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AN EXTRAPOLATION THEOREM FOR CONTRACTIONS WITH FIXED POINTS

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1. **Introduction.** In [9] de la Torre proved that if $(X, \mathcal{F}, \mu)$ is a finite measure space and T is a linear operator on a real $L_p(X, \mathcal{F}, \mu)$ for some fixed p, $1<p<\infty$, such that $\|T\|_p \le 1$ and simultaneously $\|T\|_\infty \le 1$, and also such that there exists $h \in L_p(X, \mathcal{F}, \mu)$ with $Th = h$ and $h \neq 0$ a.e., then the dominated ergodic theorem holds for T, i.e. for every $f \in L_p(X, \mathcal{F}, \mu)$ we have

$$\left\| \sup_n \frac{1}{n} \left| \sum_{i=0}^{n-1} T^i f \right| \right\|_p \le \frac{p}{p-1} \|f\|_p.$$

de la Torre proved his result, by showing that the operator S, defined by $Sf = (\operatorname{sgn} h) \cdot T(f \cdot \operatorname{sgn} h)$ for $f \in L_p(X, \mathcal{F}, \mu)$, is positive, and by applying Akcoglu's theorem [1] to S.

In this paper we shall show that such an operator may be regarded as a Dunford–Schwartz operator on $L_1(X, \mathcal{F}, \mu)$, i.e. $\|T\|_1 \le 1$ and simultaneously $\|T\|_\infty \le 1$; therefore de la Torre's result follows from Dunford and Schwartz [5] (see also Garsia [8], Chapter 2). It is important that in the present paper $(X, \mathcal{F}, \mu)$ may be σ-finite (and $L_p(X, \mathcal{F}, \mu)$ may be a complex Banach space). On the other hand, de la Torre's argument does apply for the finite measure space case only.

THEOREM. *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and T a linear operator on an $L_p = L_p(X, \mathcal{F}, \mu)$ for some fixed p, $1<p<\infty$, such that $\|T\|_p \le 1$ and simultaneously $\|Tf\|_\infty \le \|f\|_\infty$ for every $f \in L_p \cap L_\infty$. Assume that there exists $h \in L_p$, $h \neq 0$ a.e., such that $Th = h$. Then*

$$\|Tf\|_1 \le \|f\|_1 \quad \text{for every} \quad f \in L_1 \cap L_p,$$

and thus T is uniquely extended to a Dunford–Schwartz operator on L_1. Furthermore, if we set $\tau f = (\overline{\operatorname{sgn} h}) \cdot T(f \cdot \operatorname{sgn} h)$ for $f \in L_1$, then τ is a positive Dunford–Schwartz operator on L_1, and there exists $g \in L_1 \cap L_\infty$, $g > 0$ a.e., such that $\tau g = g$ and hence $T(g \cdot \operatorname{sgn} h) = g \cdot \operatorname{sgn} h$.

COROLLARY. *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and T a linear operator on an L_p for some fixed p, $1<p<\infty$, such that $\|T\|_p \le 1$ and simultaneously $\|Tf\|_1 \le \|f\|_1$ for every $f \in L_1 \cap L_p$. Assume that there exists $h \in L_p$, $h \neq 0$ a.e., such*

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that $Th=h$. Then $\|Tf\|_\infty \le \|f\|_\infty$ for every $f\in L_p\cap L_\infty$, and thus T is uniquely extended to a Dunford–Schwartz operator on L_1.

2. **Proofs.**

Proof of Theorem. Put $e(x)=\operatorname{sgn} h(x)(=h(x)/|h(x)|)$. Since L_q with $1/p+1/q=1$ is the dual space of L_p, it then follows from Hölder's inequality that

$$\begin{aligned}\|h\|_p^p &= \int he^{-1}\,|h|^{p-1}\,d\mu=\langle h, e^{-1}\,|h|^{p-1}\rangle\\ &=\langle Th, e^{-1}\,|h|^{p-1}\rangle=\langle h, T^*(e^{-1}\,|h|^{p-1})\rangle\\ &\le \|h\|_p\,\|T^*\|_q\,\|e^{-1}\,|h|^{p-1}\|_q\le \|h\|_p\,\|e^{-1}|h|^{p-1}\|_q\\ &=\|h\|_p^p,\end{aligned}$$

so that $T^*(e^{-1}\,|h|^{p-1})=e^{-1}\,|h|^{p-1}$, because there is only one function $f\in L_q$ for which $\int hf\,d\mu=\|h\|_p\,\|f\|_q=\|h\|_p^p$.

On the other hand, since $\|Tf\|_\infty\le\|f\|_\infty$ for every $f\in L_p\cap L_\infty$ (by hypothesis), T^* may be regarded as an operator on L_1, denoted by the same letter T^*, such that $\|T^*\|_1\le 1$. To see this, it suffices to notice that for every $f\in L_1\cap L_q$ we have

$$\begin{aligned}\int |T^*f|\,d\mu &= \int (T^*f)\cdot \operatorname{sgn}\overline{T^*f}\,d\mu=\lim_n \int_{A_n} (T^*f)\cdot\operatorname{sgn}\overline{T^*f}\,d\mu\\ &=\lim_n \langle T(1_{A_n}\cdot\operatorname{sgn}\overline{T^*f}), f\rangle\\ &\le \int |f|\,d\mu \quad (\text{because } \|T(1_{A_n}\cdot\operatorname{sgn}\overline{T^*f})\|_\infty\le 1)\end{aligned}$$

where $A_1\subset A_2\subset\cdots$, $\mu(A_n)<\infty$ for each $n\ge 1$, and $\lim_n A_n=X$. Now, by Chacon and Krengel [4], there exists a positive linear operator P on L_1, called the linear modulus of T^* (on L_1), such that $\|P\|_1\le 1$ and also such that for every $0\le f\in L_1$

$$Pf=\sup\{|T^*g| : g\in L_1 \quad\text{and}\quad |g|\le f\}.$$

Let C and D denote the conservative and dissipative parts (cf. [7]) of X with respect to P. Thus, for every $0\le f\in L_1$, $\sum_{k=0}^\infty P^k f(x)=0$ or ∞ a.e. on C and $\sum_{k=0}^\infty P^k f(x)<\infty$ a.e. on D. It follows that for every $f\in L_1$

$$\lim_n \frac{1}{n}\left|\sum_{k=0}^{n-1} T^{*k}f\right|\le \lim_n \frac{1}{n}\sum_{k=0}^{n-1} P^k\,|f|=0 \quad \text{a.e. on } D.$$

To see that $e^{-1}\,|h|^{p-1}=0$ a.e. on D, let $\varepsilon>0$ be given and choose $f\in L_1\cap L_q$ so that

$$\|f-e^{-1}\,|h|^{p-1}\|_q<\varepsilon.$$

Then from the fact that $\|T^*\|_q = \|T\|_p \leq 1$ we have

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} T^{*k}f - e^{-1}|h|^{p-1}\right\|_q < \varepsilon \qquad (n \geq 1),$$

and by a mean ergodic theorem (cf. [6], p. 662)

$$\lim_n \left\|\frac{1}{n}\sum_{k=0}^{n-1} T^{*k}f - \tilde{f}\right\|_q = 0$$

for some $\tilde{f} \in L_q$. Therefore $\tilde{f} = 0$ a.e. on D, and

$$\|\tilde{f} - e^{-1}|h|^{p-1}\|_q \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this implies that $e^{-1}|h|^{p-1} = 0$ a.e. on D, and thus $\mu(D) = 0$, because $|h| > 0$ a.e. (by hypothesis).

We have proved that $X = C$. Hence, by Akcoglu and Brunel [2], there exists an invariant set $\Gamma \in \mathcal{F}$ with respect to P and a function $s \in L_\infty(\Gamma)$ such that

(i) $|s| = 1$ a.e. on Γ and $T^*f = \bar{s}P(sf)$ for $f \in L_1(\Gamma)$,

(ii) if $\Delta = X - \Gamma$ then $(I - T^*)L_1(\Delta)$ is dense in $L_1(\Delta)$, in the norm topology,

(iii) a function $t \in L_\infty(\Gamma)$, with $|t| = 1$ a.e. on Γ, satisfies $T^*f = \bar{t}P(tf)$ for all $f \in L_1(\Gamma)$ if and only if there exists a function $u \in L_\infty(\Gamma)$, with $|u| = 1$ a.e. on Γ, such that $P^*u = u$ a.e. on Γ and $t = us$.

Since $X = C$, Γ and Δ are invariant sets with respect to P; thus $T^*(1_\Delta e^{-1}|h|^{p-1}) = 1_\Delta e^{-1}|h|^{p-1}$. Using this relation, we now prove that $\mu(\Delta) = 0$. To do this, let $\varepsilon > 0$ be given and take $f \in L_1(\Delta) \cap L_q(\Delta)$ with

$$\|f - 1_\Delta e^{-1}|h|^{p-1}\|_q < \varepsilon.$$

Then

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} T^{*k}f - 1_\Delta e^{-1}|h|^{p-1}\right\|_q < \varepsilon \qquad (n \geq 1)$$

and

$$\lim_n \left\|\frac{1}{n}\sum_{k=0}^{n-1} T^{*k}f - \tilde{f}\right\|_q = 0$$

for some $\tilde{f} \in L_q$. But, by (ii), we have easily that

$$\lim_n \left\|\frac{1}{n}\sum_{k=0}^{n-1} T^{*k}f\right\|_1 = 0.$$

Hence $\tilde{f} = 0$ a.e., and since $\varepsilon > 0$ was arbitrary, $1_\Delta e^{-1}|h|^{p-1} = 0$ a.e. and so $\mu(\Delta) = 0$.

Since we have observed that $X = C = \Gamma$, it follows that $Pf = sT^*(s^{-1}f)$ for all $f \in L_1$. Therefore P is also an operator on L_q such that $\|P\|_q = \|T^*\|_q \leq 1$. Hence, by Akcoglu and Chacon [3], we get

$$\|Pf\|_\infty \leq \|f\|_\infty \quad \text{for every} \quad f \in L_1 \cap L_\infty,$$

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so that the operator τ (on L_∞) adjoint to P (on L_1) satisfies $\|\tau\|_\infty \le 1$ and, for every $f \in L_1 \cap L_\infty$,

$$\tau f = P^* f = s^{-1} T(sf) \quad \text{and} \quad \|\tau f\|_1 \le \|f\|_1.$$

τ is then uniquely extended to a positive linear operator on L_1, denoted by the same letter τ, such that $\|\tau\|_1 \le 1$. (Thus T is also extended to a (unique) Dunford–Schwartz operator on L_1.) By the Riesz convexity theorem, τ may be regarded as a positive linear operator on each L_r, $1 \le r \le \infty$, such that $\|\tau\|_r \le 1$. Since $\tau\,|h| \ge |Th| = |h|$, it then follows that $\tau\,|h| = |h|$ and thus $X = C_\tau$, where C_τ denotes the conservative part of X with respect to τ.

To prove that $s^{-1}h$ is measurable with respect to the σ-field $\mathcal{I}$ of all invariant sets with respect to τ, write $s^{-1}h = (f_1 - f_2) + i(f_3 - f_4)$, where each f_k is nonnegative and $f_1 f_2 = f_3 f_4 = 0$. Since $h = Th = s\tau(s^{-1}h)$, it follows that $\tau(s^{-1}h) = s^{-1}h$ and then $\tau f_k \ge f_k \ge 0$ for each k. Hence $\tau f_k = f_k$ for each k, and (cf. [7], Chapter III) each f_k and $s^{-1}h$ are measurable with respect to $\mathcal{I}$. Using this, we next prove that

$$\tau f = \overline{(\operatorname{sgn} h)} \cdot T(f \cdot \operatorname{sgn} h) \quad \text{for all } f \in L_1.$$

To do so, we now apply Akcoglu and Brunel [2] (see (iii) above). Since $\overline{\operatorname{sgn} h} = (\overline{s^{-1}h}/|h|)s^{-1}$, it may be readily seen that it suffices to check that

$$\tau^*(\overline{s^{-1}h}/|h|) = \overline{s^{-1}h}/|h|.$$

And this is done easily, because $\overline{s^{-1}h}/|h|$ is measurable with respect to $\mathcal{I}$.

Finally we must construct a function $g \in L_1 \cap L_\infty$, $g > 0$ a.e., such that $\tau g = g$. For this purpose, put

$$B_n = \{x : |h(x)| > 1/n\}$$

for each $n \ge 1$. Then $B_n \in \mathcal{I}$, $\mu(B_n) < \infty$ and $\lim_n B_n = X$. Therefore if we set $g = \sum_{n=1}^{\infty} 2^{-n}(1 + \mu(B_n))^{-1} 1_{B_n}$, then $0 < g < 1$ a.e. and $\tau g = g$.

The proof is completed.

Proof of Corollary. This is essentially done in the proof of Theorem (see the first half of the above proof) and omitted here.

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