Can. J. Math., Vol. XL, No. 3, 1988, pp. 513-531

ON THE ANALYTIC CONTINUATION OF c-FUNCTIONS

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Introduction. Let G be a reductive Lie group; Γ a nonuniform lattice in G. Then the theory of Eisenstein series plays a major role in the spectral decomposition of $L^2(G/\Gamma)$ (cf. [**5**]). One of the most difficult aspects of the subject is the analytic continuation of the Eisenstein series along with its associated **c**-function. This was originally done by Langlands using some very difficult analysis (cf. [**5**]). Later Harish-Chandra was able to simplify somewhat the most difficult part of the continuation, the continuation to zero, by the introduction of the Maas-Selberg relation. The purpose of this note is to give a simplified account of this particular part of the theory.

Our chief tool will be the truncation operator of Arthur (cf. [**1**] and [**8**]), the systematic utilization of which has the effect of streamlining the earlier accounts, especially in so far as continuation to zero is concerned, which is reduced to an elementary manipulation.

In Section 1, the notation is set up and some standard constructions are reviewed. Section 2 is concerned with the theory of the constant term, while Section 3 deals with the truncation operator. With this preparation, we pass in Section 4 to the continuation itself. Modeling on the inner product of two rank 1 Eisenstein series, property (I.P.) is introduced. It is the analysis of property (I.P.) that provides the continuation. The material in Section 1 and Section 2 is essentially standard (cf. [**4**] and [**5**]). It is the continuity of the truncation operator, as described by property (ii) of Section 3, which, when utilized appropriately, provides a vast simplification of the previous continuation arguments.

As a general reference for any undefined terminology, see Osborne and Warner's excellent monograph, The Theory of Eisenstein Systems, Academic Press, 1981.

Acknowledgement. I would like to thank M. Scott Osborne and Garth Warner for suggesting the use of the truncation operator. Their detailed criticisms of earlier drafts of this paper have been very helpful.

1. Preliminaries. Let G be a reductive Lie group of the Harish-Chandra class; let Γ be a nonuniform lattice in G. Assume that the pair satisfies the assumption spelled out on page 62 of [**7**].

Received January 8, 1986 and in revised form June 2, 1987. Research by the author was supported in part by an Alfred P. Sloan Doctoral Dissertation Fellowship.

Let (P, S) be a Γ-cuspidal split parabolic subgroup of G with special split component A. Then $G = K \cdot P$, where K is, as usual, a fixed maximal compact subgroup of G. The Langlands decomposition is then $P = A \cdot S$, where $S = M \cdot N$ (cf. pp. 30-32 in [7]). The Lie algebra of A will be denoted by $\mathfrak{a}$ and by $\check{\mathfrak{a}}$ its dual. Then $\mathfrak{a}$ and $\check{\mathfrak{a}}$ are canonically isomorphic via the Euclidean structure induced by the Killing form. Let $x \in G$. Then x admits a decomposition

$$x = k_x a_x s_x \quad (k_x \in K, a_x \in A, s_x \in S),$$

where the factor a_x is unique. Given $\Lambda \in \check{\mathfrak{a}} \otimes \mathbf{C}$, write

$$a_x^\Lambda = e^{\Lambda(H(x))} \quad (H(x) = \log a_x)$$

and define

$$\Xi_P(x) = \inf_{\lambda \in \Sigma_P^0} a_x^\lambda,$$

where Σ_P^0 denotes the simple roots of $(\mathfrak{g}, \mathfrak{a})$. Given $\lambda \in \Sigma_P^0$ and $t > 0$, put

$$A_\lambda[t] = \{a \in A | a^\lambda \leqq t\}$$

and then set

$$A[t] = \bigcap_{\lambda \in \Sigma_P^0} A_\lambda[t].$$

If now ω is a compact neighborhood of 1 in S then

$$\mathfrak{S}_{t,\omega} = K \cdot A[t] \cdot \omega$$

is called a *Siegel domain* in $\mathfrak{S}$ (relative to (P, S; A)).

(1) Let f be a complex valued measurable function on G/Γ. f is said to be *slowly increasing* if there exists a real number r such that for every Siegel domain $\mathfrak{S}$ associated with a Γ-percuspidal parabolic subgroup P of G there is a constant $\mathfrak{c} > 0$ such that

$$|f(x)| \leqq \mathfrak{c}\Xi_P^r(x) \quad (x \in \mathfrak{S}).$$

f is said to be *rapidly decreasing* if for every real number r and for every Siegel domain $\mathfrak{S}$ associated with a Γ-percuspidal subgroup P of G there is a constant $\mathfrak{c} > 0$ such that

$$|f(x)| \leqq \mathfrak{c}\Xi_P^r(x) \quad (x \in \mathfrak{S}).$$

In either case, r is called an exponent of growth.

Let $S_r^\infty(G/\Gamma)$ be the space of slowly increasing functions f on G/Γ with exponent of growth r such that for every right invariant differential operator D on G, Df is also slowly increasing with exponent of growth r. Then the seminorms

$$|f|_{r,D} = \max_i \sup_{x \in \mathfrak{S}_i} \Xi_{P_i}^{-r}(x)|Df(x)|$$

endow $S_r^{\infty}(G/\Gamma)$ with the structure of a Fréchet space. Here, the $\mathfrak{S}_i$ are Siegel domains relative to Γ-percuspidal subgroups P_i of G such that $G = \cup \mathfrak{S}_i \cdot \Gamma$.

(2) Let f be a complex valued differentiable function on G/Γ. Then f is said to be an *automorphic form* if:

(i) f is slowly increasing,

(ii) The functions $x \mapsto f(kx)$ $(k \in K)$ span a finite dimensional vector space,

(iii) The functions $x \mapsto (Zf)(x)$ $(Z \in \mathfrak{Z})$ span a finite dimensional vector space.

Here $\mathfrak{Z}$ denotes the center of the universal enveloping algebra of the complexification of the Lie algebra of G. Denote the space of all automorphic forms by $\mathcal{U}(G/\Gamma)$.

Let (P, S) be a Γ-cuspidal split parabolic subgroup of G and let ρ denote $1/2$ the sum of the positive roots. Let

$$f \in L^1_{\mathrm{loc}}(G/\Gamma)$$

then the constant term of f along P is the function f_P on G defined by

$$f_P = d_P \cdot f^P$$

where $d_P(x) = a_x^{\rho}$ and

$$f^P(x) = \int_{N/N\cap\Gamma} f(xn)dn.$$

If $f \in \mathcal{U}(G/\Gamma)$, then for all choices of $k \in K$ and $a \in A$ the function $m \mapsto f_P(kma)$ is an element of $\mathcal{U}(M/\Gamma_M)$ (cf. Lemma 3.3 of **[5]**).

If $f_P = 0$ for all $P \neq G$, then f is called a *cusp form*. Let $\mathcal{U}_{\mathrm{cus}}(M/\Gamma_M)$ denote the space of cusp forms in $\mathcal{U}(M/\Gamma_M)$. We shall say that f_P is negligible along P, written $f_P \sim 0$, if

$$\int_{M/\Gamma_M} f_P(kma)\overline{g(m)}dm = 0$$

for every $g \in \mathcal{U}_{\mathrm{cus}}(M/\Gamma_M)$ and all choices of $k \in K$ and $a \in A$. The integral converges because every element of $\mathcal{U}_{\mathrm{cus}}(M/\Gamma_M)$ is rapidly decreasing.

Let rank(P) denote the dimension of A. Call $\mathcal{U}_q(G/\Gamma)$ the space of all automorphic forms f such that $f_P \sim 0$ for all Γ-cuspidal parabolic subgroups P whose rank differs from q.

LEMMA 1. *Let f be a slowly increasing continuous function G/Γ. Suppose that $f_P \sim 0$ for every Γ-cuspidal split parabolic subgroup P of G (including $P = G$). Then f is identically zero.*

(This is Lemma 3.7 of **[5]**.)

(3) Let $\hat{K}$ denote the unitary dual of K; let $\hat{\mathfrak{Z}}_M$ denote the characters of the center of the universal enveloping algebra of the complexification of the Lie algebra of M. If $W(A)$ denotes all automorphisms of A induced from an inner automorphism of G, then since M centralizes A, there is a natural orbit structure $W(A)\backslash\hat{\mathfrak{Z}}_M$.

Given $\delta \in \hat{K}$ and $\mathcal{O} \in W(A)\backslash\hat{\mathfrak{Z}}_M$, let $\mathcal{E}_{\text{cus}}(\delta, \mathcal{O})$ be the set of all continuous functions $\Phi:G \to \mathbf{C}$ such that:

(i) Φ is right invariant under $(\Gamma \cap P) \cdot A \cdot N$,

(ii) For every $x \in G$, the function $k \mapsto \Phi(kx)$ $(k \in K)$ belongs to $L^2(K; \delta)$,

(iii) For every $x \in G$, the function $m \mapsto \Phi(xm)$ $(m \in M)$ belongs to $L^2_{\text{cus}}(M/\Gamma_M; \mathcal{O})$.

It is known that $\mathcal{E}_{\text{cus}}(\delta, \mathcal{O})$ is a finite dimensional Hilbert space of automorphic forms with inner product

$$(\Phi, \Psi) = \int_K \int_{M/\Gamma_M} \Phi(km)\overline{\Psi(km)}dkdm.$$

Let $\mathcal{C}_P(\check{\mathfrak{a}})$ be the positive chamber in $\check{\mathfrak{a}}$. Put

$$\mathcal{T}_P(\check{\mathfrak{a}}) = -(\rho + \mathcal{C}_P(\check{\mathfrak{a}})).$$

If $\Phi \in \mathcal{E}_{\text{cus}}(\delta, \mathcal{O})$, then attached to Φ is the *Eisenstein series*

$$E(P|A:\Phi:\Lambda:x) = \sum_{\gamma \in \Gamma/\Gamma \cap P} a_{x\gamma}^{\Lambda-\rho}\Phi(x\gamma).$$

It is known that the series defining $E(P|A:\Phi:\Lambda:x)$ is absolutely uniformly convergent on compact subsets of

$$(\mathcal{T}_P(\check{\mathfrak{a}}) + \sqrt{-1}\check{\mathfrak{a}}) \times G.$$

In fact, if $\text{rank}(P) = q$ and $\text{Re}(\Lambda) \in \mathcal{T}_P(\check{\mathfrak{a}})$, then

$$E(P|A:\Phi:\Lambda:x)$$

is an element of $\mathcal{U}_q(G/\Gamma)$.

(4) Let (P_1, S_1) and (P_2, S_2) be two Γ-cuspidal split parabolic subgroups of G with special split components A_1 and A_2. (P_1, S_1) and (P_2, S_2) are said to be *associate* if there exists $\mathfrak{s} \in G$ such that

$$\mathfrak{s}A_1\mathfrak{s}^{-1} = A_2.$$

Let $W(A_2, A_1)$ denote the set of all such isomorphisms. The relation of association breaks up the Γ-cuspidal split parabolic subgroups of G into equivalence classes. Fix one such, say $\mathcal{C}$. Then the rank of $\mathcal{C}$ is the rank of any element of $\mathcal{C}$. If (P_1, S_1) and (P_2, S_2) are elements of $\mathcal{C}$, then the orbit spaces $W(A_1)\backslash\hat{\mathfrak{Z}}_{M_1}$ and $W(A_2)\backslash\hat{\mathfrak{Z}}_{M_2}$ are in canonical one-to-one correspondence. Corresponding orbits are said to be associate. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be associate orbits. Let $\mathfrak{s} \in W(A_2, A_1)$ and identify the map $\mathfrak{s}$ with the element of G that it represents. Define

$$\Gamma(\mathfrak{s}) = \Gamma \cap P_2 \mathfrak{s} P_1.$$

It can then be shown that for any Φ_1 belonging to $\mathscr{E}_{\text{cus}}(\delta, \mathscr{O}_1)$,

$$E_{P_2}(P_1|A_1:\Phi_1:\Lambda_1:x)$$
$$= \sum_{\mathfrak{s} \in W(A_2,A_1)} a_2(x)^{\mathfrak{s}\Lambda_1}(\mathbf{c}_{\text{cus}}(P_2|A_2:P_1|A_1:\mathfrak{s}:\Lambda_1)\Phi_1)(x),$$

where

$$(c_{\text{cus}}(P_2|A_2:P_1|A_1:\mathfrak{s}:\Lambda_1)\Phi_1)(x)$$
$$= a_2(x)^{(\rho_2 - \mathfrak{s}\Lambda_1)} \int_{N_2/N_2 \cap \Gamma}$$
$$\left\{ \sum_{\gamma \in \Gamma(\mathfrak{s})/\Gamma \cap P_1} a_1(xn\gamma)^{(\Lambda_1 - \rho_1)} \Phi_1(xn\gamma) \right\} dn.$$

The *c-function* $\mathbf{c}_{\text{cus}}(P_2|A_2:P_1|A_1:\mathfrak{s}:\Lambda_1)$ is a linear transformation from $\mathscr{E}_{\text{cus}}(\delta, \mathscr{O}_1)$ to $\mathscr{E}_{\text{cus}}(\delta, \mathscr{O}_2)$ which is holomorphic as a function of Λ_1 in

$$\mathscr{T}_{P_1}(\check{\mathfrak{a}}_1) + \sqrt{-1}\check{\mathfrak{a}}_1.$$

However, if (P_1, S_1) and (P_2, S_2) are of the same rank, but are not associate, then

$$E_{P_2}(P_1|A_1:\Phi_1:\Lambda_1:x) = 0$$

(cf. Lemma 4.4 on pp. 85-86 of **[5]**).

Let P_i $(1 \leqq i \leqq r)$ be a set of representatives for $G\backslash\mathscr{C}$. Then

$$\mathscr{C} = \coprod_i \mathscr{C}_i$$

where $\mathscr{C}_i = G \cdot \{P_i\} \cap \mathscr{C}$. Let $P_{i\mu}$ $(1 \leqq \mu \leqq r_i)$ be a set of representatives for $\Gamma\backslash\mathscr{C}_i$. Then

$$\{P_{i\mu}|1 \leqq i \leqq r, 1 \leqq \mu \leqq r_i\}$$

is a set of representatives for $\Gamma\backslash\mathscr{C}$. Let $k_{i\mu} \in K$ be such that

$$P_{i\mu} = k_{i\mu} P_i k_{i\mu}^{-1}.$$

If special split components are taken then

$$A_{i\mu} = k_{i\mu} A_i k_{i\mu}^{-1}.$$

Let

$$P_i = M_i \cdot A_i \cdot N_i, \; P_{i\mu} = M_{i\mu} \cdot A_{i\mu} \cdot N_{i\mu}$$

be the corresponding Langlands decompositions.

Fix a K-type δ. Let

$$\mathscr{O}_i = \{\mathscr{O}_{i\mu}|1 \leqq \mu \leqq r_i\}$$

https://doi.org/10.4153/CJM-1988-022-5 Published online by Cambridge University Press

be a collection of associate orbits. Set

$$\mathscr{E}_{\text{cus}}(\delta, \mathscr{O}_i) = \sum_{\mu} \oplus \mathscr{E}_{\text{cus}}(\delta, \mathscr{O}_{i\mu}).$$

Let $\mathscr{O}_i$ and $\mathscr{O}_j$ be associate (sets of) orbits, $1 \leqq i, j \leqq r$. Let

$$\mathfrak{s} \in W(A_j, A_i) \quad \text{and} \quad \Lambda_i \in \mathscr{T}_{P_i}(\check{\mathfrak{a}}_i) + \sqrt{-1}\check{\mathfrak{a}}_i.$$

Define

$$\mathbf{c}_{\text{cus}}(P_j|A_j:P_i|A_i:\mathfrak{s}:\Lambda_i)$$

to be the matrix

$$[\mathbf{c}_{\text{cus}}(P_{j\nu}|A_{j\nu}:P_{i\mu}|A_{i\mu}:\text{Ad}(k_{j\nu})\,\mathfrak{s}\,\text{Ad}(k_{i\mu}^{-1}):\text{Ad}(k_{i\mu})\,\Lambda_i)\,]$$

which maps $\mathscr{E}_{\text{cus}}(\delta, \mathscr{O}_i)$ to $\mathscr{E}_{\text{cus}}(\delta, \mathscr{O}_j)$. It is known that the adjoint

$$\mathbf{c}_{\text{cus}}(P_j|A_j:P_i|A_i:\mathfrak{s}:\Lambda_i)^*$$

of

$$\mathbf{c}_{\text{cus}}(P_j|A_j:P_i|A_i:\mathfrak{s}:\Lambda_i)$$

is given by

$$\mathbf{c}_{\text{cus}}(P_i|A_i:P_j|A_j:\mathfrak{s}^{-1}:-\mathfrak{s}\bar{\Lambda}_i).$$

Moreover, there exists a constant $c > 0$ and an element $H_i \in \mathfrak{a}_i$ such that

$$\|\mathbf{c}_{\text{cus}}(P_j|A_j:P_i|A_i:\mathfrak{s}:\Lambda_i)\|$$
$$\leqq c \cdot \frac{e^{(\text{Re}(\Lambda_i)-\rho_i)(H_i)}}{\prod_{\alpha \in \Sigma^0_{P_i}} |\langle \text{Re}(\Lambda_i) + \rho_i, \alpha\rangle|},$$

for all $\Lambda_i \in \mathscr{T}_{P_i}(\check{\mathfrak{a}}_i) + \sqrt{-1}\check{\mathfrak{a}}_i$ and all $\mathfrak{s} \in W(A_j, A_i)$ (cf. Lemma 4.5 on p. 86 of [**5**]). Define

$$\mathbf{E}(P_i|A_i:\Phi_i:\Lambda_i:x) = \sum_{\mu=1}^{r_i} E(P_{i\mu}|A_{i\mu}:\Phi_{i\mu}:\text{Ad}(k_{i\mu})\,\Lambda_i:x),$$

for

$$\Phi_i = (\Phi_{i\mu}) \in \mathscr{E}_{\text{cus}}(\delta, \mathscr{O}_i) \quad \text{and} \quad \Lambda_i \in \mathscr{T}_{P_i}(\check{\mathfrak{a}}_i) + \sqrt{-1}\check{\mathfrak{a}}_i.$$

It follows from the definitions that

$$\mathbf{E}_{P_{j\nu}}(P_i|A_i:\Phi_i:\Lambda_i:x)$$
$$= \sum_{\mathfrak{s} \in W(A_j,A_i)} a_{j\nu}(x)^{(\text{Ad}(k_{j\nu})\mathfrak{s}\Lambda_i)}$$
$$\times (E^{j,\nu} \circ \mathbf{c}_{\text{cus}}(P_j|A_j:P_i|A_i:\mathfrak{s}:\Lambda_i)\Phi_i)(x),$$

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where

$$E^{j,v}:\mathscr{E}_{\mathrm{cus}}(\delta, \mathscr{O}_j) \to \mathscr{E}_{\mathrm{cus}}(\delta, \mathscr{O}_{jv})$$

is the orthogonal projection.

(5) Let rank$(\mathscr{C}) = q$. Let (P, S) and (P', S') belong to $\mathscr{C}$. It follows from Lemma 81 of **[4]** that $|\rho| = |\rho'|$. Denote their common value by $|\rho|$. Fix $R > |\rho|$, $\delta \in \hat{K}$ and $\mathscr{O} \in W(A)\backslash\hat{\mathfrak{z}}_M$. Denote by $\mathscr{H}_A$ the space of Fourier-Laplace transforms of functions in $\mathscr{C}_c^\infty(\sqrt{-1}\check{\mathfrak{a}})$. Set

$$\mathscr{H}_A(\delta, \mathscr{O}) = \mathscr{H}_A \otimes \mathscr{E}_{\mathrm{cus}}(\delta, \mathscr{O}).$$

Attached to each $\Phi \in \mathscr{H}_A(\delta, \mathscr{O})$ is the *wave packet*

$$\Theta_\Phi(x) = \frac{1}{(2\pi)^q}\int_{\mathrm{Re}(\Lambda)=\Lambda_0} E(P|A:\Phi(\Lambda):\Lambda:x)|d\Lambda| \quad (\Lambda_0 \in \mathscr{T}_P(\check{\mathfrak{a}})).$$

It is known that Θ_Φ is rapidly decreasing (cf. Lemma 3.6 of **[5]**). Define $\mathscr{H}_A^2(R)$ to be the space of all bounded holomorphic functions on

$$\{\Lambda \in \check{\mathfrak{a}} \otimes \mathbf{C} |\, |\mathrm{Re}(\Lambda)| < R\}$$

that are square integrable on vertical strips. Set

$$\mathscr{H}_A^2(\delta, \mathscr{O}; R) = \mathscr{H}_A^2(R) \otimes \mathscr{E}_{\mathrm{cus}}(\delta, \mathscr{O}).$$

An elementary argument using transform theory shows that the map

$$\Phi \mapsto \Theta_\Phi:\mathscr{H}_A(\delta, \mathscr{O}) \to L^2(G/\Gamma)$$

can be extended to a map from $\mathscr{H}_A^2(\delta, \mathscr{O}; R)$ to $L^2(G/\Gamma)$. In fact, if $\Phi \in \mathscr{H}_A^2(\delta, \mathscr{O}; R)$, then the element Θ_Φ of $L^2(G/\Gamma)$ is the L^2-limit of wave packets. Define

$$\mathscr{H}_i^2(\delta, \mathscr{O}_i; R) = \prod_{\mu=1}^{r_i} \mathscr{H}_{A_{i\mu}}^2(\delta, \mathscr{O}_{i\mu}; R).$$

If $\mathscr{O}_i$ and $\mathscr{O}_j$ are collections of associate orbits and

$$\Phi_i \in \mathscr{H}_i^2(\delta, \mathscr{O}_i; R) \quad \text{and} \quad \Phi_j \in \mathscr{H}_j^2(\delta, \mathscr{O}_j; R)$$

then

$$\int_{G/\Gamma} \Theta_{\Phi_i}(x)\overline{\Theta_{\Phi_j}(x)}dx = \sum_{s \in W(A_j,A_i)} \frac{1}{(2\pi)^q}\int_{\mathrm{Re}(\Lambda_i)=\Lambda_i^0} (\mathbf{c}_{\mathrm{cus}}(P_j|A_j:P_i|A_i:s:\Lambda_i)\Phi_i(\Lambda_i), \Phi_j(-s\bar{\Lambda}_i))|d\Lambda_i|,$$

where

$$\Lambda_i^0 \in \mathscr{T}_{P_i}(\check{\mathfrak{a}}) \quad \text{and} \quad |\Lambda_i^0| < R$$

(cf. Lemma 4.6 of **[5]**).

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Let $L^2_{\mathscr{C}}(G/\Gamma;\ \delta,\ \mathscr{O})$ denote the closed subspace of $L^2(G/\Gamma)$ spanned by the Θ_Φ, where

$$\Phi \in \mathscr{H}_{A_{i\mu}}(\delta,\ \mathscr{O}_{i\mu}),\quad \mathscr{O} = \{\mathscr{O}_{i\mu}\}.$$

Then

$$L^2(G/\Gamma) = \sum_{\mathscr{C}} \sum_{\delta} \sum_{\mathscr{O}} \oplus\, L^2_{\mathscr{C}}(G/\Gamma;\ \delta,\ \mathscr{O})$$

(cf. Lemma 4.6 of **[5]**).

2. The constant term. Specialize now to rank($\mathscr{C}$) = 1. Let

$$\mathscr{O} = \{\mathscr{O}_{i\mu} | 1 \leqq i \leqq r,\ 1 \leqq \mu \leqq r_i\}$$

be a collection of associate orbits. Set

$$\mathscr{E}_{\text{cus}}(\delta,\ \mathscr{O}) = \sum_{i=1}^{r} \oplus\, \mathscr{E}_{\text{cus}}(\delta,\ \mathscr{O}_i),$$

and

$$\mathscr{H}^2(\delta,\ \mathscr{O};\ R) = \prod_{i=1}^{r} \mathscr{H}_i^2(\delta,\ \mathscr{O}_i;\ R).$$

Let $(P,\ S) \in \mathscr{C}$. Since rank($\mathscr{C}$) = 1, any element of $\mathscr{C}$ is conjugate to $P = M \cdot A \cdot N$ or to the opposite group $P^- = M \cdot A \cdot N^-$. P and P^- are conjugate if and only if $-1 \in W(A)$. Thus $r = 1$ or 2. In either case put

$$\lambda_i = \frac{\alpha_i}{|\alpha_i|},$$

where α_i is the simple root of $(P_i,\ S_i)$. Then

$$\rho_i = |\rho|\lambda_i.$$

If $r = 1$, then $W(A_1) = \{\pm 1\}$, so for $\text{Re}(z) < -|\rho|$ define

$$\mathbf{c}(z) = \mathbf{c}_{\text{cus}}(P_1|A_1{:}P_1|A_1{:}-1{:}z\lambda_1).$$

If $r = 2$ then $W(A_i) = \{1\}$ $(i = 1,\ 2)$, so for $\text{Re}(z) < -|\rho|$ define

$$\mathbf{c}(z) = \begin{pmatrix} 0, & \mathbf{c}_{\text{cus}}(P_1|A_1{:}P_2|A_2{:}s^{-1}{:}z\lambda_2) \\ \mathbf{c}_{\text{cus}}(P_2|A_2{:}P_1|A_1{:}s{:}z\lambda_1), & 0 \end{pmatrix},$$

where s is the unique element in $W(A_2,\ A_1)$.

In either case $\mathbf{c}(z)$ is a linear transformation from $\mathscr{E}_{\text{cus}}(\delta,\ \mathscr{O})$ to itself that is holomorphic for $\text{Re}(z) < -|\rho|$. Moreover, the adjoint $\mathbf{c}(z)^*$ of $\mathbf{c}(z)$ is equal to $\mathbf{c}(\bar{z})$.

If Φ and Ψ are elements of $\mathscr{H}^2(\delta, \mathscr{O}; R)$, then the inner product

$$(\Theta_\Phi, \Theta_\Psi)_{G/\Gamma}$$

is equal to

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\{(\Phi(z), \Psi(-\bar{z})) + (\mathbf{c}(z)\Phi(z), \Psi(\bar{z}))\}dz,$$

where $-R < c < -|\rho|$.

Given $z \in \mathbf{C}^\times = \mathbf{C}\backslash\{0\}$, let $\mathscr{L}(z)$ consist of all $f \in \mathscr{U}_1(G/\Gamma)$ for which

(i) $f_P = 0$ if $P \notin \mathscr{C}$ and $\operatorname{rank}(P) = 1$.

(ii) $f_{P_{i\mu}}(x) = \varphi_{i\mu}(x)e^{zt_{i\mu}(x)} + \psi_{i\mu}(x)e^{-zt_{i\mu}(x)}$

for some $\varphi = (\varphi_{i\mu})$, $\psi = (\psi_{i\mu})$ belonging to $\mathscr{E}_{\text{cus}}(\delta, \mathscr{O})$. Here,

$$t_{i\mu}(x) = \lambda_i(\operatorname{Ad}(k_{i\mu}^{-1})H_{i\mu}(x)).$$

(This definition is due to Harish-Chandra; cf. p. 91 of [**4**]).

Define $\mathscr{G}(z)$ to be the set of all pairs

$$(\varphi, \psi) \in \mathscr{E}_{\text{cus}}(\delta, \mathscr{O}) \times \mathscr{E}_{\text{cus}}(\delta, \mathscr{O})$$

for which there exists $f \in \mathscr{U}_1(G/\Gamma)$ such that (i) and (ii) hold. Lemma 1 shows that $f \leftrightarrow (\varphi, \psi)$ gives a bijective correspondence of $\mathscr{L}(z)$ with $\mathscr{G}(z)$. Let

$$(C - T): \mathscr{L}(z) \rightarrow \mathscr{G}(z)$$

denote this correspondence. If $\varphi \in \mathscr{E}_{\text{cus}}(\delta, \mathscr{O})$, then define $\tau_\varphi = 0$ if $\varphi = 0$ and $\tau_\varphi = 1$ if $\varphi \neq 0$.

LEMMA 2. *Let $\{z_n\} \subseteq \mathbf{C}^\times$. Suppose that $(f_n) \subseteq \mathscr{U}_1(G/\Gamma)$ such that*

$$f_n \in \mathscr{L}(z_n) \quad \text{and} \quad (C - T)(f_n) = (\varphi_n, \psi_n).$$

If $\varphi_n \rightarrow \varphi$, $\psi_n \rightarrow \psi$ and $z_n \rightarrow z$, then there exists $f \in \mathscr{U}_1(G/\Gamma)$ such that $f_n \rightarrow f$ uniformly on compact subsets of G/Γ. In fact, for some r, the sequence (f_n) is contained in $S_r^\infty(G/\Gamma)$ and $f_n \rightarrow f$ in $S_r^\infty(G/\Gamma)$. Moreover, if $\mathfrak{S}$ is any Siegel domain associated to a Γ-percuspidal split parabolic subgroup (P, S) of G, then there is a constant $\mathbf{c} > 0$ so that, for all $x \in \mathfrak{S}$, $|f_n(x)|$ is bounded above by

$$\mathbf{c}\{\|\varphi_n\| + \|\psi_n\|\}\left\{\sum_{i=1}^{r}\sum_{\mu=1}^{r_i}\left(\tau_{\varphi_n}\cdot e^{(\operatorname{Re}(z_n)-|\rho|)t_{i\mu}(x)} + \tau_{\psi_n}\cdot e^{-(\operatorname{Re}(z_n)+|\rho|)t_{i\mu}(x)}\right)\right\}.$$

In addition let $f \in \mathscr{L}(z)$ with $\operatorname{Re}(z) < 0$. *If*

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$$(C - T)(f) = (0, \psi),$$

then $f \in L^2(G/\Gamma)$.

(The proofs of these results can be found in [5], pp. 100-111.)

Let $f_{i\mu}$ be a bounded analytic function on

$$\{\Lambda \in \check{\mathfrak{a}}_{i\mu} \otimes \mathbf{C} | \; |\mathrm{Re}(\Lambda)| < R\}.$$

Suppose that for all $s \in W(A_{j\upsilon}, A_{i\mu})$,

$$f_{j\upsilon}(s\Lambda) = f_{i\mu}(\Lambda).$$

Set $f = (f_{i\mu})$. Define now

$$\begin{cases} f\Phi = (f_{i\mu}\Phi_{i\mu}), \; \Phi \in \mathscr{H}^2(\delta, \mathscr{O}; R), \\ T_f\Theta_\Phi = \Theta_{f\Phi}. \end{cases}$$

If $f^*_{i\mu}(\Lambda) = \overline{f_{i\mu}(-\bar{\Lambda})}$ then

$$(T_f\theta_\Phi, \theta_\Psi)_{G/\Gamma} = (\Theta_\Phi, T_{f^*}\Theta_\Psi)_{G/\Gamma}.$$

Moreover, if $\|f\|_\infty \leqq k$, then

$$\|T_f\Theta_\Phi\|_{G/\Gamma} \leqq k\|\Theta_\Phi\|_{G/\Gamma},$$

so T_f defines a bounded linear operator on $L^2(G/\Gamma)$. Observe that if $\Lambda = \Lambda_1 + \sqrt{-1}\Lambda_2$ then

$$\langle \Lambda, \Lambda \rangle = \langle \Lambda_1, \Lambda_1 \rangle - \langle \Lambda_2, \Lambda_2 \rangle + 2\sqrt{-1}\langle \Lambda_1, \Lambda_2 \rangle.$$

Hence if $\mathrm{Re}(\zeta) > R^2$ and $f^\zeta_{i\mu}(\Lambda) = (\zeta - \langle \Lambda, \Lambda \rangle)^{-1}$, then T_{f^ζ} is a bounded linear operator on $L^2_{\mathscr{C}}(G/\Gamma; \delta, \mathscr{O})$.

On the other hand, there is an essentially self-adjoint operator $\Box$ on $L^2_{\mathscr{C}}(G/\Gamma; \delta, \mathscr{O})$ characterized by the relation

$$\Box\Theta_\Phi = \Theta_{\langle ?,? \rangle \Phi} \quad (\Phi \in \mathscr{H}_{A_{i\mu}}(\delta, \mathscr{O}_{i\mu})),$$

where $(\langle ?, ? \rangle\Phi)(\Lambda) = \langle \Lambda, \Lambda \rangle \Phi(\Lambda)$. Let ω be the Casimir operator. Then ω is an essentially self-adjoint operator defined on the space of differentiable vectors in $L^2_{\mathscr{C}}(G/\Gamma; \delta, \mathscr{O})$. In fact, one has $\Box = \omega - \mathfrak{c}(\mathscr{O})$, where $\mathfrak{c}(\mathscr{O}) \in \mathbf{R}$. Since the resolvent $\mathrm{Res}(\Box, \zeta)$ of $\Box$ is equal to T_{f^ζ}, the arbitrariness of R shows that the spectrum of $\Box$ is contained in $(-\infty, |\rho|^2]$.

LEMMA 3. *Let* $\varphi, \psi \in \mathscr{E}_{\mathrm{cus}}(\delta, \mathscr{O})$. *Then* $(\mathbf{c}(\zeta)\varphi, \psi)$ *is holomorphic as a function of* ζ *on*

$$D = \{\zeta \in \mathbf{C} | \mathrm{Re}(\zeta) < 0, \zeta \notin [-|\rho|, 0)\}.$$

(This is due to Langlands, cf. pp. 127-128 of [5].)

Proof. Set

$$\Phi(\zeta) = e^{\zeta^2}\varphi \quad \text{and} \quad \Psi(\zeta) = e^{\zeta^2}\psi.$$

Then the inner product

$$(\mathrm{Res}(\square, \zeta^2)\Theta_\Phi, \Theta_\Psi)_{G/\Gamma}$$

is equal to

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{2z^2}}{\zeta^2 - z^2}\{ (\varphi, \psi) + (\mathbf{c}(z)\varphi, \psi) \}dz,$$

for $\mathrm{Re}(\zeta) < \mathfrak{c} < -|\rho|$. By shifting the line of integration to $\mathrm{Re}(z) = \mathfrak{c}_1$ with $\mathfrak{c}_1 < \mathrm{Re}(\zeta) < \mathfrak{c}$, we get

$$(\mathrm{Res}(\square, \zeta^2)\Theta_\Phi, \Theta_\Psi) = \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} \frac{e^{2z^2}}{\zeta^2 - z^2}\{ (\varphi, \psi) + (\mathbf{c}(z)\varphi, \psi) \}dz$$
$$- \frac{e^{2\zeta^2}}{2\zeta}\{ (\varphi, \psi) + (\mathbf{c}(\zeta)\varphi, \psi) \}.$$

The integral over $\mathrm{Re}(z) = \mathfrak{c}_1$ is holomorphic for $\mathfrak{c}_1 < \mathrm{Re}(\zeta) < 0$, while

$$\zeta \mapsto (\mathrm{Res}(\square, \zeta^2)\Theta_\Phi, \Theta_\Psi)$$

is holomorphic on D. Hence $(\mathbf{c}(\zeta)\varphi, \psi)$ is holomorphic for $\zeta \in D$.

Remark. The techniques used in this lemma will play an important role later on.

3. Truncation. The process of truncation was first introduced by Selberg (cf. page 183 of **[10]**). Langlands also used truncation in his continuation argument (cf. page 133 of **[5]**). However, it was Arthur who first observed that truncation can be used to define a projection operator on $L^2(G/\Gamma)$ (cf. **[1]** and **[2]**). In this section we shall review the construction of the truncation operator and then compute the inner product of two truncated Eisenstein series. Here, we are following the point of view of Osborne and Warner (cf. **[8]** and **[9]**).

Let (P, S) and (P', S') be Γ-cuspidal split parabolic subgroups of G with special split components A and A'. Suppose that $\gamma P \gamma^{-1} = P'$ for some $\gamma \in \Gamma$. Write $\gamma = kp$ ($k \in K$, $p \in P$). Define

$$I_\Gamma(P:P'):\mathfrak{a}' \to \mathfrak{a}$$

by

$$I_\Gamma(P:P')(H') = \mathrm{Ad}(k^{-1})(H') + H_P(\gamma).$$

Then

$$H_P(x\gamma) = I_\Gamma(P:P')(H_{P'}(x)).$$

As before, let $\mathscr{C}$ denote a fixed association class of rank 1 Γ-cuspidal parabolic subgroups of G. Define

$$\mathfrak{a}_{\mathscr{C}} = \left\{ \mathbf{H} \in \prod_{P \in \mathscr{C}} \mathfrak{a}_P | \mathbf{H}_{\gamma P \gamma^{-1}} = I_\Gamma(\gamma P \gamma^{-1}:P)\mathbf{H}_P \quad (\gamma \in \Gamma) \right\}.$$

Here, $\mathfrak{a}_P$ is, of course, the Lie algebra of the special split component of (P, S).

If $(P, S) \in \mathscr{C}$, let χ_P denote the characteristic function of the positive cone of (P, S). Let $z \in \mathbf{C}^\times$, $f \in \mathscr{L}(z)$ and $\mathbf{H} \in \mathfrak{a}_{\mathscr{C}}$. Define the *truncation operator*

$$(Q^{\mathbf{H}}f)(x) = f(x) - \sum_{P \in \mathscr{C}} \chi_P(\mathbf{H}_P - H_P(x)) \cdot f^P(x).$$

It follows from the definitions that

$$(Q^{\mathbf{H}}f)(x) = f(x) - \sum_{i=1}^{r} \sum_{\mu=1}^{r_i} \sum_{\gamma \in \Gamma/\Gamma \cap P_{i\mu}} \chi_{P_{i\mu}}(\mathbf{H}_{P_{i\mu}} - H_{P_{i\mu}}(x\gamma)) \cdot f^{P_{i\mu}}(x\gamma).$$

For convenience, we shall consider only those $\mathbf{H} \in \mathfrak{a}_{\mathscr{C}}$ for which

$$\alpha_{i\mu}(\mathbf{H}_{P_{i\mu}}) = \alpha_{j\nu}(\mathbf{H}_{P_{j\nu}}),$$

where $\alpha_{i\mu}$ is the simple root of $(P_{i\mu}, S_{i\mu})$. There is no loss of generality in doing this, since the set of such $\mathbf{H}$ is cofinal with respect to the natural ordering on $\mathfrak{a}_{\mathscr{C}}$. Moreover, given such an $\mathbf{H}$ there exists $N \in \mathbf{R}$ such that

$$\chi_{P_{i\mu}}(\mathbf{H}_{P_{i\mu}} - H_{P_{i\mu}}(x)) = \chi_{(-\infty,-N)}(t_{i\mu}(x)).$$

$Q^{\mathbf{H}}$ possesses the following properties.

(i) $\lim_{\mathbf{H}\to-\infty} Q^{\mathbf{H}}f = f$ uniformly on compact subsets of G/Γ.

(ii) $Q^{\mathbf{H}}:S_r^\infty(G/\Gamma) \to L^2(G/\Gamma)$ is continuous.

(iii) For $\mathbf{H} \ll 0$, $Q^{\mathbf{H}} \circ Q^{\mathbf{H}} = Q^{\mathbf{H}}$ and the closure of $Q^{\mathbf{H}}$ in $L^2(G/\Gamma)$ is an orthogonal projection.

The proofs of these results, for adelic groups, are due to Arthur (cf. [**1**]). (The fact that $Q^{\mathbf{H}}$ extends to an orthogonal projection is only a remark.) For real groups and for lattices with more than one cusp, the proofs are due to Osborne and Warner (cf. [**8**]).

We shall now compute the L^2-inner product of two truncated rank 1 Eisenstein series.

Let $\boldsymbol{\varphi} \in \mathscr{E}_{\text{cus}}(\delta, \mathscr{O})$ and $\text{Re}(z) < -|\rho|$. Define

$$\mathbf{E}(\boldsymbol{\varphi}, z) = \sum_{i=1}^{r} \sum_{\mu=1}^{r_i} E(P_{i\mu}|A_{i\mu}:\boldsymbol{\varphi}_{i\mu}:z(\text{Ad}(k_{i\mu})\,\lambda_i)).$$

Observe that

$$\mathbf{E}_{P_{i\mu}}(\boldsymbol{\varphi}, z) = \boldsymbol{\varphi}_{i\mu}e^{zt_{i\mu}} + (\mathbf{c}(z)\boldsymbol{\varphi})_{i\mu}e^{-zt_{i\mu}}.$$

It follows that $\mathbf{E}(\varphi, z)$ belongs to $\mathscr{L}(z)$. Rewrite $(Q^{\mathbf{H}}\mathbf{E}(\varphi, z))(x)$ in the form

$$\sum_{i=1}^{r} \sum_{\mu=1}^{r_i} \sum_{\gamma \in \Gamma/\Gamma \cap P_{i\mu}} \chi_{[-N,\infty)}(t_{i\mu}(x\gamma))\varphi_{i\mu}(x\gamma)e^{(z-|\rho|)t_{i\mu}(x\gamma)}$$

$$- \sum_{i=1}^{r} \sum_{\mu=1}^{r_1} \sum_{\gamma \in \Gamma/\Gamma \cap P_{i\mu}} \chi_{(-\infty,-N)}(t_{i\mu}(x\gamma))(\mathbf{c}(z)\varphi)_{i\mu}(x\gamma)e^{-(z+|\rho|)t_{i\mu}(x\gamma)}.$$

Let $\lambda, \mu \in \mathbf{C}^{\times}$ such that $\operatorname{Re}(\lambda) < -|\rho|$. Since $R > |\rho|$ is arbitrary, it may be assumed that $\operatorname{Re}(\lambda) < -R$. Suppose that $g \in \mathscr{L}(\mu)$ such that

$$(C - T)(g) = (\varphi', \psi').$$

Let $\zeta \in \mathbf{C}$ with $|\operatorname{Re}(\zeta)| < R$. Compute the Fourier transforms:

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} (\chi_{[-N,\infty)}(t)e^{(\lambda-|\rho|)t})e^{-(\zeta-|\rho|)t}dt = \frac{e^{(\zeta-\lambda)N}}{\zeta - \lambda} \\ \displaystyle\int_{-\infty}^{\infty} (\chi_{(-\infty,-N)}(t)e^{-(\lambda+|\rho|)t})e^{-(\zeta-|\rho|)t}dt = -\frac{e^{(\zeta+\lambda)N}}{\zeta + \lambda}. \end{cases}$$

Hence,

$$\Phi(\zeta) = \frac{e^{(\zeta-\lambda)N}}{\zeta - \lambda}\varphi + \frac{e^{(\zeta+\lambda)N}}{\zeta + \lambda}\mathbf{c}(\lambda)\varphi$$

belongs to $\mathscr{H}^2(\delta, \mathscr{O}; R)$ and

$$Q^{\mathbf{H}}\mathbf{E}(\varphi, \lambda) = \Theta_{\Phi}.$$

Moreover, by an elementary argument, it follows that

$$\begin{aligned}(Q^{\mathbf{H}}\mathbf{E}(\varphi, \lambda), Q^{\mathbf{H}}g)_{G/\Gamma} &= (\Theta_{\Phi}, g)_{G/\Gamma} \\ &= (\Phi(-\bar{\mu}), \varphi') + (\Phi(\bar{\mu}), \psi').\end{aligned}$$

Thus

$$\begin{aligned}(Q^{\mathbf{H}}\mathbf{E}(\varphi, \lambda), Q^{\mathbf{H}}g)_{G/\Gamma} = {}& \frac{1}{\lambda + \bar{\mu}}\{e^{(\lambda+\bar{\mu})N}(\mathbf{c}(\lambda)\varphi, \psi') - e^{-(\lambda+\bar{\mu})N}(\varphi, \varphi')\} \\ &+ \frac{1}{\lambda - \bar{\mu}}\{e^{(\lambda-\bar{\mu})N}(\mathbf{c}(\lambda)\varphi, \varphi') - e^{-(\lambda-\bar{\mu})N}(\varphi, \psi')\}.\end{aligned}$$

It should be noted that a formula of this sort first appeared on p. 242 of Langlands original paper on Eisenstein series in [**6**]. A more general formula for the inner product of truncated cuspidal Eisenstein series of arbitrary rank appears on p. 247 of [**6**]. However, the first detailed proof is due to Arthur in [**1**].

From the form of the inner product, after setting $g = \mathbf{E}(\varphi', \zeta')$, one sees that the function

$$(\zeta, \bar{\zeta}') \mapsto (Q^{\mathbf{H}}\mathbf{E}(\varphi, \zeta), Q^{\mathbf{H}}\mathbf{E}(\varphi', \zeta'))_{G/\Gamma}$$

is holomorphic for $(\zeta, \zeta') \in D \times D$. Therefore

$$\sum_{n=0}^{\infty} \left\| \frac{\partial^n Q^{\mathbf{H}}\mathbf{E}(\varphi, \zeta)}{\partial \zeta^n} \right\|^2 \frac{|\zeta - \zeta_0|^{2n}}{(n!)^2}$$

must converge in the largest circle about ζ_0 ($\zeta_0 \in D$) which does not meet the real or imaginary axis. Hence so does

$$\sum_{n=0}^{\infty} \left\| \frac{\partial^n Q^{\mathbf{H}}\mathbf{E}(\varphi, \zeta)}{\partial \zeta^n} \right\| \frac{|\zeta - \zeta_0|^{n}}{n!},$$

which certainly provides the contribution of $Q^{\mathbf{H}}\mathbf{E}(\varphi, \zeta)$ to D as a distribution. Since

$$\lim_{\mathbf{H} \to -\infty} Q^{\mathbf{H}}\mathbf{E}(\varphi, \zeta) = \mathbf{E}(\varphi, \zeta)$$

uniformly on compact sets, it follows that $\mathbf{E}(\varphi, \zeta)$ is holomorphic on D in the sense of distributions. By using a standard result,

$$\xi \mapsto \mathbf{E}(\varphi; \zeta): D \to \mathscr{C}^{\infty}(G/\Gamma)$$

is holomorphic (cf. appendix to chapter 4 in [7]). This argument is a variation of the one of Langlands on p. 242 of [6].

Fix $\zeta = x + iy$ such that $x < 0$ and $y \neq 0$. It follows from the inner product formula that

$$\|Q^{\mathbf{H}}\mathbf{E}(\varphi, \zeta)\|^2_{G/\Gamma} = \frac{1}{2x}\{e^{2xN}\|\mathbf{c}(\zeta)\varphi\|^2 - e^{-2xN}\|\varphi\|^2\} + \operatorname{Re}\left\{\frac{e^{2iyN}(\mathbf{c}(\zeta)\varphi, \varphi)}{iy}\right\}.$$

So, if $\|\varphi\| = 1$ and $\|\mathbf{c}(\zeta)\varphi\| = \|\mathbf{c}(\zeta)\|$, then

$$-\frac{e^{2xN}\|\mathbf{c}(\zeta)\|^2 - e^{-2xN}}{2|x|} + \frac{\|\mathbf{c}(\zeta)\|}{|y|} \geqq 0.$$

This gives

$$\|\mathbf{c}(\zeta)\| \leqq e^{-2xN}\left(\left|\frac{x}{y}\right| + \sqrt{1 + \left(\frac{x}{y}\right)^2}\right).$$

In particular, every nonzero point of the imaginary axis has a neighborhood, on which, when intersected with the left half plane, $\mathbf{c}(\zeta)$ is bounded.

https://doi.org/10.4153/CJM-1988-022-5 Published online by Cambridge University Press

4. Continuation. Let $\lambda, \mu \in \mathbf{C}^{\times}$.

Property (I.P.). A pair $(f, g) \in \mathscr{L}(\lambda) \times \mathscr{L}(\mu)$ will be said to have *property* (I.P.) if the following inner product formula is valid:

$$(Q^{\mathbf{H}}f, Q^{\mathbf{H}}g)_{G/\Gamma}$$

$$= \frac{1}{\lambda + \bar{\mu}}\{e^{(\lambda+\bar{\mu})N}(\psi, \psi') - e^{-(\lambda+\bar{\mu})N}(\varphi, \varphi')\}$$

$$+ \frac{1}{\lambda - \bar{\mu}}\{e^{(\lambda-\bar{\mu})N}(\psi, \varphi') - e^{-(\lambda+\bar{\mu})N}(\varphi, \psi')\}.$$

Here, of course,

$$(C - T)(f) = (\varphi, \psi) \quad \text{and} \quad (C - T)(g) = (\varphi', \psi').$$

Define

$$\mathscr{L}_0(\lambda) = \{f \in \mathscr{L}(\lambda) \mid \forall \mu \neq \pm\bar{\lambda}, \forall g \in \mathscr{L}(\mu),$$
$$(f, g) \text{ has property (I.P.)}\}.$$

LEMMA 4. *If* $\operatorname{Re}(z) < 0$ *and if* $\mathbf{c}(\zeta)\varphi$ *and* $\mathbf{E}(\varphi, \zeta)$ *are both holomorphic at* $\zeta = z$ *for every* $\varphi \in \mathscr{E}_{\text{cus}}(\delta, \mathscr{O})$, *then the map*

$$\varphi \mapsto \mathbf{E}(\varphi, z): \mathscr{E}_{\text{cus}}(\delta, \mathscr{O}) \rightarrow \mathscr{L}(z)$$

is an isomorphism. In particular, $\mathscr{L}_0(z) = \mathscr{L}(z)$.

Proof. It follows from Lemma 1 and the subsequent discussion that the map in question is linear and injective. Therefore, it will be enough to prove surjectivity. Suppose that $f \in \mathscr{L}(z)$ has the property that $(C - T)(f) = (\varphi, \psi)$. Then

$$(C - T)(f - \mathbf{E}(\varphi, z)) = (0, \psi - \mathbf{c}(z)\varphi).$$

By Lemma 2, $f - \mathbf{E}(\varphi, z)$ belongs to $L^2(G/\Gamma)$. Moreover, by an elementary computation,

$$\operatorname{Res}(\square, \zeta^2)(f - \mathbf{E}(\varphi, z)) = \frac{1}{\zeta^2 - z^2}(f - \mathbf{E}(\varphi, z)).$$

Referring back to the proof of Lemma 3, we see that $\mathbf{c}(\zeta)\varphi$ analytic at $\zeta = z$ forces $\operatorname{Res}(\square, \zeta^2)$ to be analytic at $\zeta = z$. Hence $f = \mathbf{E}(\varphi, z)$.

Let $z \in [-|\rho|, 0)$. Suppose that (ζ_n) and (ζ'_n) are two sequences in D converging to z so that, as operators, $\mathbf{c}(\zeta_n) \rightarrow \mathbf{c}$ and $\mathbf{c}(\zeta'_n) \rightarrow \mathbf{c}'$. By Lemma 2 there is a real number r so that in $S_r^{\infty}(G/\Gamma)$,

$$\mathbf{E}(\varphi, \zeta_n) \rightarrow f \quad \text{and} \quad \mathbf{E}(\varphi', \zeta'_n) \rightarrow g$$

for some $f, g \in S_r^{\infty}(G/\Gamma)$. From property (I.P.) and the fact that

$$Q^{\mathbf{H}}: S_r^{\infty}(G/\Gamma) \rightarrow L^2(G/\Gamma)$$

https://doi.org/10.4153/CJM-1988-022-5 Published online by Cambridge University Press

is continuous, it follows that

$$\begin{aligned}
&(Q^{\mathbf{H}}\mathbf{E}(\boldsymbol{\varphi}, \zeta_n), Q^{\mathbf{H}}g) \\
&= \lim_{m\to\infty} (Q^{\mathbf{H}}\mathbf{E}(\boldsymbol{\varphi}, \zeta_n), Q^{\mathbf{H}}\mathbf{E}(\boldsymbol{\varphi}', \zeta'_m)) \\
&= \frac{1}{\zeta_n + z}\{e^{(\zeta_n+z)N}(\mathbf{c}(\zeta_n)\boldsymbol{\varphi}, \mathbf{c}'\boldsymbol{\varphi}') - e^{-(\zeta_n+z)N}(\boldsymbol{\varphi}, \boldsymbol{\varphi}')\} \\
&+ \frac{1}{\zeta_n - z}\{e^{(\zeta_n-z)N}(\mathbf{c}(\zeta_n)\boldsymbol{\varphi}, \boldsymbol{\varphi}') - e^{-(\zeta_n-z)N}(\boldsymbol{\varphi}, \mathbf{c}'\boldsymbol{\varphi}')\}.
\end{aligned}$$

Letting $n \to \infty$ shows that

$$(\mathbf{c}\boldsymbol{\varphi}, \boldsymbol{\varphi}') = (\boldsymbol{\varphi}, \mathbf{c}'\boldsymbol{\varphi}') \quad \text{for all } \boldsymbol{\varphi}, \boldsymbol{\varphi}' \in \mathscr{E}_{\text{cus}}(\delta, \mathscr{O}).$$

On the other hand, since

$$\lim_{n\to\infty} (Q^{\mathbf{H}}\mathbf{E}(\boldsymbol{\varphi}, \zeta_n), Q^{\mathbf{H}}\mathbf{E}(\boldsymbol{\varphi}', \zeta_n))$$

is finite,

$$(c\boldsymbol{\varphi}, \boldsymbol{\varphi}') = (\boldsymbol{\varphi}, \mathbf{c}\boldsymbol{\varphi}').$$

Therefore $\mathbf{c} = \mathbf{c}'$.

Suppose now that $(\zeta_n) \subseteq D$ with $\zeta_n \to z \in [-|\rho|, 0)$ and $v_n = \|\mathbf{c}(\zeta_n)\| \to \infty$ and $v_n^{-1}\mathbf{c}(\zeta_n) \to \mathbf{c}$ as operators. By Lemma 2 there exists $f \in L^2(G/\Gamma)$ such that

$$\mathbf{E}(v_n^{-1}\boldsymbol{\varphi}, \zeta_n) \to f$$

uniformly on compact subsets of G/Γ with

$$(C - T)(f) = (0, \mathbf{c}\boldsymbol{\varphi}).$$

Since

$$\text{Res}(\Box, \zeta)f = \frac{1}{\zeta - z^2}f,$$

it follows that $\Box f = z^2 f$. Suppose that (z_n) is a sequence in $[-|\rho|, 0)$ converging to $z \neq 0$ such that there exists $(f_n) \subseteq L^2(G/\Gamma)$ with $\Box f_n = z_n^2 f_n$,

$$(C - T)(f_n) = (0, \psi_n) \quad \text{and} \quad \psi_n \to \psi \neq 0.$$

By Lemma 2, $f_n \to f$ in $L^2(G/\Gamma)$ for some $f \in L^2(G/\Gamma)$. However, for $z_n \neq z_m$, f_n is orthogonal to f_m. This forces all but finitely many of the z_n to be equal to z. Hence the set of points in $[-|\rho|, 0)$ where $\mathbf{c}(\zeta)$ is not continuous must be discrete. Let $\{z_n\}$ denote all such points. Suppose that $\text{Re}(z) < 0$ and $z \neq z_n$. It follows that $\mathbf{c}(\zeta)$ and $\mathbf{E}(\boldsymbol{\varphi}, \zeta)$ can be analytically continued to $\zeta = z$. Moreover, $\mathscr{L}_0(z) = \mathscr{L}(z)$. (This argument is due to

Langlands; cf. pp. 129-131 of [5]. It should also be noted that the argument of the previous paragraph could be incorporated into the present one, by considering the difference $\mathbf{E}(\boldsymbol{\varphi}, \zeta_n) - \mathbf{E}(\boldsymbol{\varphi}, \zeta_n').$)

From the proof of Lemma 3, there exists a function $g(\zeta)$, holomorphic in some neighborhood of $[-|\rho|, 0)$, such that

$$\begin{aligned}&(\mathrm{Res}(\square, \zeta^2)\Theta_{\Phi}, \Theta_{\Phi'})_{G/\Gamma}\\ &= g(\zeta) - \frac{e^{2\zeta^2}}{2\zeta}\{ (\boldsymbol{\varphi}, \boldsymbol{\varphi}') + (\mathbf{c}(\zeta)\boldsymbol{\varphi}, \boldsymbol{\varphi}') \},\end{aligned}$$

where

$$\Phi = e^{z^2}\boldsymbol{\varphi} \quad \text{and} \quad \Phi' = e^{z^2}\boldsymbol{\varphi}'.$$

Now

$$\|\mathrm{Res}(\square, \zeta^2)\| \leqq |\mathrm{Im}(\zeta^2)|^{-1} = |2z_n y|^{-1}$$

if $\zeta = z_n + iy$ and $y \in \mathbf{R}^{\times}$. Hence

$$(\mathbf{c}(\zeta)\boldsymbol{\varphi}, \boldsymbol{\varphi}') = O(|y|^{-1}),$$

so that $(\mathbf{c}(\zeta)\boldsymbol{\varphi}, \boldsymbol{\varphi}')$ has a simple pole at z_n.

Suppose that (ζ_n) and (ζ_n') are sequences in D that converge to iy $(y \in \mathbf{R}^{\times})$ such that $\mathbf{c}(\zeta_n) \to \mathbf{c}$ and $\mathbf{c}(\zeta_n') \to \mathbf{c}'$ as operators. From property (I.P.) and the fact that $\zeta_n + \bar{\zeta}_n \to 0$, it follows that

$$\|\mathbf{c}\boldsymbol{\varphi}\|^2 = \lim_{n\to\infty} \|\mathbf{c}(\zeta_n)\boldsymbol{\varphi}\|^2 = \|\boldsymbol{\varphi}\|^2 \quad \text{for all } \boldsymbol{\varphi} \in \mathscr{E}_{\mathrm{cus}}(\delta, \mathscr{O}).$$

Hence $\mathbf{c}$ is unitary. On the other hand,

$$(\mathbf{c}\boldsymbol{\varphi}, \mathbf{c}'\boldsymbol{\varphi}') = \lim_{n\to\infty} (\mathbf{c}(\zeta_n)\boldsymbol{\varphi}, \mathbf{c}(\zeta_n')\boldsymbol{\varphi}') = (\boldsymbol{\varphi}, \boldsymbol{\varphi}')$$

for all $\boldsymbol{\varphi}, \boldsymbol{\varphi}' \in \mathscr{E}_{\mathrm{cus}}(\delta, \mathscr{O})$, by property (I.P.). Thus

$$(\boldsymbol{\varphi}, \boldsymbol{\varphi}') = (\boldsymbol{\varphi}, \mathbf{c}^{-1}\mathbf{c}'\boldsymbol{\varphi}'),$$

forcing $\mathbf{c} = \mathbf{c}'$.

If $y \in \mathbf{R}^{\times}$, define

$$\mathbf{c}(iy) = \lim_{\substack{\zeta\to iy\\ \zeta\in D}} \mathbf{c}(\zeta).$$

Since $\mathbf{c}(\zeta)^* = \mathbf{c}(\bar{\zeta})$ and

$$\mathbf{c}(iy)^{-1} = \lim_{\substack{\zeta\to iy\\ \zeta\in D}} \mathbf{c}(\zeta)^*,$$

$\mathbf{c}(iy)\mathbf{c}(-iy) = I$. If $\mathrm{Re}(\zeta) > 0$, define $\mathbf{c}(\zeta) = \mathbf{c}(-\zeta)^{-1}$. This provides the continuation of $\mathbf{c}(\zeta)$ to $\mathbf{C}^{\times}$ as a meromorphic function.

Define

$$\mathbf{E}(\varphi, iy) = \lim_{\substack{\zeta \to iy \\ \zeta \in D}} \mathbf{E}(\varphi, \zeta) \quad (y \in \mathbf{R}^{\times}).$$

$$\begin{aligned}(C - T)\mathbf{E}(\varphi, iy) &= (\varphi, \mathbf{c}(iy)\varphi) \\ &= (\mathbf{c}(-iy)\mathbf{c}(iy)\varphi, \mathbf{c}(iy)\varphi) \\ &= (C - T)\mathbf{E}(\mathbf{c}(iy)\varphi, -iy),\end{aligned}$$

so Lemma 1 gives

$$\mathbf{E}(\varphi, iy) = \mathbf{E}(\mathbf{c}(iy)\varphi, -iy).$$

Define $\mathbf{E}(\varphi, \zeta) = \mathbf{E}(\mathbf{c}(\zeta)\varphi, -\zeta)$ when $\mathrm{Re}(\zeta) > 0$. This provides the continuation of $\mathbf{E}(\varphi, \zeta)$ to $\mathbf{C}^{\times}$ as a meromorphic function so that $\mathbf{E}(\varphi, \zeta)$ and $\mathbf{c}(\zeta)\varphi$ have the same poles. Moreover, $\mathbf{E}(\varphi, z) \in \mathscr{L}_0(z)$ whenever $\mathbf{E}(\varphi, \zeta)$ is holomorphic at $\zeta = z$.

We shall now prove that $\mathbf{c}(\zeta)$ (and hence $\mathbf{E}(\varphi, \zeta)$) is holomorphic at $\zeta = 0$. Suppose that there exists a sequence $(\zeta_n) \subseteq \mathbf{C}^{\times}$ such that

$$|\zeta_1| > |\zeta_2| > |\zeta_3| > \ldots, \zeta_n \to 0$$

and for each n there exists $\varphi_n \in \mathscr{E}_{\mathrm{cus}}(\delta, \mathscr{O})$ such that $\mathbf{c}(\zeta_n)\varphi_n \to 0$ but $\varphi_n \to \varphi \neq 0$. By Lemma 2, there exists a constant r and a function $f \in S_r^{\infty}(G/\Gamma)$ so that

$$\mathbf{E}(\varphi_n, \zeta_n) \to f \quad \text{in } S_r^{\infty}(G/\Gamma).$$

Hence

$$Q^{\mathbf{H}}\mathbf{E}(\varphi, \zeta_n) \to Q^{\mathbf{H}}f \quad \text{in } L^2(G/\Gamma).$$

Consider property (I.P.), $n \neq m$:

$$\begin{aligned}&(Q^{\mathbf{H}}\mathbf{E}(\varphi_n, \zeta_n), Q^{\mathbf{H}}\mathbf{E}(\varphi_m, \zeta_m)) \\ &= \frac{1}{\zeta_n + \bar{\zeta}_m}\{e^{(\zeta_n + \bar{\zeta}_m)N}(\mathbf{c}(\zeta_n)\varphi_n, \mathbf{c}(\zeta_m)\varphi_m) - e^{-(\zeta_n + \bar{\zeta}_m)N}(\varphi_n, \varphi_m)\} \\ &+ \frac{1}{\zeta_n - \bar{\zeta}_m}\{e^{(\zeta_n - \bar{\zeta}_m)N}(\mathbf{c}(\zeta_n)\varphi_n, \varphi_m) - e^{-(\zeta_n - \bar{\zeta}_m)N}(\varphi_n, \mathbf{c}(\zeta_m)\varphi_m)\}.\end{aligned}$$

Let $m \to \infty$ to obtain

$$(Q^{\mathbf{H}}\mathbf{E}(\varphi_n, \zeta_n), Q^{\mathbf{H}}f) = \frac{1}{\zeta_n}\{e^{\zeta_n N}(\mathbf{c}(\zeta_n)\varphi_n, \varphi) - e^{-\zeta_n N}(\varphi_n, \varphi)\}.$$

Let $n \to \infty$ to obtain a contradiction. It follows from the functional equation $\mathbf{c}(\zeta)\mathbf{c}(-\zeta) = I$ that neither the zeros nor the poles of $\mathbf{c}(\zeta)$ accumulate at the origin. However, the argument supra shows that $\|\mathbf{c}(\zeta)\|$

is bounded for all ζ in some neighborhood of the origin. Hence the origin is a removable singularity.

Let $z_1, \ldots, z_n$ be the poles of $\mathbf{c}(\zeta)$ in $[-|\rho|, 0)$. The formula for the inner product of two wave packets then takes the form:

$$(\Theta_\Phi, \Theta_\Psi)_{G/\Gamma} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{ (\Phi(iy), \Psi(iy)) + (\mathbf{c}(iy)\Phi(iy), \Psi(-iy)) \} dy$$

$$- \sum_{v=1}^{n} (\mathbf{c}_{\text{res}}(z_v)\Phi(z_v), \Psi(z_v)),$$

where $\mathbf{c}_{\text{res}}(z_v)$ is the residue of $\mathbf{c}(\zeta)$ at $\zeta = z_v$.

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