscillatory in Ω, then (2) is also nodally oscillatory in Ω.

PROOF. If (1) is nodally oscillatory in Ω, then for every positive number r the region $\{x \in \Omega : |x| > r\}$ contains a nodal domain G for the differential operator L. Thus, there exists a nontrivial real-valued function u in $C^{2m}(G) \cap H_m^0(G)$ such that (1) holds.

Furthermore, (13), (18) (*i.e.*, Lemma 2.2) and (11) together imply that for every real-valued ϕ in $C_0^\infty(G)$ we have:

$$\begin{aligned}\int_G \phi\ell\phi\,dx &= f_G[\ell;\phi] + \int_G \phi^2 a_{0,0}(x)\,dx\\ &\leq c_5|\phi|_{m,G}^2 + \int_G |\phi|^2[a_{0,0}(x) + c_6]\,dx\\ &\leq c_5|\phi|_{m,G,w}^2 + \int_G |\phi|^2[a_{0,0}(x)+c_6]\,dx.\end{aligned} \tag{21}$$

From (21), (19) and (20) we deduce that for every $\phi \in C_0^\infty(G;R^1)$ we have

$$\int_G \phi\ell\phi\,dx \leq c_4|\phi|_{m,G,w}^2 + \int_G |\phi|^2[A_{0,0}(x) - c_2]\,dx. \tag{22}$$

We also note that (12), (17) (*i.e.*, Lemma 2.1), (8) and (11) imply that for every $\phi \in C_0^\infty(G;R^1)$ we have

$$\begin{aligned}\int_G \phi L\phi\,dx &= f_G[L;G] + \int_G \phi^2 A_{0,0}(x)\,dx\\ &\geq c_1E_0\|\phi\|_{m,G}^2 + \int_G |\phi|^2[A_{0,0}(x)-c_2]\,dx\\ &\geq c_1E_0|\phi|_{m,G}^2 + \int_G |\phi|^2[A_{0,0}(x)-c_2]\,dx\\ &\geq c_0^{-1}c_1E_0|\phi|_{m,G,w}^2 + \int_G |\phi|^2[A_{0,0}(x)-c_2]\,dx\\ &= c_4|\phi|_{m,G,w}^2 + \int_G |\phi|^2[A_{0,0}(x)-c_2]\,dx.\end{aligned} \tag{23}$$

It follows, from (22) and (23), that for every real-valued ϕ in $C_0^\infty(G)$ we have

$$\int_G \phi \ell \phi \, dx \leq \int_G \phi L \phi \, dx. \tag{24}$$

Since u is in $H_m^0(G) \cap C^{2m}(G)$ and satisfies (1), and since $C_0^\infty(G)$ is dense in $H_m^0(G)$, it follows from (24) that

$$\int_G u \ell u \, dx \leq \int_G u L u \, dx = 0. \tag{25}$$

A standard variational argument [6, 3] may now be employed to find a nonempty open set $G' \subseteq G$ such that zero is the smallest eigenvalue of the boundary-value problem

$$\ell y = \lambda y, \quad y \in H_m^0(G') \cap C^{2m}(G'). \tag{26}$$

Thus, we have shown that for every positive number r the equation $\ell y = 0$ has a nontrivial solution y, with a nodal domain $G' \subseteq G \subseteq \{x \in \Omega : |x| > r\}$. The proof of Theorem 3.1 is now complete.

Before formulating and proving our next comparison theorem, we recall some ideas and results from [2].

Suppose that the functions $\rho_1, \dots, \rho_N$ introduced in Remark 1.1 have the property that for each $j \in \{1, \dots, N\}$ we have $\rho_j \in C^{N-j}\big(J; (0, \infty)\big)$. Define the quasiderivatives $L_0 v, \dots, L_N v$ in the usual way:

$$L_0 v = \rho_0 v, \quad L_j v = \rho_j \frac{d}{dt}(L_{j-1} v) \quad (1 \leq j \leq N). \tag{27}$$

Furthermore, let $\Gamma := \{i_1, \dots, i_k\}$ and $\Lambda := \{j_1, \dots, j_{N-k}\}$ be subsets of $\{0, 1, \dots, N-1\}$ such that $0 < i_1 < i_2 < \cdots < i_k \leq N-1$ and $0 \leq j_1 < j_2 < \cdots < j_{n-k} \leq N-1$.

For any point a in the interval J, the *first right extremal point* $\theta_1(\Gamma, \Lambda; a)$ for (3) is defined to be the first point $s \in J \cap (a, \infty)$ for which there exists a nontrivial solution of (3) satisfying the boundary conditions

$$\begin{cases} L_i v(a) = 0, & (i \in \Gamma) \\ L_j v(s) = 0, & (j \in \Lambda) \end{cases} \tag{28}$$

Similarly, we can define $\tilde{\theta}_1(\Gamma, \Lambda; a)$, the first right extremal point for (4).

The differential equation (3) is said to be (Γ, Λ)-*disconjugate* on J iff for every $a \in J$ the first right extremal point $\theta_1(\Gamma, \Lambda; a)$ is nonexistent.

The pair (Γ, Λ) is said to be *admissible* iff for every integer $b \in \{1, \dots, N-1\}$, at least b members of the sequence $(i_1, \dots, i_k, \dots, j_1, \dots, k\})$ are less than b.

The following known criterion for (Γ, Λ) to be admissible will be needed in the proof of our next comparison theorem.

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PROPOSITION 3.2 (SEE [2, P. 216]). *The pair* (Γ, Λ) *is admissible if, and only if, for every pair of points a and s in J satisfying* $a < s$, *there exists no nontrivial solution of the differential equation* $L_N y = 0$ *satisfying (28).*

REMARK 3.3. We now prove a comparison theorem which extends [2, Theorem 2.5]. To facilitate the statement of the theorem, we introduce some additional notation. We define the differential operators M_0 and M_1 as follows:

$$M_0 u := (-1)^m c_4 \Delta^m u + [A_{0,0}(x) - c_2]u, \tag{29}$$

$$M_1 u := (-1)^m c_5 \Delta^m u + [a_{0,0}(x) + c_6]u. \tag{30}$$

Suppose that there exists $r_0 \geq 0$ such that the interior of J is the open interval (r_0, ∞), and suppose that there exists $x^0 \in R^n$ such that $\Omega \supset \{x \in R^n : |x - x^0| \geq r_0\}$. For any positive r, let $S_r = \{x \in R^n : |x| = r\}$. Define the real-valued functions h_j ($2 \leq j \leq 7$) as follows:

$$h_2(r) = \min\{A_{0,0}(x) - c_2 : x \in S_r\} \text{ whenever } r \in J, \tag{31}$$

$$h_3(x) = h_2(|x|) \text{ whenever } x \in \Omega, \tag{32}$$

$$h_4(r) = \max\{a_{0,0}(x) + c_6 : x \in S_r\} \text{ whenever } r \in J, \tag{33}$$

$$h_5(x) = h_4(|x|) \text{ whenever } x \in \Omega, \tag{34}$$

$$h_6(r) = \min\{0, h_2(r)\} \text{ whenever } r \in J, \tag{35}$$

$$h_7(x) = h_6(|x|) \text{ whenever } x \in \Omega. \tag{36}$$

Following [2, p. 216], we will suppose that $h_6(r)$ is not identically zero and that $h_4(r) < 0$.

Define the differential operators M_3 and M_4 as follows:

$$M_3 u := (-1)^m c_5 \Delta^m u + h_5(x)u \tag{37}$$

and

$$M_4 u := (-1)^m c_4 \Delta^m u + h_7(x)u. \tag{38}$$

Let $\Delta_{|x|}$ denote the radially symmetric form of the Laplace operator. In other words, if $|x| = r$, then

$$\Delta_r = r^{n-1} \frac{d}{dr}\left(r^{1-n} \frac{d}{dr}\right). \tag{39}$$

Furthermore, let M_5 and M_6 denote the radially symmetric forms of the differential operators M_3 and M_4, respectively. In other words, let

$$M_5 v = (-1)^m c_5 \Delta_r^m v + h_4(r)v \tag{40}$$

$$M_6 v = (-1)^m c_4 \Delta_r^m v + h_6(r)v, \tag{41}$$

where h_4 and h_6 are defined in (33) and (35), respectively.

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THEOREM 3.4. *Suppose that for all $r \in J$ we have*

$$\int_r^\infty t^{1-n}|h_4(t)|\,dt \geq \int_r^\infty t^{1-n}|h_6(t)|\,dt. \tag{42}$$

If (1) is nodally oscillatory in Ω, then so is (2).

PROOF. Let G be any nonempty open subset of Ω, and let ϕ be any real-valued function in $C_0^\infty(G)$. Then (23), (9), (7) and (29) imply that

$$\begin{aligned}\int_G \phi L\phi\,dx &\geq c_4|\phi|^2_{m,G,w} + \int_G |\phi|^2[A_{0,0}(x) - c_2]\,dx \\ &= \int_G \phi M_0\phi\,dx.\end{aligned} \tag{43}$$

If (1) is nodally oscillatory in Ω, then (43) and the arguments following (25) in the proof of Theorem 3.1 together imply that the equation

$$M_0u = 0 \tag{44}$$

is nodally oscillatory in Ω.

Furthermore, (29), (31), (32), (35), (36) and (38) imply that

$$\begin{aligned}\int_G \phi M_0\phi\,dx &= (-1)^m c_4 \int_G \phi\Delta^m\phi + \int_G [A_{0,0}(x) - c_2]|\phi|^2\,dx \\ &\geq \int_G [(-1)^m c_4\phi\Delta^m\phi + h_7(x)|\phi|^2]\,dx \\ &= \int_G \phi M_4\phi\,dx.\end{aligned} \tag{45}$$

Since (44) is nodally oscillatory in Ω, therefore (45) and the arguments following (25) in the proof of Theorem 3.1 together imply that the equation

$$M_4u = 0 \tag{46}$$

is nodally oscillatory in Ω. In other words, for every positive number r_1, the region $\{x \in \Omega : |x| > r_1\}$ contains a nodal domain for (46). Let $J_1 := J \cap (r_1, \infty)$. Then we can employ the method of spherical means (as in the proof of [3, Theorem 4.1]) to show that the ordinary differential equation

$$M_6v = 0 \tag{47}$$

is (Γ, Λ)-nondisconjugate on J_1 in the case where $\Gamma = \Lambda = \{0, 1, \ldots, m-1\}$. (See (41) for the definition of M_6.) Because of the representation (39), we can choose

$$\begin{cases} N = 2m, \quad \rho_N(r) = r^{n-1},\ \rho_{N-1}(r) = r^{1-n}, \ldots, \rho_1(r) = r^{1-n},\ \rho_0(r) = 1, \\ L_N = \Delta_r^m \end{cases} \tag{48}$$

in (5). Since (47) is (Γ, Λ)-disconjugate on J_1 in the case where $\Gamma = \Lambda = \{0, 1, \ldots, m-1\}$, it follows from (42) and [2, Theorem 2.5] that either the pair (Γ, Λ) is inadmissible or the ordinary differential equation

$$M_5v = 0 \tag{49}$$

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is (Γ, Λ)-nondisconjugate on J_1 (See (40) for the definition of M_5). But if the pair (Γ, Λ) is inadmissible, then it follows from Proposition 3.2 that there will exist two points a and s in J_1, with $a < s$, such that the boundary-value problem consisting of the ordinary differential equation

$$L_n v_0 = 0 \text{ on } J_1 \tag{50}$$

and the boundary conditions (28) (with v replaced by v_0) has at least one nontrivial solution. It follows from (48) and (39) that the boundary-value problem consisting of the partial differential equation

$$\Delta^m u_0 = 0 \text{ in } \Omega_{a,s} := \{x \in \Omega : a < |x| < s\} \tag{51}$$

and the boundary conditions

$$\Delta^k u_0 = 0 \text{ on } \partial\Omega_{a,s} \quad (0 \le k \le m-1) \tag{52}$$

has at least one radially-symmetric nontrivial solution. (Here, $\Delta^0 u_0 := u_0$.) But since $\Delta^m u_0 := \Delta(\Delta^{m-1} u_0)$ whenever $m \geq 1$, it follows from the maximum principle that any solution of (51) and (52) has the property that

$$\Delta^{m-1} u_0 = 0 \text{ throughout } \Omega_{a,s}. \tag{53}$$

Furthermore, (52) implies that

$$\Delta^k u_0 = 0 \text{ on } \partial\Omega_{a,s} \quad (0 \le k \le m-2). \tag{54}$$

Continuing recursively, we deduce eventually that $u_0 = 0$ throughout $\Omega_{a,s}$. This contradicts the nontrivialness of u_0, and shows that the pair (Γ, Λ) cannot be inadmissible. Thus, we have proved that (49) is (Γ, Λ)-nondisconjugate on J_1 in the case where $\Gamma = \Lambda = \{0, 1, \ldots, m-1\}$. In other words, there exist points a and s (in J_1) and a real-valued function $v \in C^{2m}(J_1)$ such that (49) and (28) hold.

Introducing spherical polar coordinates in the usual way [8, p. 58], we note that, for every multi-index β, if $|x| = r$, then the expression $x^\beta / r^{|\beta|}$ is independent of r. It follows from the Chain Rule and the final statement in the last paragraph above that there exist points a and s (in J_1) and a radially-symmetric, real-valued, C^{2m} function $u: x \longrightarrow v(|x|)$ such that

$$M_3 u = M_5 v = 0 \tag{55}$$

throughout the spherical shell $\Omega_{a,s}$, and

$$D^\alpha u|_{\partial\Omega_{a,s}} = \frac{x^\alpha}{r^{|\alpha|}} \left(\frac{\partial}{\partial r}\right)^{|\alpha|} v|_{\partial\Omega_{a,s}} = 0 \text{ whenever } 0 \le |\alpha| \le m-1. \tag{56}$$

But (56) implies, because of [1, Lemma 9.10], that $u \in H_m^0(\Omega_{a,s})$. Since r_1 was chosen arbitrarily, we have therefore shown that (55) is nodally oscillatory in Ω.

Furthermore, (37), (34), (33) and (30) imply that if G is any nonempty open subset of Ω, and if ϕ is any real-valued function in $C_0^\infty(G)$, then

$$\begin{aligned}\int_G \phi M_3 \phi \, dx &= \int_G [(-1)^m c_5 \phi \Delta^m \phi + h_5(x)|\phi|^2] \, dx \\ &\geq \int_G \left[(-1)^m c_5 \phi \Delta^m \phi + [a_{0,0}(x) + c_6]|\phi|^2\right] dx \\ &= \int_G \phi M_1 \phi \, dx.\end{aligned} \tag{57}$$

Since (55) is nodally oscillatory in Ω, therefore (57) and a familiar argument imply that the partial differential equation

$$M_1 u = 0 \tag{58}$$

is nodally oscillatory in Ω.

Finally, (30), (7), (9) and (21) imply that if G is any nonempty open subset of Ω, and if ϕ is any real-valued function in $C_0^\infty(G)$, then

$$\begin{aligned}\int_G \phi M_1 \phi \, dx &= \int_G \left[(-1)^m c_5 \phi \Delta^m \phi + [a_{0,0}(x) + c_6]|\phi|^2\right] dx \\ &= c_5 |\phi|^2_{m,G,w} + \int_G [a_{0,0}(x) + c_6]|\phi|^2 \, dx \\ &\geq \int_G \phi \ell \phi \, dx.\end{aligned} \tag{59}$$

Since (58) is nodally oscillatory in Ω, (59) implies that (2) is nodally oscillatory in Ω.

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