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ABSOLUTE APPROXIMATE RETRACTS AND AR-SPACES

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1. Introduction. A subset A of a topological space X is an *approximate retract* of X if for every neighborhood U of A in X there is a retract R of X such that $A \subset R \subset U$. A compactum X is an *absolute approximate retract* (AAR-space) if whenever X is embedded as a subset of a compactum Z, then X is an approximate retract of Z. These concepts were first defined in [**2**] where it is shown that every AAR-space is a contractible Peano continuum. In [**3**] an example is given to show that there exists a contractible LC^∞ compactum which is not an AAR-space.

The purpose of this paper is to show that every AAR-space is locally contractible, and to give an example of a contractible and locally contractible compactum which is not an AAR-space.

2. Preliminaries. The terminology used in this paper may be found in [**1**]. In particular, Hilbert space will be denoted by E^ω and the diameter of a set M will be denoted by $\delta(M)$. The following two constructions will be used in the sequel.

(1) The cap over a compactum. Let X be a compact subset of E^ω and let S be a compact segment in E^ω which is disjoint from X. The construction of the cap of X and S, denoted by cap XS, may be found in [**5**, p. 42].

(2) A special plane continuum C. The infinite ray A, the 1-dimensional plane continuum B, and $C = A \cup B$ shall be as defined in [**3**, p. 491].

3. The results. In the proof of the following theorem, we make use of techniques which have been used in [**3**] and [**4**].

THEOREM 1. *Every* AAR-*space is locally contractible.*

Proof. Let X be an AAR-space and suppose that X is not locally contractible at a point p. Then there is a neighborhood U of p in X which contains a decreasing sequence $V_1, V_2, \ldots$ of compact neighborhoods of p such that $\lim_{i\to\infty}\delta(V_i) = 0$ and no V_i is contractible in U. Consider a sequence of disjoint continua $Y_1, Y_2, \ldots$ with $\lim_{i\to\infty}\delta(Y_i) = 0$ obtained by first taking the disjoint union $\bigcup_{i=1}^\infty V_i$ and then letting $Y_i =$ cap V_iA_i

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for $i = 1, 2, \ldots$. Let $Y = \bigcup_{i=1}^{\infty} Y_i$. Form the compactum M obtained by taking the disjoint union of X and Y, and then identifying V_i in X with $V_i \times \{0\}$ in Y for $i = 1, 2, \ldots$. We now use the special continuum C defined in Section 2 to attach the infinite ray A to M. Since the construction used to attach A to M will subsequently be referred to in the paper, it shall be denoted by (*).

Let $g: B \to M$ be a homeomorphism into M such that

(1) $g(I) \subset Y$ and $g(0) = p$.

(2) $g(B_i) \subset Y_i$ and $g(A_i) = A_i$. (*)

Define Z_1 to be the continuum obtained by taking the disjoint union of C and M, and then identifying each point $z \in B$ with its image $g(z) \in M$.

Let W be a neighborhood of X in Z_1 such that $W \cap C$ consists of all the points in C whose distance from the x-axis is less than 1/6. Then W contains all but finitely many of the sets Y_i, and no connected subset of W containing X contains a point of A. Since X is an AAR-space, there is a retract R of Z_1 such that $X \subset R \subset W$. Then, since R is a retract of the connected space Z_1, it follows that R is a connected subset of W which contains X. Let $r: Z_1 \to R$ be a retraction of Z_1 onto R.

First we show that $A_1 \cap R = \emptyset$ for all $i = 1, 2, \ldots$. To see this, suppose $A_j \cap R \neq \emptyset$ for some j. It then follows that R contains a non-locally connected subcontinuum of Y_j which contains both V_j and A_j. Then r must map a subinterval of A onto a nonlocally connected subset of Y_j, which is impossible. Therefore $A_i \cap R = \emptyset$ for all $i = 1, 2, \ldots$ and, hence,

$$R \subset Z_1 - \left(A \cup \bigcup_{i=1}^{\infty} A_i\right).$$

Define a retraction $s: R \to X$ by

$$s(y) = \begin{cases} y & \text{if } y \in R \cap X, \\ (x, 0) & \text{if } y = (x, t) \in R \cap \left(\bigcup_{i=1}^{\infty} Y_i\right). \end{cases}$$

Let $r_1 = sr$. Then $r_1: Z_1 \to X$ is a retraction such that $r_1(Y_i) \subset U$ for all but finitely many of the Y_i. To facilitate notation, we shall assume that $r_1(Y_i) \subset U$ for $i = 1, 2, \ldots$. Since no V_j is contractible in U, it follows that $\delta(r_1(A_1)) \neq 0$ for $i = 1, 2, \ldots$. Hence, we may assume that $\delta(r_1(A_i)) = \lambda_i$ where $\lambda_1, \lambda_2, \ldots$ is a sequence of positive numbers which converges to 0.

Let a_i denote the midpoint of A_i, and let $0 = t_{i,0} < t_{i,1} < \ldots < t_{i,j} < \ldots$ be a sequence of numbers in $[0, \infty)$ such that

$$\lim_{j \to \infty} V_i \times \{t_{i,j}\} = a_i.$$

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Set $b_i = r_1(a_i)$, $B_{i,j} = r_1(V_i \times \{t_{i,j}\})$. Let $Y_{i,j} = \text{cap}\,(B_{i,j}\, A_{i,j})$ denote a null sequence of caps satisfying the following properties.

(1) $Y_{i,j} \cap X = B_{i,j}$.
(2) $Y_{i,j} \cap Y_{i,k} \subset X$ if $j \neq k$.
(3) $\lim_{j\to\infty} Y_{i,j} = b_i$.

Let $M_i = X \cup \bigcup_{i=1}^{\infty} Y_{i,j}$ and, as in (*), obtain a continuum $Z_{2,i}$ by attaching an infinite ray to M_i. Then, using arguments similar to previous arguments, there is a sequence of retractions $r_{2,i}: Z_{2,i} \to X$, $i = 1, 2, \ldots$, each of which maps infinitely many sets of the form $Y_{i,j}$ into U. The process can be continued so that in the n^{th} stage we obtain retractions

$$r_{n,i_1,\ldots,i_{n-1}}: Z_{n,i_1,\ldots,i_{n-1}} \to X,$$

each of which maps infinitely many of the sets of the form $Y_{i_1,i_2,\ldots,i_n}$ into U. Consequently, there is a sequence

$$V_{j_1} = B_{j_1,0}, B_{j_1,j_2}, B_{j_1,j_2,j_3}, \ldots, B_{j_1,j_2,\ldots,j_n}, \ldots$$

of sets lying in U, a point $q \in U$, and a sequence of homotopies $\{F_n\}_{n=1}^{\infty}$ with values in U such that the following properties are satisfied.

(1) $\lim_{n\to\infty} B_{j_1,j_2,\ldots,j_n} = \{q\}$.
(2) F_1 is a homotopy which deforms V_{j_1} onto B_{j_1,j_2} and, for $n > 1$, F_n deforms $B_{j_1,j_2,\ldots,j_n}$ onto $B_{j_1,j_2,\ldots,j_n,j_n+1}$.
(3) $\lim_{n\to\infty} \delta(\text{Im } F_n) = 0$.

It is possible to construct a homotopy F with values in U which deforms V_{j_1} to the point q. This contradiction shows that X must be locally contractible and the proof is complete.

Proposition 1 in [**2**, p. 410] together with the above theorem shows that every AAR-space is contractible and locally contractible. Thus every finite dimensional AAR-space is an AR-space [**1**, p. 122]. Since every AR-space is an AAR-space, we have the following theorem.

THEOREM 2. *A finite dimensional compactum X is an* AAR-*space if and only if X is an* AR-*space.*

Since a retract of an AR-space is an AR-space [**1**, p. 101], the following result would be a consequence of Theorem 2 for the finite dimensional case.

PROPOSITION. *Every retract of an* AAR-*space is an* AAR-*space.*

Proof. Let X be a retract of an AAR-space Y and let $r: Y \to X$ be a retraction of Y onto X. Let Z be a compactum and suppose $e: X \to Z$ is an embedding of X into Z. Form the identification space M obtained by taking the disjoint union of Y and Z, and then identifying each point

$x \in X$ with $e(x) \in Z$. Let U be a neighborhood of X in Z. Then $Y \cup U$ is a neighborhood of Y in M. Thus there is a retract Q of M such that $Y \subset Q \subset Y \cup U$. Let $R = Q \cap Z$. Then $X \subset R \subset U$. Define a retraction $f: Q \to R$ from Q onto R by

$$f(x) = \begin{cases} x & \text{if } x \in Q \cap Z, \\ r(x) & \text{if } x \in Y. \end{cases}$$

Let $g: M \to Q$ be a retraction of M onto Q and let $h = fg|Z$. Then it is easy to check that $h: Z \to R$ is a retraction of Z onto R. Thus X is an approximate retract of Z and, hence, X is an AAR-space.

An example of a contractible LC^∞ compactum which is not an AAR-space is given in [**3**]. We now mention that a well-known example due to Borsuk [**1**, p. 126] is in fact an example of a contractible and locally contractible compactum which is not an AAR-space.

Example. Consider the following subsets of the Hilbert cube Q^ω (for notation see [**1**, p. 10]):

$$X_0 = \{x = \{x_i\} | x_1 = 0\},$$

$$B_k = \{x = \{x_i\} | 1/(k+1) \leqq x_1 \leqq 1/k \text{ and } x_i = 0 \text{ for } i > k\} \quad \text{for } k = 1, 2, \ldots .$$

The boundary Bd B_k of B_k is a $(k-1)$-sphere which we shall denote by X_k for $k = 1, 2, \ldots$. Let $X = X_0 \cup \bigcup_{k=1}^\infty X_k$. Then, if Y denotes the cone over X with vertex p, Y is a contractible and locally contractible compactum which is not an ANR-space [**1**, p. 126]. We now show that Y is not an AAR-space by constructing a compactum Z containing Y such that Y is not an approximate retract of Z. Since the proof is analogous to that found in [**3**, p. 491], we omit the details.

Let $C_k = \text{cap } X_k A_k$, $k = 1, 2, \ldots$, denote a sequence of caps in Hilbert space E^ω such that the following properties are satisfied.

(1) $C_k \cap Y = X_k \times \{0\} = X_k$ for $k = 1, 2, \ldots$.

(2) $C_i \cap C_j = X_i \cap X_j$ for $i, j = 1, 2, \ldots$.

(3) $\lim_{k\to\infty} C_k = X_0$.

Define $M = Y \cup \bigcup_{k=1}^\infty C_k$. Let Z denote the continuum obtained by using (*) to attach an infinite ray A to M. Let U denote the neighborhood of Y in Z which consists of all the points in Z whose distance from Y is less than 1/6. Suppose R is a retract of Z such that $Y \subset R \subset U$. Then the following facts may be easily verified.

(1) C_k cannot be retracted onto $X_k = X_k \times \{0\}$ for $k = 1, 2, \ldots$.

(2) Only finitely many of the sets C_k, $k = 1, 2, \ldots$, can be retracted into Y.

(3) R is a connected subset which contains a set of the form C_j, but R contains no point of A.

(4) Any retraction of Z onto R must map a subinterval of A onto a nonlocally connected subset of C_j, which is a contradiction.

In view of the fact that the class of AAR-spaces is properly contained in the class of contractible and locally contractible compacta, it is appropriate to ask the following question:

Question. Does the class of AAR-spaces coincide with the class of AR-spaces?

References

1. K. Borsuk, *Theory of retracts*, Monografie Matematyczne *44* (Warszawa, 1967).
2. J. R. Martin, *A generalization of absolute retracts*, Proc. Amer. Math. Soc. *52* (1975), 409–413.
3. ——— *An example of a contractible* LC^∞ *compactum which is not an absolute approximate retract*, Bull. Ac. Pol. Sci. *25* (1977), 489–492.
4. ——— *Absolute fixed point sets and* AR-*spaces*, Fund. Math. (to appear).
5. ——— *Absolute fixed point sets in compacta*, Colloq. Math. *39* (1978), 41–44.

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