ON THE INDEX OF A SYMMETRIC FORM

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Let E be a finite dimensional vector space over a finite field of characteristic $p > 0$; $\dim E = n$. Let (x, y) be a symmetric bilinear form in E. The radical E_0 of this form is the subspace consisting of all the vectors x which satisfy $(x, y) = 0$ for every $y \in E$. The rank r of our form is the codimension of the radical. Thus

$$r = n - \dim E_0 .$$

If V is any subspace of E, the vectors $x \in E$ for which $(x, v) = 0$ for all $v \in V$ form a vector space N, the subspace normal to V. If $E_0 \subset V$, V will also be the subspace normal to N and

(1) $$\dim V + \dim N = \dim E + \dim E_0 = 2n - r .$$

The subspace V is called totally isotropic if $V \subset N$. The maximum dimension i of a totally isotropic subspace is called the index of our form [Jonathan Wild, Canad. Math. Bull. 1 (1958), 180]. We wish to show that

(2) $$i = \left[n - \frac{r}{2}\right] \quad \text{if } p = 2$$

and

(3) $$n - 1 - \frac{r}{2} \leq i \leq n - \frac{r}{2} \quad \text{if } p > 2 .$$

The bracket indicates the largest integer not greater than $n - \frac{r}{2}$. The second formula implies : Let $p > 2$. Then

$$i = \left[n - \frac{r}{2}\right] \quad \text{if } r \text{ is odd} ,$$

$$i = n - \frac{r}{2} \quad \text{or} \quad i = n - \frac{r}{2} - 1 \quad \text{if } r \text{ is even} .$$

In order to prove (2), we consider a totally isotropic subspace V of maximum dimension i and the subspace N normal to V. Since $V \subset N$ there exists a subspace M such that

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(4) $$N = V \dotplus M .$$

By (4) and (1)

(5) $\dim M = \dim N - \dim V = 2n - r - 2 \dim V = 2n - r - 2i$.

Let $x \in M$. Since $x \in N$, we have $(x, v) = 0$ for every $v \in V$.

Suppose the vector x is isotropic, i.e.

(6) $$(x, x) = 0 .$$

Consider the space W spanned by V and x. Since $x \in N$, (4) implies that V is a proper subspace of W. Any two vectors w and w' of W permit representations

$$w = v + \lambda x \quad , \quad w' = v' + \lambda' x \quad ; \quad v, v' \in V \quad .$$

Thus

$$(w, w') = (v + \lambda x , v' + \lambda' x) = (v, v') + \lambda(x, v') + \lambda'(v, x) + \lambda \lambda'(x, x)$$

$$= 0 + \lambda . 0 + \lambda' . 0 + \lambda \lambda' . 0 = 0 .$$

Hence W would be a totally isotropic subspace of a dimension greater than i. Since this is impossible, (6) is false. Thus M contains no isotropic vectors.

It has been observed by P. Scherk that any two-space over a finite field of characteristic two and any three-space over a finite field of characteristic $p > 2$ must contain isotropic vectors [Canad. Math. Bull. 2 (1959), 45-46] . Hence

(7) $0 \leqslant \dim M \leqslant 1$ if $p = 2$ and $0 \leqslant \dim M \leqslant 2$ if $p > 2$.

Combining (7) with (5) we obtain

$$0 \leq 2n - r - 2i \leq 1 \qquad \text{if } p = 2$$

$$0 \leq 2n - r - 2i \leq 2 \qquad \text{if } p > 2$$

or

$$n - \frac{r+1}{2} \leqslant i \leqslant n - \frac{r}{2} \quad \text{if } p = 2,$$

$$n - \frac{r}{2} - 1 \leqslant i \leqslant n - \frac{r}{2} \quad \text{if } p > 2.$$

This proves (2).

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