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# A NOTE ON A COMPARISON RESULT FOR ELLIPTIC EQUATIONS

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In a recent paper [**2**], Bushard established and applied a comparison theorem for positive solutions to the equation:

$$\sum_{i=1}^{n} D_i[p_i(x, u)D_iu] + q(x, u)u = 0,$$

in an arbitrary bounded domain $D$ of Euclidean $n$-space $R^n$. The proof of these results depended on the absence of mixed derivatives of $u$ in the equation considered. The purpose of this note to extend some of the results of [**2**] to a more general second order equation. Our theorems will also not require any regularity of the boundary of $D$ and, furthermore, will permit relaxation of some of the strict inequalities found in the results of [**2**]. This is achieved by assuming that the coefficients of our equation are somewhat more regular than was the case in [**2**].

Let $x = (x_1, \ldots, x_n)$ denote the points of $R^n$ and let $D_i$ denote differentiation with respect to $x_i$ for $i = 1, \ldots, n$. We consider the operator $L$ formally defined by:

$$Lu = -\sum_{i,j=1}^{n} D_i[a_{ij}(x, u)D_ju] + 2\sum_{j=1}^{n} b_j(x, u)D_ju + c(x, u)u.$$

The coefficients $a_{ij}(x, u)$ are assumed to be of class $C^1[\bar{D} \times [0, \infty)]$ and the coefficients $b_j(x, u)$, $c(x, u)$ are assumed in $C^0[\bar{D} \times [0, \infty)]$ for $i, j = 1, \ldots, n$. The matrix $(a_{ij}(x, u))$ is assumed uniformly positive definite symmetric in $D \times [0, \infty)$. The domain of $L$ is defined to be the set $C^2(D) \wedge C^0(\bar{D})$ but we shall also require that any function $v$, in $C^2(D) \wedge C^0(\bar{D})$ and positive in $D$, encountered in the sequel satisfy:

$$\lim_{\eta \to 0^+} \left[ \int_D \frac{[L(v) - L(v + \eta)]_+}{v + \eta} \right] = 0.$$

This condition can be satisfied by making simple requirements on the coefficients of $L$ and on the derivatives of $v$. It is, for example, sufficient to further assume that the first derivatives of $a_{ij}(x, u)$ and the coefficients $b_j(x, u)$, $c(x, u)$ satisfy a uniform Lipschitz condition with respect to $u$ on the compacta of $\bar{D} \times [0, \infty)$ and that $D_iv$, $D_{ij}v$ be in $L^2(D)$, for $i, j = 1, \ldots, n$. More general conditions can also be given.

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We do not postulate any conditions on the sign of $c(x, u)$ or on the sign of $\partial/\partial u\,[c(x, u)u]$. This is a departure from the assumptions usually encountered in the literature. Instead, we assume that the $(n + 1) \times (n + 1)$ matrix:

$$\begin{bmatrix} (a_{ij}(x, u)) & (b_j(x, u))^T \\ (b_j(x, u)) & c(x, u) + H(x, u) \end{bmatrix}$$

is nondecreasing in $u$ (as a form) for $(x, u)$ in $D \times [0, \infty)$, where $H(x, u) = \sum_{i,j=1}^n a^{ij}(x, u)b_i(x, u)b_j(x, u)$ and $(a^{ij}) = (a_{ij})^{-1}$. It is also useful to introduce the following expressions: let $u, v$ be of class $C^2(D)$, $v > 0$ in $D$, $\epsilon > 0$. We define:

$$P(u, v) = \sum_{i,j} [a_{ij}(x, v) - a_{ij}(x, u)]D_iuD_ju + 2 \sum_j \{b_j(x, v) - b_j(x, u)\} \cdot uD_ju + [H(x, v) - H(x, u) + c(x, v) - c(x, u)]u^2 + \sum_{i,j} D_i\left[\frac{u}{v}\{a_{ij}(x, u)D_juv - a_{ij}(x, v)uD_jv\}\right],$$

$$Q(u, v) = v^2\left\{\sum_{i,j} a_{ij}(x, v)D_i\left(\frac{u}{v}\right)D_j\left(\frac{u}{v}\right) + 2 \sum_j b_j(x, v)\left(\frac{u}{v}\right)D_j\left(\frac{u}{v}\right) + H(x, v)\left(\frac{u}{v}\right)^2\right\},$$

$$K(u, v, \epsilon) = \sum_{i,j} D_i\left\{(v + \epsilon)[-a_{ij}(x, v + \epsilon) + a_{ij}(x, v)]D_ju + \frac{\epsilon(v + \epsilon)}{v} a_{ij}(x, v)D_jv\right\}.$$

We observe that the function $H(x, v)$ is chosen so that the form $Q[u, v]$ is nonnegative [**4**]. We also note that for any function $\phi$ and set $E$ under consideration in the sequel we shall denote by $J_\epsilon(\phi)$ the standard mollified $C^\infty$ function of $\phi$, and by $\chi_E$ the characteristic function of $E$.

LEMMA 1. *Let $u$, $v$ be in $C^\infty(R^n)$, with $v$ positive, and let the closure of the set $\{x|u(x) > v(x)\}$ be contained in $D$. It then follows that:*

$$\int_{\{x|u(x)>v(x)\}} \left\{Q[u, v] - ul(u) + \frac{u^2}{v} L(v)\right\} \leqq 0,$$

*where*

$$l(u) = -\sum_{i,j} D_i[a_{ij}(x, u)D_ju] + 2 \sum_j b_j(x, u)D_ju + \{H(x, u) + c(x, u)\}u.$$

*Proof.* Let $G = \{x|x \text{ in } D,\ u(x) > v(x)\}$ and assume that $G$ is not empty. Then $G$ is a bounded open set in $R^n$ with $u = v$ on $\partial G$. By Sard's Theorem [**3**], there exists a sequence $\{\epsilon(n)\}$ of positive numbers such that: (i) $\epsilon(n) \to 0$

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as $n \to \infty$ ; (ii) grad $(u - v) \neq 0$ on the surfaces $u - v = \epsilon(n)$. We set $G(n) = \{x|u - v > \epsilon(n)\}$ and observe that $G(n)$ has a $C^1$ boundary and that, consequently, the Divergence Theorem may be applied to $G(n)$. We next note that for $x$ in $G_n$ the following identity [**4**], which is an extension of the well-known Picone identity, is valid:

$$(1) \quad P(u, v) = Q(u, v) - ul(u) + \frac{u^2}{v} Lv.$$

We next add $K(u, v, \epsilon(n))$ to both sides of (1) and integrate the resulting left side over $G(n)$. By the Divergence Theorem it follows that:

$$(2) \quad \int_{G(n)} P(u, v) + K(u, v, \epsilon(n)) \leqq \int_{G(n)} \Bigg\{ \sum_{i,j} D_i\left[\frac{u}{v}\{a_{ij}(x, u)D_j uv - a_{ij}(x, v)uD_j v\}\right] + \sum_{i,j} D_i\left[(v + \epsilon)(-a_{ij}(x, v + \epsilon) + a_{ij}(x, v))D_j u + \frac{\epsilon(v + \epsilon)}{v} a_{ij}(x, v)D_j v\right]\Bigg\}$$

$$= \int_{\partial G(n)} (v + \epsilon) \sum_{i,j} a_{ij}(x, v) \frac{[D_i(u - v)][D_j(v - u)]}{|\text{grad } (v - u)|} \leqq 0.$$

Integrating the right hand side of (1) and using (2) we obtain:

$$\int_{G(n)} \left\{ Q(u, v) - ulu + \frac{u^2}{v} Lv \right\} + \int_{G(n)} K(u, v, \epsilon(n)) \leqq 0.$$

In view of the hypothesis on the coefficients we observe that:

$$\lim_{n \to 0} \left[ \int_{G(n)} K(u, v, \epsilon(n)) \right] = 0.$$

Consequently,

$$\int_G \left\{ Q(u, v) - ulu + \frac{u^2}{v} Lv \right\} \leqq 0.$$

THEOREM. *Let $u, v$ be in $C^2(D) \wedge C^0(\bar{D})$. Assume that:*
(i) $v \geqq u \geqq 0$ *on* $\partial D$;
(ii) $v > 0$, $u > 0$ *in* $D$;
(ii) $Lv \geqq 0$, $lu \leqq 0$ *in* $D$.
*Then either $v \geqq u$ in $D$ or, in each component of $\{x|u(x) > v(x)\}$, we have $v = e^w u$ for some $C^1$ function $w$ such that* grad $w = (\sum_{j=1}^n a^{ij}(x, v)b_j(x, v))$.

*Proof*. Let $G = \{x|x \text{ in } D, u(x) > v(x)\}$ and assume that $G$ is not empty. Let $\eta > 0$ be chosen sufficiently small and set $M = \sup (|\text{grad } (u - v)|)$ on $\{x|u(x) > v(x) + \eta/2\}$. Next, extend $u, v$ outside $D$ by setting them equal to zero, and apply Lemma 1 to the functions:

$$W = J_\epsilon[u], \quad V = J_\epsilon[v + \eta + M\epsilon].$$

We find for $\epsilon > 0$ and sufficiently small:

$$\int_{\{x|W>V\}} Q[W, V] - Wl(W) + \frac{W^2}{V} L(V) \leqq 0.$$

Since

$$\lim_{\epsilon \to 0} \chi_{\{x|W>V\}}(x) = \chi_{\{x|u>v+\eta\}}(x)$$

pointwise, and in view of the properties of mollified functions, we let $\epsilon$ approach zero and obtain:

$$\int_{\{x|u>v+\eta\}} \left\{ Q[u, v + \eta] - ulu + \frac{u^2}{v + \eta} L(v + \eta) \right\} \leqq 0,$$

and, therefore,

$$\int_{\{x|u>v+\eta\}} Q[u, v + \eta] \leqq \int_{\{x|u>v+\eta\}} \frac{u^2}{v + \eta} [L(v) - L(v + \eta)]_+ \leqq c_0 \int_D \frac{[L(v) - L(v + \eta)]_+}{v + \eta}$$

for some constant $c_0$. If we choose any compact subset $R$ of $G$, it follows that for $\eta$ sufficiently small the following inequality holds:

$$\int_R Q[u, v + \eta] \leqq \int_{\{x|u>v+\eta\}} Q[u, v + \eta] \leqq c_0 \int_D \frac{[L(v) - L(v + \eta)]_+}{v + \eta}.$$

As $\eta$ approaches zero, we find:

$$\int_R Q[u, v] = 0$$

and, consequently, we conclude that $(D_1(u/v), \ldots, D_n(u/v), u/v)$ must lie in the kernel of the matrix:

$$\begin{bmatrix} (a_{ij}(x, v)) & (b_j(x, v))^T \\ (b_j(x, v)) & H(x, v) \end{bmatrix}.$$

Following the procedure of [**1**], we can now conclude that $v = ue^w$ in each component of $G$, for some function $w$ such that grad $(w) = (\sum_{j=1}^n a^{ij}(x, v) b_j(x, v))$.

COROLLARY 1. *Let $b_j \equiv 0$. Then $l = L$ and if the set $G = \{x|u(x) > v(x)\}$ is not empty, then $u = v = 0$ on $\partial D$. Furthermore, in this case $G = D$ and $u$ and $v$ are linearly dependent.*

*Proof.* For this special case, we find that: $w = H = 0$. It follows from the theorem that in any component of $G$, $u$ and $v$ must be linearly dependent. Since

$u = v$ on $\partial G$ we must have $u = v$ in $G$ unless $u = v = 0$ on the boundary of some component $G_1$ of $G$. Since $u, v$ are positive in $D$ it follows that $G_1 = D$, and the corollary is proven.

We can now state our uniqueness result which extends the results of [**2**] in the allowed boundary data as well as in the type of equation:

COROLLARY 2. *Let* $b_j \equiv 0$. *Then the problem:*

$$(3)\quad \begin{cases} Lu = 0 & \text{in } D, \\ u = \phi \geqq 0 & \text{on } \partial D, \end{cases}$$

*has at most one linearly independent positive solution. If* $\phi \not\equiv 0$, *then problem* (3) *has at most one positive solution.*

Corollary 2 cannot be improved upon, in the sense that it is possible to construct problems such as (3) with $\phi \equiv 0$ which have infinitely many linearly dependent positive solutions. It is not difficult, however, to give other conditions, besides $\phi \not\equiv 0$, to guarantee uniqueness of the positive solution, and as an example we state:

COROLLARY 3. *Assume that the above structure holds and further assume that either the matrix* $(a_{ij}(x, \xi))$ *or the function* $c(x, \xi)$ *is strictly increasing in* $\xi$. *Then problem* (3) *has at most one positive solution.*

We conclude by considering the following example to illustrate the above results. Motivated by the type of function which arises in reactor theory problems, cf. [**2**], we consider the problem:

$$(4)\quad \begin{cases} -\sum_{i,j=1}^{n} D_i[a_{ij}(x, u)D_j u] - \lambda \exp\left(\frac{-1}{|u|}\right) = 0 & \text{in } D, \\ u = \phi > \frac{1}{4} & \text{on } \partial D, \end{cases}$$

where $(a_{ij})$ and $D$ are as above and $\lambda > 0$. Applying Corollary 2 we conclude that problem (4) has at most one positive solution.

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