# A CHARACTERIZATION OF SPECTRAL OPERATORS ON HILBERT SPACES

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Let $H$ be a complex Hilbert space and denote by $B(H)$ the Banach algebra of all bounded linear operators on $H$. In [**5**; **6**] J. Ph. Labrousse proved that every operator $S \in B(H)$ which is spectral in the sense of N. Dunford (see [**3**]) is similar to a $T \in B(H)$ with the following property

$$\lim_{n\to\infty} \left\| \sum_{j=0}^{n} \binom{n}{j} T^j T^* (-T)^{n-j} \right\|^{1/n} = 0. \tag{1}$$

Conversely, he showed that given an operator $S \in B(H)$ such that its essential spectrum (in the sense of [**5**; **6**]) consists of at most one point and such that $S$ is similar to a $T \in B(H)$ with the property (1), then $S$ is a spectral operator. This led him to the conjecture that an operator $S \in B(H)$ is spectral if and only if it is similar to a $T \in B(H)$ with property (1). The purpose of this note is to prove this conjecture in the case of operators which are decomposable in the sense of C. Foias (see [**2**]).

For the convenience of the reader we first recall some notations and definitions. For $T \in B(H)$ let $\sigma(T)$ be the spectrum of $T$ and denote by $\mathrm{Lat}(T)$ the family of all closed subspaces of $H$ which are invariant for $T$. Recall ([**1**; **7**; **8**]) that $T$ is *decomposable* if and only if for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$ there are $X, Y \in \mathrm{Lat}(T)$ such that $X + Y = H$ and $\sigma(T \mid X) \subset U$, $\sigma(T \mid Y) \subset V$. Then $T$ has the *single valued extension property* ([**2**; **3**]), i.e. for every $H$-valued function $f: D_f \to H$ which is locally analytic in an open set $D_f \subset \mathbb{C}$ and satisfies $(z - T)f(z) \equiv 0$ on $D_f$ we have $f \equiv 0$ on $D_f$. If $T \in B(H)$ has the single valued extension property then, for $x \in H$. $\rho_T(x)$ is the set of all $z \in \mathbb{C}$ such that there exists an open neighborhood $U$ of $z$ and a locally analytic function $x_T: U \to H$ with $(w - T)x_T(w) \equiv x$ on $U$. The *local spectrum of $T$ at $x$* is then $\sigma_T(x) := \mathbb{C} \backslash \rho_T(x)$ see [**2**; **3**]). As usual, we put for $M \subset \mathbb{C}: H_T(M) := \{x \in H : \sigma_T(x) \subset M\}$. If $T$ is decomposable and $M$ is closed, then $H_T(M) \in \mathrm{Lat}(T)$ and $\sigma(T \mid H_T(M)) \subset M \cap \sigma(T)$ (cf. [**2**; **1**]). We can now state our main result.

THEOREM. *$S \in B(H)$ is spectral if and only if $S$ is a decomposable operator which is similar to a $T \in B(H)$ with property* (1).

*Proof.* If $S$ is spectral, then $S$ is obviously decomposable and is (by [**5**, Theorem 2]) similar to a $T \in B(H)$ with property (1).

Conversely, let now $S$ be a decomposable operator which is similar to a $T \in B(H)$ with property (1). We shall show that $T = N + Q$, where $N \in B(H)$ is normal and $Q \in B(H)$ is a quasinilpotent operator commuting with $T$. Then $T$ and hence also $S$ is spectral.

First, let us remark that $T$ is decomposable as it is similar to the decomposable operator $S$. If $F \subset \mathbb{C}$ is closed, we denote by $P(F)$ the orthogonal projection with

$P(F)H=H_T(F)$. As $T$ satisfies (1) we obtain from [**2**, Theorem 2.3.3] that $H_T(F)\in \mathrm{Lat}(T^*)$ and therefore (because of $H_T(F)\in \mathrm{Lat}(T)$),

$$P(F)T=TP(F) \quad \text{for all closed } F\subset\mathbb{C}. \tag{2}$$

Again we may apply [**2**, Theorem 2.3.3] and conclude that for all closed $F_1, F_2\subset\mathbb{C}$ we have $P(F_1)H_T(F_2)\subset H_T(F_2)$ and therefore $P(F_1)P(F_2)=P(F_2)P(F_1)$ (as $P(F_1)$ and $P(F_2)$ are orthogonal projections). Hence, $P(F_1)P(F_2)$ is the orthogonal projection with range $H_T(F_1)\cap H_T(F_2)=H_T(F_1\cap F_2)$ and we have proved

$$P(F_1)P(F_2)=P(F_1\cap F_2) \quad \text{for all closed } F_1, F_2\subset\mathbb{C}. \tag{3}$$

In our next step we show the following.

For all closed $F_1, F_2\subset\mathbb{C}$ with $F_1\subset F_2$, $P(F_2)-P(F_1)$ is the orthogonal projection with range $\overline{H_T(F_2\backslash F_1)}$. (4)

*Proof.* First, if $x\in H_T(F_2\backslash F_1)$, then $\sigma_T(x)\cap F_1=\varnothing$ and $x\in H_T(\sigma_T(x))\cap H_T(F_2)$. Because of $H_T(\varnothing)=\{0\}$ we obtain

$$(P(F_2)-P(F_1))x=x-P(F_1)P(\sigma_T(x))x=x-P(F_1\cap\sigma_T(x))x=x.$$

This shows that $\overline{H_T(F_2\backslash F_1)}\subset(P(F_2)-P(F_1))H$. Let now $x$ be an arbitrary element of $(P(F_2)-P(F_1))H\ominus\overline{H_T(F_2\backslash F_1)}$ and consider open sets $U_n$, $V_n\subset\mathbb{C}$ with $U_n\cup V_n=\mathbb{C}$ and $\overline{V_n}\cap F_1=\varnothing$ for all $n\in\mathbb{N}$ such that $\bigcap_{n=1}^{\infty}\overline{U_n}=F_1$. By the decomposability of $T$ we have for fixed $n\in\mathbb{N}$ elements $x_1\in H_T(\overline{U_n})$, $x_2\in H_T(\overline{V_n})$ with $x=x_1+x_2$. As $x\in(P(F_2)-P(F_1))H\subset P(F_2)H$ (because of (3) and $F_1\subset F_2$), we obtain

$$\begin{aligned} x&=P(F_2)x=P(F_2)P(\overline{U_n})x_1+P(F_2)P(\overline{V_n})x_2\\ &=P(F_2\cap\overline{U_n})x_1+P(F_2\cap\overline{V_n})x_2. \end{aligned}$$

Now, $F_2\cap\overline{V_n}\subset F_2\backslash F_1$ and therefore $H_T(F_2\cap\overline{V_n})\subset\overline{H_T(F_2\backslash F_1)}$. As $x\in\overline{H_T(F_2\backslash F_1)}^{\perp}$, we conclude that

$$\begin{aligned} 0&=P(F_2\cap\overline{V_n})x=P(F_2\cap\overline{V_n})P(F_2\cap\overline{U_n})x_1+P(F_2\cap\overline{V_n})x_2\\ &=P(F_2\cap\overline{V_n}\cap\overline{U_n})x_1+P(F_2\cap\overline{V_n})x_2. \end{aligned}$$

Hence, $P(F_2\cap\overline{V_n})x_2\in H_T(F_2\cap\overline{V_n}\cap\overline{U_n})\subset H_T(\overline{U_n})$, so that $x=P(F_2\cap\overline{U_n})x_1+P(F_2\cap\overline{V_n})x_2\in H_T(\overline{U_n})$ and therefore,

$$x\in\bigcap_{n=1}^{\infty}H_T(\overline{U_n})=H_T\left(\bigcap_{n=1}^{\infty}\overline{U_n}\right)=H_T(F_1).$$

From this and (3) we obtain

$$x=(P(F_2)-P(F_1))x=(P(F_2)-P(F_1))P(F_1)x=0$$

and (4) is proved.

Put $r:=\|T\|+1$ and consider the square

$$R:=R_{0,0}^{(0)}:=\{z\in\mathbb{C}:-r\le\operatorname{Re} z, \operatorname{Im} z<r\}.$$

We shall now construct a homomorphism $\Phi$ from the Banach algebra $C(\bar{R})$ of all continuous complex valued functions on $\bar{R}$ to $B(H)$. For $n \in \mathbb{N}$ and $0 \leq j \leq 2^n$ we introduce the sets

$$H_j^n := \{z \in \mathbb{C} : \operatorname{Re} z \geq -r + j2^{1-n}r\},$$
$$K_j^n := \{z \in \mathbb{C} : \operatorname{Im} z \geq -r + j2^{1-n}r\},$$

and for $0 \leq j, k \leq 2^n - 1$, $R_{j,k}^{(n)} := (H_j^n \backslash H_{j+1}^n) \cap (K_k^n \backslash K_{k+1}^n)$.

For $n \in \mathbb{N} \cup \{0\}$, $0 \leq j, k \leq 2^n - 1$ we put

$$z_{j,k}^{(n)} := -r + j2^{1-n}r + i(-r + k2^{1-n}r),$$

and

$$P_{j,k}^{(n)} := (P(H_j^n) - P(H_{j+1}^n))(P(K_k^n) - P(K_{k+1}^n)).$$

Because of (2), (3), and (4), the mappings $P_{j,k}^{(n)}$ are orthogonal projections commuting with $T$ such that

$$P_{j,k}^{(n)} P_{p,q}^{(n)} = 0 \quad \text{if} \quad (j, k) \neq (p, q), \tag{5}$$

and

$$P_{j,k}^{(n)} = \sum_{p,q=0}^{2^{m-n}-1} P_{j2^{m-n}+p,k2^{m-n}+q}^{(m)} \quad \text{for} \quad m \geq n \text{ and } 0 \leq j, k \leq 2^n - 1. \tag{6}$$

Moreover,

$$P_{0,0}^{(0)} = I \tag{7}$$

as $H \supset P_{0,0}^{(0)} H \supset P(\sigma(T))H = H_T(\sigma(T)) = H$ because of (4) and $\sigma(T) \subset \operatorname{int} R$. For $n \in \mathbb{N}$ we define now $\Phi_n : C(\bar{R}) \to B(H)$ by

$$\Phi_n(f) := \sum_{j,k=0}^{2^n-1} f(z_{j,k}^{(n)}) P_{j,k}^{(n)} \quad \text{for} \quad f \in C(\bar{R}).$$

For arbitrary $\varepsilon > 0$ there exists (by the continuity of $f$ on $\bar{R}$) an $n \in \mathbb{N}$ such that for all $z, w \in \bar{R}$ with $|z - w| < 2^{2-n}r$ we have $|f(z) - f(w)| < \varepsilon$. For arbitrary $x \in H$ and $m \geq n$ we therefore obtain, (using (5), (6), (7), diam $R_{j,k}^{(n)} \leq \sqrt{2} \cdot 2^{1-n}r < 2^{2-n}r$, and the fact that $z_{j,k}^{(n)}$, $z_{j2^{m-n}+p,k2^{m-n}+q}^{(m)} \in R_{j,k}^{(n)}$ for $0 \leq p, q \leq 2^{m-n} - 1$)

$$\begin{aligned}
\|(\Phi_n(f) - \Phi_m(f))x\|^2 &= \sum_{j,k=0}^{2^n-1} \sum_{p,q=0}^{2^{m-n}-1} |f(z_{j,k}^{(n)}) - f(z_{j2^{m-n}+p,k2^{m-n}+q}^{(m)})| \, \|P_{j2^{m-n}+p,k2^{m-n}+q}^{(m)} x\|^2 \\
&\leq \varepsilon^2 \sum_{j,k=0}^{2^n-1} \sum_{p,q=0}^{2^{m-n}-1} \|P_{j2^{m-n}+p,k2^{m-n}+q}^{(m)} x\|^2 \\
&\leq \varepsilon^2 \|x\|^2.
\end{aligned}$$

Therefore, $(\Phi_n(f))_{n=1}$ is a Cauchy sequence in $B(H)$. We define now $\Phi(f) := \lim_{n \to \infty} \Phi_n(f)$.

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Then $\Phi: C(\bar{R}) \to B(H)$ is a continuous homomorphism with $\Phi(1) = I$ as the mappings $\Phi_n: C(\bar{R}) \to B(H)$ are continuous homomorphisms with $\Phi_n(1) = I$ (because of (5) and (7)). Moreover, the operators $\Phi(f)$, $f \in C(\bar{R})$, are normal and commute with $T$ (as this is true for all $\Phi_n(f)$). We define now the normal operator $N \in B(H)$ as $N := \Phi(Z)$, where $Z: \bar{R} \to \mathbb{C}$ denotes the function with $Z(z) = z$ for all $z \in \bar{R}$. In order to complete the proof of the theorem we have to show that

$$Q := T - N \text{ is a quasinilpotent operator.} \tag{8}$$

*Proof.* Fix an arbitrary closed set $F \subset \mathbb{C}$. If $x \in H_T(F)$, then by (3) and (4), $P_{j,k}^{(n)} x \in H_T(\overline{R_{j,k}^{(n)}} \cap F)$ and therefore $P_{j,k}^{(n)} x = 0$ if $\overline{R_{j,k}^{(n)}} \cap F = \varnothing$ ($n \in \mathbb{N}$, $0 \le j, k \le 2^{n-1}$). This implies $x = \sum_F^{(n)} P_{j,k}^{(n)} x$, where $\sum_F^{(n)}$ $\left(\text{resp. } \bigcup_F^{(n)}\right)$ means that the sum (resp. union) has to be taken with respect to all $j, k \in \{0, 1, \ldots, 2^n - 1\}$ such that $\overline{R_{j,k}^{(n)}} \cap F \neq \varnothing$. We obtain

$$\begin{aligned} H_T(F) &\subset \bigcap_{n=1}^{\infty} \sum_F^{(n)} P_{j,k}^{(n)} H \subset \bigcap_{n=1}^{\infty} \sum_F^{(n)} H_T(\overline{R_{j,k}^{(n)}}) \\ &\subset \bigcap_{n=1}^{\infty} H_T\left(\bigcup_F^{(n)} \overline{R_{j,k}^{(n)}}\right) = H_T\left(\bigcap_{n=1}^{\infty} \bigcup_F^{(n)} \overline{R_{j,k}^{(n)}}\right) = H_T(F) \end{aligned} \tag{9}$$

Fix now an arbitrary $x \in H_N(F)$. If $n \in \mathbb{N}$, $0 \le j, k \le 2^n - 1$ with $\overline{R_{j,k}^{(n)}} \cap F = \varnothing$, then there exists a function $f \in C(\bar{R})$ with $\operatorname{supp}(f) \cap \overline{R_{j,k}^{(n)}} = \varnothing$ and $f \equiv 1$ in $U \cap \bar{R}$ for an open neighborhood U of $F$. Then, by [**2**, Proposition 3.1.17] and the construction of $\Phi$, $x = \Phi(f)x = (I - P_{j,k}^{(n)})\Phi(f)x$. Thus, $P_{j,k}^{(n)} x = 0$ and we obtain $x = \sum_F^{(n)} P_{j,k}^{(n)} x$. Moreover, we have by the construction of $\Phi$ and by the fact that $\Phi$ is a $C(\bar{R})$-functional calculus for $N$ that $P_{j,k}^{(n)} H \subset H_T(\overline{R_{j,k}^{(n)}})$ for all $n \in \mathbb{N}$ and $0 \le j, k \le 2^n - 1$. Therefore, by (9) and as in (9),

$$H_N(F) \subset \bigcap_{n=1}^{\infty} \sum_F^{(n)} P_{j,k}^{(n)} H = H_T(F) \subset \bigcap_{n=1}^{\infty} \sum_F^{(n)} H_n(\overline{R_{j,k}^{(n)}}) \subset H_N(F).$$

Hence, $H_N(F) = H_T(F)$ for all closed $F \subset \mathbb{C}$, so that $T$ and $N$ are quasinilpotent equivalent by [**2**, Theorem 2.2.2]. As $T$ and $N$ commute, this implies that $Q := T - N$ is a quasinilpotent operator commuting with $T$ and the normal operator $N$. This completes the proof of (8) and of the whole theorem.

It is a well known fact that every operator $S \in B(H)$ with $\dim \sigma(T) = 0$ is decomposable. This follows easily by means of the analytic functional calculus and [**4**, B on p. 54]. Hence, we obtain the following results.

Corollary. *If $S \in B(H)$ with* $\dim \sigma(S) = 0$, *then $S$ is a spectral operator if and only if $S$ is similar to an operator $T \in B(H)$ with property* (1).

Corollary (cf. [**5**, Theorems 3 and 5] and [**6**, Proposition 5.5.4]). *Let $S \in B(H)$ be an operator such that its essential spectrum (in the sense of* [**5**; **6**]) *is empty or consists of one point. Then $S$ is a spectral operator if and only if it is similar to an operator $T \in B(H)$ with property* (1).

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*Proof.* If the essential spectrum of $S$ consists of at most one point, then dim $\sigma(S)=0$ by [**6**, Proposition 5.1.1 and 5.5.1]. Therefore, we may apply the preceding corollary.

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