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A CHARACTERIZATION OF BANACH FUNCTION SPACES ASSOCIATED WITH MARTINGALES

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Abstract. Let $(\Omega,\ \Sigma,\ \mathbb{P})$ be a nonatomic probability space and let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ be a filtration. If $f = (f_n)_{n\in\mathbb{Z}_+}$ is a uniformly integrable $\mathcal{F}$-martingale, let $\mathcal{A}_{\mathcal{F}}f = (\mathcal{A}_{\mathcal{F}}f_n)_{n\in\mathbb{Z}_+}$ denote the martingale defined by $\mathcal{A}_{\mathcal{F}}f_n = \mathbb{E}[|f_\infty||\mathcal{F}_n]$ $(n\in\mathbb{Z}_+)$, where $f_\infty = \lim_n f_n$ a.s. Let X be a Banach function space over Ω. We give a necessary and sufficient condition for X to have the property that $S(f)\in X$ if and only if $S(\mathcal{A}_{\mathcal{F}}f)\in X$, where $S(f)$ stands for the square function of $f = (f_n)$.

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1. Introduction. Let $(\Omega,\ \Sigma,\ \mathbb{P})$ be a nonatomic probability space and let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ be a filtration; i.e., an increasing sequence of sub-σ-algebras of Σ. If $f = (f_n)_{n\in\mathbb{Z}_+}$ is a uniformly integrable $\mathcal{F}$-martingale, we let $\mathcal{A}f \equiv \mathcal{A}_{\mathcal{F}}f = (\mathcal{A}_{\mathcal{F}}f_n)_{n\in\mathbb{Z}_+}$ denote the $\mathcal{F}$-martingale defined by

$$\mathcal{A}f_n \equiv \mathcal{A}_{\mathcal{F}}f_n = \mathbb{E}[|f_\infty||\mathcal{F}_n] \qquad (n\in\mathbb{Z}_+),$$

where $f_\infty = \lim_{n\to\infty} f_n$ almost surely (a.s.) on Ω. If $f = (f_n)_{n\in\mathbb{Z}_+}$ is a martingale, we denote by $S(f)$ the square function of f. Let us recall Burkholder's inequality: if $1 < p < \infty$, then there are positive constants c_p and C_p such that

$$c_p\,\|f_\infty\|_p \le \|S(f)\|_p \le C_p\,\|f_\infty\|_p$$

for all uniformly integrable martingales $f = (f_n)$ (with the convention that $\|x\|_p = \infty$ unless $x\in L_p$). It then follows that $S(f)\in L_p$ if and only if $S(\mathcal{A}f)\in L_p$. There are similar results for other function spaces. For example, let L_Φ be the Orlicz space generated by an N-function Φ satisfying the Δ_2- and ∇_2-conditions. (See e.g. [**13**, p. 22].) Then $S(f)\in L_\Phi$ if and only if $S(\mathcal{A}f)\in L_\Phi$. This follows from the Burkholder-Davis-Gundy inequality and the Doob inequality in L_Φ ([**9**, p. 89, p. 96]).

Now let X be a Banach function space over Ω. (See Definition 1 below.) Our aim is to find a necessary and sufficient condition for X to have the property that $S(f)\in X$ if and only if $S(\mathcal{A}f)\in X$. (See Theorem 1.)

Such a problem concerning the maximal function $M(f) = \sup_n |f_n|$ of f has been studied. As in [**7**], we can prove that the following statements are equivalent.

(i) $M(f)\in X$ if and only if $M(\mathcal{A}f)\in X$.

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(ii) X is rearrangement-invariant and can be renormed with a rearrangement-invariant norm for which the upper Boyd index is less than 1.

2. Preliminaries. We shall deal with martingales on a (fixed) *nonatomic* probability space $(\Omega, \Sigma, \mathbb{P})$. Let I denote the interval $(0, 1]$ and let μ be Lebesgue measure on the σ-algebra $\mathfrak{M}$ of measurable subsets of I. In order to deal with the two probability spaces $(\Omega, \Sigma, \mathbb{P})$ and $(I, \mathfrak{M}, \mu)$ at the same time, we shall work with an arbitrary nonatomic probability space $(R, \mathfrak{R}, \lambda)$ throughout this section.

Let X and Y be Banach spaces of (equivalence classes of) random variables on R. We write $X \hookrightarrow Y$ to mean that X is continuously embedded in Y; i.e., $X \subset Y$ and $\|x\|_Y \le c\,\|x\|_X$, for all $x \in X$ and some positive constant c.

Definition 1. A real Banach space $(X, \|\cdot\|_X)$ of random variables on R is called a *Banach function space* if it has the following properties:
(B1) $L_\infty \hookrightarrow X \hookrightarrow L_1$;
(B2) $x \in X,\ |y| \le |x|$ a.s. $\Longrightarrow y \in X,\ \|y\|_X \le \|x\|_X$;
(B3) $x_n \in X,\ 0 \le x_n \uparrow x$ a.s., $\sup_n \|x_n\|_X < \infty$
$$\Longrightarrow x \in X,\ \|x\|_X = \sup_n \|x_n\|_X.$$

From (B2) it follows that $x \in X$ if and only if $|x| \in X$, and also that $\|x\|_X = \||x|\|_X$ for all $x \in X$.

Let x be a random variable on R. The *nonincreasing rearrangement* of x is the function $x^*(t)$ on $I = (0, 1]$ defined by

$$x^*(t) = \inf\{s > 0 \mid \lambda(|x| > s) \le t\} \qquad (t \in I).$$

Notice that x^* is a unique right-continuous nonincreasing function on I that has the same distribution (with respect to μ) as $|x|$.

Let x and y be random variables on R. The inequality

$$\int_R |xy|\, d\lambda \le \int_0^1 x^*(s) y^*(s)\, ds \tag{1}$$

is fundamental and called the *Hardy-Littlewood inequality*. (See, for example, [**2**, p. 44].) In particular, if $A \in \mathfrak{R}$, then

$$\int_A |x|\, d\lambda \le \int_0^{\lambda(A)} x^*(s)\, ds. \tag{2}$$

Again let x and y be random variables on R. We write $y \prec x$ to mean that

$$\int_0^t y^*(s)\, ds \le \int_0^t x^*(s)\, ds \quad \text{for all } t \in I.$$

Note that if $y \prec x$ and $x \prec y$, then $x^* = y^*$ on I: in this case, we write $x \simeq_d y$. Thus $x \simeq_d y$ if and only if x and y are identically distributed.

Definition 2. Let X be a Banach function space equipped with the norm $\|\cdot\|_X$. We say that X is *rearrangement-invariant* (r.i.) if

(R1) $$x \in X,\ x \simeq_d y \Longrightarrow y \in X.$$

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We say that X is *equipped with a rearrangement-invariant norm* (or an r.i. norm) if

(R2) $$x,\ y \in X,\ x \simeq_d y \Longrightarrow \|x\|_X = \|y\|_X.$$

Using (B2), (B3), and (R2), we can easily verify that if X is equipped with an r.i. norm, then the space X is r.i. The converse is false in general. However, if X is r.i., then there exists an r.i. norm $|||\cdot|||_X$ on X such that $\|\cdot\|_X \approx |||\cdot|||_X$ (i.e., these norms are equivalent). See [**10**, p. 138] for details.

Since the underlying probability space Ω is nonatomic, we can replace (R1) by

(R1′) $$x \in X,\ y \prec x \Longrightarrow y \in X,$$

and (R2) by

(R2′) $$x,\ y \in X,\ y \prec x \Longrightarrow \|y\|_X \le \|x\|_X.$$

For details, see [**10**, Section 11].

Now let us recall the Luxemburg representation theorem. If X is an r.i. space equipped with an r.i. norm $\|\cdot\|_X$, then there exists a unique r.i. space $(\widehat{X},\ \|\cdot\|_{\widehat{X}})$ over I equipped with an r.i. norm such that

(i) $x \in X \Longleftrightarrow x^* \in \widehat{X}$,

(ii) $\|x\|_X = \|x^*\|_{\widehat{X}}$ for all $x \in X$.

We call $\widehat{X}$ the *Luxemburg representation* of X. See [**2**, pp. 62–64].

Now we recall the definition of Boyd indices. For each positive number s, the *dilation operator* D_s, acting on the space of measurable functions on I, is defined as follows: if $t \in I$, then

$$(D_s\varphi)(t) = \begin{cases} \varphi(st) & \text{if } st \in I, \\ 0 & \text{otherwise.} \end{cases}$$

If Y is an r.i. space over I equipped with an r.i. norm, then each D_s is a bounded linear operator from Y into Y and $\|D_s\|_{B(Y)} \le 1 \vee s^{-1}$, where $\|D_s\|_{B(Y)}$ denotes the operator norm of D_s: $Y \to Y$. The *lower* and *upper Boyd indices* are defined by

$$\alpha_Y = \sup_{0<s<1} \frac{\log \|D_{s^{-1}}\|_{B(Y)}}{\log s} = \lim_{s\to 0+} \frac{\log \|D_{s^{-1}}\|_{B(Y)}}{\log s}$$

and

$$\beta_Y = \inf_{1<s<\infty} \frac{\log \|D_{s^{-1}}\|_{B(Y)}}{\log s} = \lim_{s\to\infty} \frac{\log \|D_{s^{-1}}\|_{B(Y)}}{\log s},$$

respectively. If X is an r.i. space over Ω equipped with an r.i. norm, then the Boyd indices of X are defined as $\alpha_X = \alpha_{\widehat{X}}$ and $\beta_X = \beta_{\widehat{X}}$. Moreover, if X is an arbitrary r.i. space over Ω, then the Boyd indices of X are defined to be those of $(X,\ |||\cdot|||_X)$, where $|||\cdot|||_X$ is an r.i. norm such that $\|\cdot\| \approx |||\cdot|||_X$.

For any r.i. space X, we have $0 \le \alpha_X \le \beta_X \le 1$. See [**3**] or [**2**, p. 149]. For example, $\alpha_{L_\infty} = \beta_{L_\infty} = 0$, and $\alpha_{L_p} = \beta_{L_p} = 1/p$ whenever $1 \le p < \infty$.

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Let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ be a filtration. If $f = (f_n)_{n\in\mathbb{Z}_+}$ is an $\mathcal{F}$-martingale, we let

$$\Delta_0 f = f_0, \quad \Delta_n f = f_n - f_{n-1} \quad (n = 1, 2, \ldots), \quad \text{and} \quad S(f) = \left\{ \sum_{n=0}^{\infty} (\Delta_n f)^2 \right\}^{1/2}.$$

Given a Banach function space X over Ω, we denote by $\mathcal{H}_{\mathcal{F}}(X)$ the vector space consisting of all $\mathcal{F}$-martingales $f = (f_n)$ such that $S(f) \in X$. Since $X \hookrightarrow L_1$, every martingale in $\mathcal{H}_{\mathcal{F}}(X)$ is uniformly integrable. If we set $\|f\|_{\mathcal{H}_{\mathcal{F}}(X)} = \|S(f)\|_X$ for $f \in \mathcal{H}_{\mathcal{F}}(X)$, then $\mathcal{H}_{\mathcal{F}}(X)$ forms a Banach space with this norm; see [**12**].

3. Main results. From now on we shall consider a fixed Banach function space $(X, \|\cdot\|_X)$ over Ω, and adopt the convention that $\|x\|_X = \infty$ unless $x \in X$. We denote by $\mathbb{F}$ the collection of all filtrations $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ such that $\Sigma = \sigma(\bigcup_{n=0}^{\infty} \mathcal{F}_n)$.

THEOREM 1. *The following are equivalent.*

(i) *For any* $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$,

$$f = (f_n) \in \mathcal{H}_{\mathcal{F}}(X) \Longleftrightarrow \mathcal{A}_{\mathcal{F}} f = (\mathcal{A}_{\mathcal{F}} f_n) \in \mathcal{H}_{\mathcal{F}}(X).$$

(ii) *There are positive constants c and C, depending only on X, such that*

$$c\,\|f_\infty\|_X \le \|S(f)\|_X \le C\,\|f_\infty\|_X, \tag{3}$$

for all uniformly integrable martingales f.

(iii) *X is rearrangement-invariant and can be renormed with a rearrangement-invariant norm for which* $0 < \alpha_X \le \beta_X < 1$.

It was shown by Antipa [**1**] that (iii) implies (ii). See also [**5**], [**6**] and [**11**]. Furthermore we see from our convention that (ii) implies (i). Indeed if (ii) holds, then

$$S(f) \in X \Longleftrightarrow f_\infty \in X \Longleftrightarrow |f_\infty| \in X \Longleftrightarrow S(\mathcal{A}_{\mathcal{F}} f) \in X.$$

Thus, to prove Theorem 1, it suffices to show that (i) implies (iii). To this end, we shall prove Propositions 1, 2, and 3 below. Incidentally, we can prove directly that (ii) implies (iii), as in [**8**].

PROPOSITION 1. *If X satisfies the condition that*

$$f \in \mathcal{H}_{\mathcal{F}}(X) \Longrightarrow \mathcal{A}_{\mathcal{F}} f \in \mathcal{H}_{\mathcal{F}}(X), \tag{4}$$

for any $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, *then X is rearrangement-invariant.*

PROPOSITION 2. *Suppose that X is rearrangement-invariant. If X satisfies* (4) *for any* $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, *then* $\beta_X < 1$.

PROPOSITION 3. *Suppose that X is rearrangement-invariant. If $\beta_X < 1$ and if X satisfies the condition that*

$$\mathcal{A}_{\mathcal{F}} f \in \mathcal{H}_{\mathcal{F}}(X) \Longrightarrow f \in \mathcal{H}_{\mathcal{F}}(X),$$

for any $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$, *then* $\alpha_X > 0$.

4. Proof of Proposition 1. We begin with a lemma.

LEMMA 1. *The following are equivalent.*

(i) *X is rearrangement-invariant.*

(ii) *Let x and y be nonnegative integer-valued random variables such that $x \simeq_d y$ and $x \wedge y = 0$ a.s. If $x \in X$, then $y \in X$.*

Proof. It suffices to show that (ii) implies (R1) of Definition 2. Suppose that $x \simeq_d y$ and $x \in X$. We must show that $y \in X$. If $x \in L_\infty$, then $y \in L_\infty \subset X$. Hence we deal with the case in which $x \notin L_\infty$. Choose an integer n so large that $\mathbb{P}(x \geq n) \leq 1/3$. If we set

$$x' = \sum_{j=n}^{\infty} j 1_{\{j \leq x < j+1\}} \quad \text{and} \quad y' = \sum_{j=n}^{\infty} j 1_{\{j \leq y < j+1\}},$$

then $x' \leq x \in X$ and $x' \simeq_d y'$. Since $\mathbb{P}(x' = 0,\ y' = 0) = \mathbb{P}(x < n,\ y < n) \geq 1/3$ and the set $\{x' = 0,\ y' = 0\}$ contains no atom, we can find a random variable z such that $z \simeq_d x'$ and $\{z > 0\} \subset \{x' = 0,\ y' = 0\}$. (See [**4**, p. 44].) From (ii) we see first that $z \in X$ and then that $y' \in X$. Since $y \leq n + 1 + y' \in X$, we conclude that $y \in X$, completing the proof. □

Proof of Proposition 1. It suffices to show that (ii) of Lemma 1 holds. Let $\{c_j\}_{j=1}^{\infty}$ be a sequence of integers such that $0 < c_1 < c_2 < \cdots$; let $\{A_j\}_{j=1}^{\infty}$ and $\{B_j\}_{j=1}^{\infty}$ be pairwise disjoint sequences of sets in Σ such that

$$\left(\bigcup_{j=1}^{\infty} A_j\right) \cap \left(\bigcup_{j=1}^{\infty} B_j\right) = \emptyset \quad \text{and} \quad \mathbb{P}(A_j) = \mathbb{P}(B_j) \ \text{ for all } j = 1, 2, \ldots$$

We must show that if $x := \sum_{j=1}^{\infty} c_j 1_{A_j} \in X$, then $y := \sum_{j=1}^{\infty} c_j 1_{B_j} \in X$. Setting $\Lambda_0 = \Omega$ and $\Lambda_n = \bigcup_{j=n}^{\infty} (A_j \cup B_j)$ for $n \geq 1$, we define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ by

$$\mathcal{F}_n = \sigma\{\Lambda \setminus \Lambda_n \mid \Lambda \in \Sigma\} \qquad (n \in \mathbb{Z}_+). \tag{5}$$

For each $j \in \mathbb{Z}_+$ we divide A_j into two parts with the same measure; that is, let A_{j1} and A_{j2} be measurable subsets of A_j such that

$$A_j = A_{j1} \cup A_{j2}, \quad A_{j1} \cap A_{j2} = \emptyset, \quad \text{and} \quad \mathbb{P}(A_{j1}) = \mathbb{P}(A_{j2}).$$

Let $x_k = \sum_{j=1}^{\infty} c_j 1_{A_{jk}}$ $(k = 1, 2)$, let $f_\infty = x_1 - x_2$, and let $f = (f_n)_{n \in \mathbb{Z}_+}$ be the martingale defined by

$$f_n = \mathbb{E}[f_\infty \mid \mathcal{F}_n] = f_\infty 1_{\Omega \setminus \Lambda_n} \qquad (n \in \mathbb{Z}_+). \tag{6}$$

Then, since $\Delta_0 f = f_0 \equiv 0$ and $\Delta_n f = f_\infty 1_{\Lambda_{n-1} \setminus \Lambda_n}$ $(n \geq 1)$, we see that $S(f) = |f_\infty| = x \in X$; that is, $f \in \mathcal{H}_{\mathcal{F}}(X)$. Hence $\mathcal{A}f = \mathcal{A}_{\mathcal{F}} f \in \mathcal{H}_{\mathcal{F}}(X)$ or equivalently $S(\mathcal{A}_{\mathcal{F}} f) \in X$, by hypothesis. Observe that

$$\mathcal{A}f_n = \mathbb{E}[x \mid \mathcal{F}_n] = \frac{1_{\Lambda_n}}{\mathbb{P}(\Lambda_n)} \int_{\Lambda_n} x \, d\mathbb{P} + x 1_{\Omega \setminus \Lambda_n} \qquad (n \in \mathbb{Z}_+).$$

Then we have

$$\begin{aligned} \Delta_{n+1} \mathcal{A}f = {} & \left\{ \frac{1}{\mathbb{P}(\Lambda_{n+1})} \int_{\Lambda_{n+1}} x \, d\mathbb{P} - \frac{1}{\mathbb{P}(\Lambda_n)} \int_{\Lambda_n} x \, d\mathbb{P} \right\} 1_{\Lambda_{n+1}} \\ & + \left\{ x - \frac{1}{\mathbb{P}(\Lambda_n)} \int_{\Lambda_n} x \, d\mathbb{P} \right\} 1_{\Lambda_n \setminus \Lambda_{n+1}} \qquad (n \in \mathbb{Z}_+). \end{aligned}$$

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Since $B_n \subset \Lambda_n \setminus \Lambda_{n+1}$ and $x = 0$ on B_n, we can deduce that

$$|\Delta_{n+1}\mathcal{A}f|\,1_{B_n} = \frac{1_{B_n}}{\mathbb{P}(\Lambda_n)}\left|\int_{\Lambda_n} x\,d\mathbb{P}\right| = \frac{1_{B_n}}{\mathbb{P}(\Lambda_n)}\sum_{j=n}^{\infty} c_j\,\mathbb{P}(A_j)$$
$$\geq \frac{c_n 1_{B_n}}{\mathbb{P}(\Lambda_n)}\sum_{j=n}^{\infty}\mathbb{P}(A_j) = \frac{c_n}{2}\,1_{B_n} \qquad (n = 1, 2, \ldots).$$

Consequently,

$$y = \sum_{n=1}^{\infty} c_n 1_{B_n} \leq 2\sum_{n=1}^{\infty}|\Delta_{n+1}\mathcal{A}f|\,1_{B_n} = 2\left\{\sum_{n=1}^{\infty}(\Delta_{n+1}\mathcal{A}f)^2 1_{B_n}\right\}^{1/2} \leq 2S(\mathcal{A}f).$$

Since $S(\mathcal{A}_{\mathcal{F}}f) \in X$, we conclude that $y \in X$ as desired. □

5. Proofs of Propositions 2 and 3. Let $\mathcal{P}$ and $\mathcal{Q}$ be the linear operators on $L_1(I)$ defined respectively by

$$(\mathcal{P}\varphi)(t) = \frac{1}{t}\int_0^t \varphi(s)\,ds \quad \text{and} \quad (\mathcal{Q}\varphi)(t) = \int_t^1 \frac{\varphi(s)}{s}\,ds \qquad (t \in I).$$

It is easy to verify that

$$\mathcal{P}\mathcal{Q}\varphi = \mathcal{P}\varphi + \mathcal{Q}\varphi \tag{7a}$$

and

$$\mathcal{Q}\mathcal{P}\varphi = \mathcal{P}\varphi + \mathcal{Q}\varphi - \int_0^1 \varphi(s)\,ds, \tag{7b}$$

for all $\varphi \in L_1(I)$. Let us recall Shimogaki's Theorem. In terms of Boyd indices, it can be expressed as follows.

Shimogaki's Theorem ([**14**]; cf. [**3**]). *Let Y be a rearrangement-invariant space over I. Then*

(i) $\beta_Y < 1$ *if and only if $\mathcal{P}$ is a bounded linear operator from Y into Y;*
(ii) $\alpha_Y > 0$ *if and only if $\mathcal{Q}$ is a bounded linear operator from Y into Y.*

The next lemma is a variant of Shimogaki's result. Before stating it, we introduce some notation.

Notation. Let Y be an r.i. space over I. We denote by $\mathfrak{D}_Y$ the collection of all nonnegative, nonincreasing, and right-continuous functions $\varphi \in Y$ such that $\mu(\varphi \neq 0) \leq 1/2$.

Lemma 2. *Let Y be a rearrangement-invariant space over I. Then*

(i) $\beta_Y < 1$ *if and only if* $\mathcal{P}(\mathfrak{D}_Y) \subset Y$,
(ii) $\alpha_Y > 0$ *if and only if* $\mathcal{Q}(\mathfrak{D}_Y) \subset Y$.

Proof. (i) If $\mathcal{P}(Y) \subset Y$, then the graph $\{(\varphi, \mathcal{P}\varphi) \mid \varphi \in Y\}$ is closed in $Y \times Y$, since $Y \hookrightarrow L_1$. Hence $\mathcal{P}$ is a bounded linear operator if and only if $\mathcal{P}(Y) \subset Y$. Therefore, in view of Shimogaki's Theorem, it suffices to show that if $\mathcal{P}(\mathfrak{D}_Y) \subset Y$, then $\mathcal{P}(Y) \subset Y$.

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Suppose that $\mathcal{P}(\mathfrak{D}_Y) \subset Y$. Given $\psi \in Y$, we choose $\lambda > 0$ so large that $\mu(|\psi| > \lambda) \leq 1/2$, and let $\varphi = \psi^* 1_{\{\psi^* > \lambda\}}$. Then $\varphi \in \mathfrak{D}_Y$ and therefore $\mathcal{P}\varphi \in Y$. On the other hand, by the Hardy-Littlewood inequality (2), we have that

$$\begin{aligned} |(\mathcal{P}\psi)(t)| &\leq \frac{1}{t}\int_0^t |\psi(s)|\, ds \leq \frac{1}{t}\int_0^t \psi^*(s)\, ds \\ &\leq \frac{1}{t}\int_0^t \{\varphi(s) + \lambda\}\, ds = (\mathcal{P}\varphi)(t) + \lambda \qquad (t \in I). \end{aligned}$$

Since $\mathcal{P}\varphi + \lambda \in Y$, we conclude that $\mathcal{P}\psi \in Y$, as desired.

(ii) As in the proof of (i), we see that $\mathcal{Q}$ is a bounded linear operator from Y into Y if and only if $\mathcal{Q}(Y) \subset Y$. Hence it suffices to show that if $\mathcal{Q}(\mathfrak{D}_Y) \subset Y$, then $\mathcal{Q}(Y) \subset Y$.

Suppose that $\mathcal{Q}(\mathfrak{D}_Y) \subset Y$. Given $\psi \in Y$, we let $\varphi_1 = \psi^* 1_{(0,1/2)}$ and $\varphi_2 = \psi^* 1_{[1/2,1]}$. Then $\varphi_1 \in \mathfrak{D}_Y$ and hence $\mathcal{Q}\varphi_1 \in Y$. As for φ_2, it is easy to see that $\mathcal{Q}\varphi_2 \leq 2\,\|\psi\|_1$ on I. Therefore $\mathcal{Q}\varphi_2 \in L_\infty(I) \subset Y$. Thus $\mathcal{Q}\psi^* = \mathcal{Q}\varphi_1 + \mathcal{Q}\varphi_2 \in Y$. On the other hand, by the Hardy-Littlewood inequality (1), we have that

$$\int_0^t (\mathcal{Q}|\psi|)(s)\, ds = \int_0^1 \frac{t \wedge s}{s}\, |\psi(s)|\, ds \leq \int_0^1 \frac{t \wedge s}{s}\, \psi^*(s)\, ds = \int_0^t (\mathcal{Q}\psi^*)(s)\, ds,$$

for all $t \in I$. This can be written as $\mathcal{Q}|\psi| \prec \mathcal{Q}\psi^*$. Since $\mathcal{Q}\psi^* \in Y$, we conclude from (R1′) that $|\mathcal{Q}\psi| \leq \mathcal{Q}|\psi| \in Y$. This completes the proof. □

In order to prove Propositions 2 and 3, we need one more lemma.

LEMMA 3. *If x is a nonnegative integrable random variable on Ω, then there exists a family $\{A(t) \mid t \in I\}$ of sets in Σ satisfying the following conditions:*

(i) $A(s) \subset A(t)$ *whenever* $0 < s < t \leq 1$;
(ii) $\mathbb{P}(A(t)) = t$ *for all* $t \in I$;
(iii) $\{x > x^*(t)\} \subset A(t) \subset \{x \geq x^*(t)\}$;
(iv) $\int_{A(t)} x\, d\mathbb{P} = \int_0^t x^*(s)\, ds$ *for all* $t \in I$.

See [**2**, p. 46] for a proof.

Proof of Proposition 2. We may assume that X is equipped with an r.i. norm. In view of Lemma 2, we show that $\mathcal{P}\varphi \in \widehat{X}$ whenever $\varphi \in \mathfrak{D}_{\widehat{X}}$, where $\widehat{X}$ is the Luxemburg representation of X. If $\varphi \in L_\infty(I)$, then $\mathcal{P}\varphi \in L_\infty(I) \subset \widehat{X}$. Hence we may assume $\varphi \notin L_\infty(I)$. Because Ω is nonatomic and $\mu(\varphi \neq 0) \leq 1/2$, there are nonnegative random variables x and y such that $x \wedge y = 0$ a.s. and $x^* = y^* = \varphi$ on I. (See [**4**, p. 44].) Then $x, y \in X$, since $x^* = y^* \in \widehat{X}$. By Lemma 3, there are increasing families $\{A(t) \mid 0 < t \leq 1/2\}$ and $\{B(t) \mid 0 < t \leq 1/2\}$ of sets in Σ such that

$$\mathbb{P}(A(t)) = \mathbb{P}(B(t)) = t \qquad (0 < t \leq 1/2), \tag{8a}$$

$$\{x > x^*(t)\} \subset A(t) \subset \{x \geq x^*(t)\} \quad (0 < t \leq 1/2), \tag{8b}$$

$$\{y > x^*(t)\} \subset B(t) \subset \{y \geq x^*(t)\} \quad (0 < t \leq 1/2), \tag{8c}$$

and

$$\int_{A(t)} x\, d\mathbb{P} = \int_{B(t)} y\, d\mathbb{P} = \int_0^t x^*(s)\, ds \qquad (0 < t \leq 1/2). \tag{8d}$$

We define a sequence of numbers in the interval $(0,\ 1/2]$ by setting

$$t_0 = \mu(\varphi \neq 0) = \sup\{t \in I \mid x^*(t) > 0\},$$
$$t_n = \sup\{t \in I \mid (\mathcal{P}x^*)(t) > 2(\mathcal{P}x^*)(t_{n-1})\} \qquad (n = 1, 2, \ldots).$$

Then, since $\mathcal{P}x^*$ is continuous and $(\mathcal{P}x^*)(t) \to \infty$ as $t \to 0+$,

$$(\mathcal{P}x^*)(t_n) = 2(\mathcal{P}x^*)(t_{n-1}) \qquad (n = 1, 2, \ldots). \tag{9}$$

This implies that $t_n \downarrow 0$. Note that $A(t_0) \cap B(t_0) = \{x > 0\} \cap \{y > 0\} = \emptyset$ a.s. Setting $\Lambda_n = A(t_n) \cup B(t_n)$ for each $n \in \mathbb{Z}_+$, we define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ again by (5). Let $f_\infty = x - y$ and let $f = (f_n)$ be the martingale defined by (6). Then, since $\Delta_n f = f_\infty 1_{\Lambda_{n-1} \setminus \Lambda_n}$ $(n = 1, 2, \ldots)$, we see that $S(f) = |f_\infty| = x + y \in X$. Therefore $S(\mathcal{A}f) \in X$ by hypothesis. On the other hand, by (8d) we have that

$$\begin{aligned}\mathcal{A}f_n &= \frac{1_{\Lambda_n}}{\mathbb{P}(\Lambda_n)} \int_{\Lambda_n} (x + y)\, d\mathbb{P} + |f_\infty| 1_{\Omega \setminus \Lambda_n} \\ &= (\mathcal{P}x^*)(t_n) 1_{\Lambda_n} + |f_\infty| 1_{\Omega \setminus \Lambda_n} \qquad (n \in \mathbb{Z}_+).\end{aligned}$$

Hence by (9),

$$\Delta_n \mathcal{A}f = (\mathcal{P}x^*)(t_{n-1}) 1_{\Lambda_n} + \{|f_\infty| - (\mathcal{P}x^*)(t_{n-1})\} 1_{\Lambda_{n-1} \setminus \Lambda_n} \qquad (n \in \mathbb{Z}_+).$$

As a result,

$$(\mathcal{P}x^*)(t_{n+1}) 1_{\Lambda_n} = 4(\mathcal{P}x^*)(t_{n-1}) 1_{\Lambda_n} \leq 4\,|\Delta_n \mathcal{A}f| \qquad (n = 1, 2, \ldots). \tag{10}$$

We also have $(\mathcal{P}x^*)(t_1) 1_{\Lambda_0} = 2(\mathcal{P}x^*)(t_0) 1_{\Lambda_0} = 2\mathcal{A}f_0$. Thus (10) remains valid for $n = 0$. Since $(\mathcal{P}x^*)(2t_{n+1}) \leq (\mathcal{P}x^*)(t_{n+1})$, it follows from (10) that

$$\sum_{n=0}^{\infty} (\mathcal{P}x^*)(2t_{n+1}) 1_{\Lambda_n \setminus \Lambda_{n+1}} \leq \left\{ \sum_{n=0}^{\infty} (\mathcal{P}x^*)(t_{n+1})^2\, 1_{\Lambda_n} \right\}^{1/2} \leq 4S(\mathcal{A}f) \in X. \tag{11}$$

Observe that

$$\left(\sum_{n=0}^{\infty} (\mathcal{P}x^*)(2t_{n+1}) 1_{\Lambda_n \setminus \Lambda_{n+1}} \right)^* (t) = \sum_{n=0}^{\infty} (\mathcal{P}x^*)(2t_{n+1})\, 1_{[2t_{n+1},\, 2t_n)}(t),$$

for all $t \in I$. This, together with (11), implies that

$$\begin{aligned}(\mathcal{P}\varphi)(t) &= (\mathcal{P}x^*)(t) \leq (\mathcal{P}x^*)(t \wedge (2t_0)) \\ &\leq \sum_{n=0}^{\infty} (\mathcal{P}x^*)(2t_{n+1})\, 1_{[2t_{n+1},\, 2t_n)}(t) + (\mathcal{P}x^*)(2t_0) \\ &\leq 4(S(\mathcal{A}f))^*(t) + \frac{1}{2t_0} \int_0^1 \varphi(s)\, ds,\end{aligned}$$

for all $t \in I$. Since the function on the right-hand side belongs to $\widehat{X}$, so is $\mathcal{P}\varphi$. This completes the proof. $\square$

The proof of Proposition 3 is similar to the proof of Proposition 2.

Proof of Proposition 3. By Lemma 2, it suffices to show that $\mathcal{Q}\varphi \in \widehat{X}$ whenever $\varphi \in \mathfrak{D}_{\widehat{X}}$. To this end, we may assume that $\varphi \not\equiv 0$. Since Ω is nonatomic and $\{\mathcal{Q}\varphi \neq 0\} \subset (0,\ 1/2)$, we can find nonnegative random variables x and y such that $x^* = y^* = \mathcal{Q}\varphi$ and $x \wedge y = 0$ a.s. Let $\{A(t) \mid 0 < t \leq 1/2\}$ and $\{B(t) \mid 0 < t \leq 1/2\}$ be increasing families of sets in Σ satisfying (8a)–(8d). Now we define a sequence in $(0,\ 1/2]$ by setting

$$t_0 = \mu(\mathcal{Q}\varphi \neq 0) = \sup\{t \in I \mid x^*(t) > 0\};$$
$$t_n = \sup\{t \in I \mid (\mathcal{P}x^*)(t) > (\mathcal{P}x^*)(t_{n-1}) + 1/n\} \qquad (n = 1, 2, \ldots).$$

Then, since $(\mathcal{P}x^*)(t) \geq x^*(t) \to \infty$ as $t \to 0+$ and $\mathcal{P}x^*$ is continuous,

$$(\mathcal{P}x^*)(t_n) = (\mathcal{P}x^*)(t_{n-1}) + \frac{1}{n} \qquad (n = 1, 2, \ldots).$$

Hence $t_n \downarrow 0$. We also have $A(t_0) \cap B(t_0) = \emptyset$ a.s. As before, let $\Lambda_n = A(t_n) \cup B(t_n)$ for $n \in \mathbb{Z}_+$ and define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ by (5). Let $f_\infty = x - y$ and let $f = (f_n)$ be the martingale defined by (6). Then $S(f) = |f_\infty| = x + y \geq x$ and therefore $\mathcal{Q}\varphi = x^* \leq (S(f))^*$ on I. Thus the proof will be complete if we can show that $(S(f))^* \in \widehat{X}$.

As observed before, $\mathcal{A}f_n = (\mathcal{P}x^*)(t_n)1_{\Lambda_n} + |f_\infty|1_{\Omega\setminus\Lambda_n}$, and therefore

$$\Delta_n \mathcal{A}f = \frac{1_{\Lambda_n}}{n} + \{|f_\infty| - (\mathcal{P}x^*)(t_{n-1})\}1_{\Lambda_{n-1}\setminus\Lambda_n} \qquad (n = 1, 2, \ldots). \tag{12}$$

Since $x^*(t_{n-1}) \leq x \leq x^*(t_n)$ on the set $A(t_{n-1}) \setminus A(t_n)$ by (8b), we find that

$$\begin{aligned} -\frac{1}{n} &\leq (\mathcal{P}x^*)(t_n) - x^*(t_n) - \frac{1}{n} \\ &= (\mathcal{P}x^*)(t_{n-1}) - x^*(t_n) \\ &\leq (\mathcal{P}x^*)(t_{n-1}) - x \\ &\leq (\mathcal{P}x^*)(t_{n-1}) - x^*(t_{n-1}) \qquad \text{on } A(t_{n-1}) \setminus A(t_n). \end{aligned}$$

As a result,

$$|x - (\mathcal{P}x^*)(t_{n-1})| \leq (\mathcal{P}x^*)(t_{n-1}) - x^*(t_{n-1}) + \frac{1}{n} \qquad \text{on } A(t_{n-1}) \setminus A(t_n).$$

In the same way, we see that

$$|y - (\mathcal{P}x^*)(t_{n-1})| \leq (\mathcal{P}x^*)(t_{n-1}) - x^*(t_{n-1}) + \frac{1}{n} \qquad \text{on } B(t_{n-1}) \setminus B(t_n).$$

Since $\mathcal{P}x^* - x^* = \mathcal{P}\mathcal{Q}\varphi - \mathcal{Q}\varphi = \mathcal{P}\varphi$ by (7a), it follows that

$$||f_\infty| - (\mathcal{P}x^*)(t_{n-1})| \leq (\mathcal{P}\varphi)(t_{n-1}) + \frac{1}{n} \qquad \text{on } \Lambda_{n-1} \setminus \Lambda_n.$$

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Combining this with (12) gives

$$|\Delta_n \mathcal{A} f| \leq \frac{1}{n} + (\mathcal{P}\varphi)(t_{n-1}) 1_{\Lambda_{n-1}\setminus\Lambda_n} \qquad (n = 1, 2, \ldots).$$

Moreover

$$|\Delta_0 \mathcal{A} f| = |\mathcal{A} f_0| \equiv (\mathcal{P}x^*)(t_0) = \frac{1}{t_0}\|x^*\|_1 = \frac{1}{t_0}\,\|\mathcal{Q}\varphi\|_1 = \frac{1}{t_0}\,\|\varphi\|_1.$$

Therefore

$$S(\mathcal{A} f) \leq \left\{ \left(\frac{1}{t_0}\,\|\varphi\|_1\right)^2 + \sum_{n=1}^{\infty} \left(\frac{1}{n} + (\mathcal{P}\varphi)(t_{n-1}) 1_{\Lambda_{n-1}\setminus\Lambda_n}\right)^2 \right\}^{1/2}$$

$$\leq \frac{1}{t_0}\,\|\varphi\|_1 + \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2} + \sum_{n=1}^{\infty} (\mathcal{P}\varphi)(t_{n-1}) 1_{\Lambda_{n-1}\setminus\Lambda_n}.$$

Because

$$\left(\sum_{n=1}^{\infty} (\mathcal{P}\varphi)(t_{n-1}) 1_{\Lambda_{n-1}\setminus\Lambda_n}\right)^*(t) = \sum_{n=1}^{\infty} (\mathcal{P}\varphi)(t_{n-1}) 1_{[2t_n,\, 2t_{n-1})}(t) \leq (\mathcal{P}\varphi)(t/2) = (D_{1/2}\mathcal{P}\varphi)(t)$$

for all $t \in I$, we obtain

$$(S(\mathcal{A} f))^*(t) \leq \frac{1}{t_0}\,\|\varphi\|_1 + \frac{\pi}{\sqrt{6}} + (D_{1/2}\mathcal{P}\varphi)(t) \qquad (t \in I).$$

Since $\varphi \in \widehat{X}$ and $\beta_{\widehat{X}} = \beta_X < 1$, Shimogaki's Theorem yields that $\mathcal{P}\varphi \in \widehat{X}$ and hence $D_{1/2}\mathcal{P}\varphi \in \widehat{X}$. Consequently, $(S(\mathcal{A} f))^* \in \widehat{X}$, or equivalently $S(\mathcal{A} f) \in X$. The hypothesis implies that $S(f) \in X$ and hence that $(S(f))^* \in \widehat{X}$. This completes the proof. □

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