Canad. Math. Bull. Vol. **60** (1), 2017 pp. 122–130
http://dx.doi.org/10.4153/CMB-2016-052-x

A Homological Property and Arens Regularity of Locally Compact Quantum Groups

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Abstract. We characterize two important notions of amenability and compactness of a locally compact quantum group $\mathbb{G}$ in terms of certain homological properties. For this, we show that $\mathbb{G}$ is character amenable if and only if it is both amenable and co-amenable. We finally apply our results to Arens regularity problems of the quantum group algebra $L^1(\mathbb{G})$. In particular, we improve an interesting result by Hu, Neufang, and Ruan.

1 Introduction and Preliminaries

The class of locally compact quantum groups was first introduced and studied by Kustermans and Vaes [10, 11]. Recall that a quadruple $\mathbb{G} = (L^\infty(\mathbb{G}), \Gamma, \varphi, \psi)$ is called a (*von Neumann algebraic*) *locally compact quantum group*, where $L^\infty(\mathbb{G})$ is a von Neumann algebra with identity element 1 and a co-multiplication

$$\Gamma\colon L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G}).$$

Moreover, φ and ψ are normal faithful semifinite left and right Haar weights on $L^\infty(\mathbb{G})$, respectively. Here $\overline{\otimes}$ denotes the von Neumann algebra tensor product. The predual of $L^\infty(\mathbb{G})$ is denoted by $L^1(\mathbb{G})$. Then the pre-adjoint of Γ induces an associative completely contractive multiplication $*$ on $L^1(\mathbb{G})$ given by $\langle f * g, x\rangle = \langle f \otimes g, \Gamma x\rangle$ for all $f, g \in L^1(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$. Therefore, $L^1(\mathbb{G})$ is a Banach algebra under the multiplication. Moreover, the module actions of $L^1(\mathbb{G})$ on $L^\infty(\mathbb{G})$ are given by

$$f \cdot x := (\mathrm{id} \otimes f)(\Gamma x) \quad \text{and} \quad x \cdot f := (f \otimes \mathrm{id})(\Gamma x)$$

for all $f \in L^1(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$.

Locally compact quantum groups allow for the development of a duality theory that extends Pontryagin duality for locally compact abelian groups. The dual quantum group of $\mathbb{G} = (L^\infty(\mathbb{G}), \Gamma, \phi, \psi)$ is denoted by $\widehat{\mathbb{G}} = (L^\infty(\widehat{\mathbb{G}}), \widehat{\Gamma}, \widehat{\phi}, \widehat{\psi})$ and $\widehat{\mathbb{G}}$ is a locally compact quantum group. In particular, $\widehat{\widehat{\mathbb{G}}} = \mathbb{G}$; for more details see [10, 11].

Moreover, it is worthwhile to mention that there exist two classical locally compact quantum groups $\mathbb{G}_a = (L^\infty(G), \Gamma_a, \phi_a, \psi_a)$ and $\mathbb{G}_s = (VN(G), \Gamma_s, \phi_s, \psi_s)$ obtained

Received by the editors August 12, 2015; revised November 28, 2015.
Published electronically October 4, 2016.
Author M. R. G. was supported in part by a grant from IPM (No. 93470067). Author R. N.-I. was supported in part by a grant from IPM (No. 94430417). Author M. N. was supported in part by a grant from IPM (No. 93470058).
AMS subject classification: **46L89**, 43A07, 46H20, 46M10, 58B32.
Keywords: amenability, Arens regularity, co-amenability, locally compact quantum group, homological property.

from a locally compact group G, where $L^\infty(G)$ is the usual Lebesgue space with predual $L^1(G)$ and $VN(G)$ is the group von Neumann algebra of G with predual $A(G)$, the Fourier algebra of G; also, $\widehat{\mathbb{G}}_a = \mathbb{G}_s$. For more details see [15].

The concepts of amenability and co-amenability for locally compact quantum groups were introduced by Bédos and Tuset [2]. Recall that $\mathbb{G}$ is called *amenable* if there exists a *left invariant mean* M on $L^\infty(\mathbb{G})$, *i.e.*, a bounded linear functional with $\|M\| = M(1) = 1$ such that $M(x \cdot f) = f(1)M(x)$ for all $x \in L^\infty(\mathbb{G})$ and $f \in L^1(\mathbb{G})$. Right invariant means are defined similarly. For further works related to amenability of locally compact quantum groups see [3,16,18,19]. We denote by $\mathrm{sp}(L^1(\mathbb{G}))$ the set of all non-zero characters on $L^1(\mathbb{G})$. Then

$$\mathrm{sp}(L^1(\mathbb{G})) = \{y \in L^\infty(\mathbb{G}) : \Gamma(y) = y \otimes y \text{ and } y \text{ is invertible}\}.$$

We note that $\mathrm{sp}(L^1(\mathbb{G}))$ is a group, called the *intrinsic group*, and every element of this group is unitary; see [7, Theorem 3.9].

The paper is organized as follows. In Section 2, for a locally compact quantum group $\mathbb{G}$ and for each $y \in \mathrm{sp}(L^1(\mathbb{G}))$, we study y-amenability and character amenability of $\mathbb{G}$. As an application of this result, we show that $\mathbb{G}$ is co-amenable and amenable if and only if $\mathbb{G}$ is character amenable. In Section 3, we obtain a homological characterization of amenability and compactness of $\mathbb{G}$. In Section 4, among the other things, we prove that if $\mathbb{G}$ is amenable and co-amenable, then the quantum group algebra $L^1(\mathbb{G})$ is Arens regular if and only if $\mathbb{G}$ is finite, which in particular improves an interesting result by Hu, Neufang, and Ruan [6, Theorem 3.10].

2 Character Amenability

For a non-zero character φ on a Banach algebra $\mathcal{A}$, Kaniuth, Lau, and Pym [8, 9] introduced and investigated a notion of amenability for $\mathcal{A}$ called *φ-amenability*. In fact, $\mathcal{A}$ is called φ-amenable if there exists a bounded linear functional M on $\mathcal{A}^*$ satisfying $M(\varphi) = 1$ and $M(x \cdot a) = \varphi(a)M(x)$ for all $a \in \mathcal{A}$ and $x \in \mathcal{A}^*$. Around the same time, Monfared [13] introduced and studied *character amenability* of Banach algebras. He defined the Banach algebra $\mathcal{A}$ to be character amenable if it is φ-amenable for all non-zero character φ on $\mathcal{A}$ and has a bounded right approximate identity.

In harmonic analysis, the interest in character amenability arises from the fact that for a locally compact group G, the character amenability of both the group algebra $L^1(G)$ and the Fourier algebra $A(G)$ are completely determined by the amenability of G; see [13].

For $y \in \mathrm{sp}(L^1(\mathbb{G}))$, we simply say that $\mathbb{G}$ is *y-amenable* if $L^1(\mathbb{G})$ is y-amenable, *i.e.*, there is a functional $M \in L^\infty(\mathbb{G})^*$ satisfying $M(y) = 1$ and $M(x \cdot f) = M(x)f(y)$ for all $x \in L^\infty(\mathbb{G})$ and $f \in L^1(\mathbb{G})$. We call any such M a *left invariant y-mean*. Similarly we can define a right invariant y-mean. A functional $M \in L^\infty(\mathbb{G})^*$ is invariant y-mean if it is both left and right invariant y-mean. We also say that $\mathbb{G}$ is *character amenable* if $\mathbb{G}$ is co-amenable and y-amenable for all $y \in \mathrm{sp}(L^1(\mathbb{G}))$. Recall that $\mathbb{G}$ is called *co-amenable* if $L^1(\mathbb{G})$ has a bounded approximate identity.

In the sequel, we show that $\mathbb{G}$ is character amenable if and only if it is both amenable and co-amenable. To this end, we need the following lemma. First, for each $y \in$

$\mathrm{sp}(L^1(\mathbb{G}))$ we define $L_y\colon L^1(\mathbb{G}) \to L^1(\mathbb{G})$ by $L_y(f)(x) = \langle f, y^{-1}x\rangle$ for all $x \in L^\infty(\mathbb{G})$ and $f \in L^1(\mathbb{G})$.

Lemma 2.1 *Let $\mathbb{G}$ be a locally compact quantum group. Then $y(x\cdot f) = (yx)\cdot L_y(f)$, for all $y \in \mathrm{sp}(L^1(\mathbb{G}))$, $x \in L^\infty(\mathbb{G})$, and $f, g \in L^1(\mathbb{G})$.*

Proof First we show that $L_y(f * g) = L_y(f) * L_y(g)$ for all $y \in \mathrm{sp}(L^1(\mathbb{G}))$ and $f, g \in L^1(\mathbb{G})$. Indeed, for each $x \in L^\infty(\mathbb{G})$ we have

$$\begin{aligned}\langle x, L_y(f * g)\rangle &= \langle \Gamma(y^{-1}x), f \otimes g\rangle = \langle (y^{-1} \otimes y^{-1})\Gamma(x), f \otimes g\rangle \\ &= \langle \Gamma(x), L_y(f) \otimes L_y(g)\rangle = \langle x, L_y(f) * L_y(g)\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle y(x\cdot f), g\rangle &= \langle x\cdot f, L_{y^{-1}}(g)\rangle = \langle x, f * L_{y^{-1}}(g)\rangle \\ &= \langle x, L_{y^{-1}}(L_y(f) * g)\rangle = \langle (yx)\cdot L_y(f), g\rangle.\end{aligned}$$

This shows that $y(x\cdot f) = (yx)\cdot L_y(f)$ for all $y \in \mathrm{sp}(L^1(\mathbb{G}))$, $x \in L^\infty(\mathbb{G})$, and $f \in L^1(\mathbb{G})$. ■

A standard argument, used in the proof of [12, Theorem 4.1] on F-algebras, a class of Banach algebras including all convolution quantum group algebras, shows that y-amenability of $\mathbb{G}$ is equivalent to the existence of a left invariant y-mean on $L^\infty(\mathbb{G})$ with norm one; see also [8, Remark 1.3]. In particular, 1-amenability coincides with amenability of $\mathbb{G}$.

Theorem 2.2 *Let $\mathbb{G}$ be a locally compact quantum group. Then the following statements are equivalent.*

(i) *$\mathbb{G}$ is amenable.*
(ii) *$\mathbb{G}$ is y-amenable for all $y \in \mathrm{sp}(L^1(\mathbb{G}))$.*
(iii) *$\mathbb{G}$ is y_0-amenable for some $y_0 \in \mathrm{sp}(L^1(\mathbb{G}))$.*

Proof (i) ⇒ (ii). Suppose that $M \in L^\infty(\mathbb{G})^*$ is a left invariant mean, and $y \in \mathrm{sp}(L^1(\mathbb{G}))$. Define $\widetilde{M} \in L^\infty(\mathbb{G})^*$ by $\widetilde{M}(x) = M(y^{-1}x)$, for all $x \in L^\infty(\mathbb{G})$. Then by Lemma 2.1 we have

$$\begin{aligned}\widetilde{M}(x\cdot f) &= M(y^{-1}(x\cdot f)) = M((y^{-1}x)\cdot L_{y^{-1}}(f)) \\ &= M(y^{-1}x)L_{y^{-1}}(f)(1) = \widetilde{M}(x)f(y),\end{aligned}$$

for all $x \in L^\infty(\mathbb{G})$ and $f \in L^1(\mathbb{G})$. Moreover, it is easy to check that $\|\widetilde{M}\| = \widetilde{M}(y) = 1$. Therefore, $\mathbb{G}$ is y-amenable. That (iii) implies (i) is established in exactly the same way. Since $1 \in \mathrm{sp}(L^1(\mathbb{G}))$, the implication that (ii) ⇒ (iii) is trivial. ■

The results below are immediate by Theorem 2.2 together with the fact that the existence of a left invariant mean and the existence of an invariant mean are both equivalent to $\mathbb{G}$ being amenable; see [3].

Corollary 2.3 *Let $\mathbb{G}$ be a locally compact quantum group and let $y \in \mathrm{sp}(L^1(\mathbb{G}))$. Then the following statements are equivalent.*

(i) *$\mathbb{G}$ is y-amenable.*
(ii) *There exists an invariant y-mean on $L^\infty(\mathbb{G})$.*

Corollary 2.4 *Let $\mathbb{G}$ be a locally compact quantum group. Then $\mathbb{G}$ is character amenable if and only if $\mathbb{G}$ is amenable and co-amenable.*

Recall from [2, Proposition 3.1] that the compactness of $\mathbb{G}$ is equivalent to the existence of an invariant mean in $L^1(\mathbb{G})$. We have the following characterization for compact quantum groups whose proof is similar to that of Theorem 2.2, and so the proof is omitted.

Proposition 2.5 *Let $\mathbb{G}$ be a locally compact quantum group. Then the following statements are equivalent.*

(i) *$\mathbb{G}$ is compact.*
(ii) *There is an invariant y-mean in $L^1(\mathbb{G})$ for all $y \in \operatorname{sp}(L^1(\mathbb{G}))$.*
(iii) *There is an invariant y_0-mean in $L^1(\mathbb{G})$ for some $y_0 \in \operatorname{sp}(L^1(\mathbb{G}))$.*

As a consequence of Proposition 2.5, we have the following result, which is interesting in its own right.

Theorem 2.6 *Let $\mathbb{G}$ be locally compact quantum group and suppose that $\mathcal{A}$ is a Banach algebra such that $\phi\colon \mathcal{A} \to L^1(\mathbb{G})$ is a weakly compact homomorphism with dense range. If $y \in \operatorname{sp}(L^1(\mathbb{G}))$ and $\mathcal{A}$ is $y \circ \phi$-amenable, then $\mathbb{G}$ is compact.*

Proof Suppose that $\varphi = y \circ \phi$. Then by assumption and [8, Theorem 1.4] there is a bounded net (a_α) in $\mathcal{A}$ such that $\|aa_\alpha - \varphi(a)a_\alpha\| \to 0$ for all $a \in \mathcal{A}$ and $\varphi(a_\alpha) = 1$ for all α. Since ϕ is weakly compact, we can assume that the net $(\phi(a_\alpha))$ is weakly convergent to some element f in $L^1(\mathbb{G})$. Thus, for each $a \in \mathcal{A}$, the net $(\phi(a) * \phi(a_\alpha))$ weakly converges to $\phi(a) * f$, which implies that $\phi(a) * f = y(\phi(a))f$. Since $\phi(\mathcal{A})$ is norm dense in $L^1(\mathbb{G})$, it follows that $g * f = g(y)f$ for all $g \in L^1(\mathbb{G})$. Hence, $L^1(\mathbb{G})$ has an invariant y-mean and consequently $\mathbb{G}$ is compact by Proposition 2.5. ∎

Example 2.7 Let $\mathbb{Z}$ be the additive group of integers and for each $0 < \alpha < 1$, define the weight function ω_α on $\mathbb{Z}$ by $\omega_\alpha(n) = (1 + |n|)^\alpha$ for all $n \in \mathbb{Z}$. Let

$$L^1(\mathbb{Z}, \omega_\alpha) := \Big\{ f = (f(n))_{n\in\mathbb{Z}} : \|f\|_\alpha := \sum_{n\in\mathbb{Z}} f(n)\omega(n) < \infty \Big\}.$$

Then $L^1(\mathbb{Z}, \omega_\alpha)$ is a Banach algebra with respect to the convolution product and the norm $\|\cdot\|_\alpha$. For any z in the circle group $\mathbb{T}$, define $y_z\colon L^1(\mathbb{Z}_a) \to \mathbb{C}$ by

$$y_z(f) = \sum_{n\in\mathbb{Z}} f(n)z^n$$

for all $f \in L^1(\mathbb{Z}_a)$. Then $\operatorname{sp}(L^1(\mathbb{Z}_a)) = \operatorname{sp}(L^1(\mathbb{Z}, \omega_\alpha)) = \{y_z : z \in \mathbb{T}\}$; see, for example, [14, p. 291, Exercise 17]. It is clear that $L^1(\mathbb{Z}, \omega_\alpha) \subseteq L^1(\mathbb{Z}_a)$ and $\|f\|_1 \le \|f\|_\alpha$ for all $f \in L^1(\mathbb{Z}, \omega_\alpha)$. Moreover, for each $f \in L^1(\mathbb{Z}, \omega_\alpha)$, there is $N \in \mathbb{N}$ for which

$$|f(n)|(1 + |n|)^\alpha \le \frac{1}{1 + |n|}$$

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for all $|n| > N$. Therefore,

$$\sum_{|n|>N} |f(n)| \le \sum_{|n|>N} \frac{1}{(1+|n|)^{1+\alpha}}.$$

This shows that the inclusion map $\iota: L^1(\mathbb{Z}, \omega_\alpha) \to L^1(\mathbb{Z}_a)$ is a compact homomorphism with dense range. Since $\mathbb{Z}_a$ is not compact, it follows from Theorem 2.6 that $L^1(\mathbb{Z}, \omega_\alpha)$ is not $y_z \circ \iota$-amenable for all $z \in \mathbb{T}$.

3 Homological Characterization of Amenability and Compactness

For $y_0 \in \operatorname{sp}(L^1(\mathbb{G}))$, we say that $\mathbb{G}$ is *y-biflat* [*y-biprojective*] if $L^1(\mathbb{G})$ is y-biflat [y-biprojective] in the sense of [17], *i.e.*, there exists a bounded $L^1(\mathbb{G})$-bimodule morphism $\rho: L^1(\mathbb{G}) \to (L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G}))^{**}$ [$\rho: L^1(\mathbb{G}) \to L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G})$] such that for each $f \in L^1(\mathbb{G})$, $\langle(\pi^{**}\circ\rho)(f), \widetilde{y}\rangle = \langle f, y\rangle$ [$\langle(\pi\circ\rho)(f), y\rangle = \langle f, y\rangle$], where $\widetilde{y}$ is the unique extension of y on $L^1(\mathbb{G})^{**}$. Moreover, define $I_{0,y}(\mathbb{G})$ to be the y-augmentation ideal in $L^1(\mathbb{G})$,*i.e.*, $I_{0,y}(\mathbb{G}) = \{f \in L^1(\mathbb{G}) : f(y) = 0\}$. Now in this section, we give some characterizations of amenability and compactness in terms of certain homological properties of $L^1(\mathbb{G})$.

Lemma 3.1 *Let $\mathbb{G}$ be a locally compact quantum group and let $y \in \operatorname{sp}(L^1(\mathbb{G}))$. Then the linear span of $L^1(\mathbb{G}) * I_{0,y}(\mathbb{G})$ is dense in $I_{0,y}(\mathbb{G})$.*

Proof Given $\varepsilon > 0$ and $h \in I_{0,y}(\mathbb{G})$, then $L_{y^{-1}}(h) \in I_{0,1}(\mathbb{G})$. Since the linear span of $L^1(\mathbb{G}) * I_{0,1}(\mathbb{G})$ is dense in $I_{0,1}(\mathbb{G})$ by [1, Theorem 4.4], we can find $f_1, \dots, f_n \in L^1(\mathbb{G})$ and $g_1, \dots, g_n \in I_{0,1}(\mathbb{G})$ such that

$$\Big\| \sum_{i=1}^n f_i * g_i - L_{y^{-1}}(h) \Big\| < \varepsilon.$$

As shown in the proof of Lemma 2.1, we have $L_y(\sum_{i=1}^n f_i * g_i) = \sum_{i=1}^n L_y(f_i) * L_y(g_i)$. Moreover, it is clear that $L_y(g_i) \in I_{0,y}(\mathbb{G})$ for $1 \le i \le n$. It follows that

$$\begin{aligned}\Big\| \sum_{i=1}^n L_y(f_i) * L_y(g_i) - h \Big\| &= \Big\| L_y\Big(\sum_{i=1}^n f_i * g_i\Big) - L_y(L_{y^{-1}}(h)) \Big\| \\ &\le \Big\| \sum_{i=1}^n f_i * g_i - L_{y^{-1}}(h) \Big\| < \varepsilon,\end{aligned}$$

which gives the result. ∎

We end this section with our main result.

Theorem 3.2 *Let $\mathbb{G}$ be a locally compact quantum group and let $y \in \operatorname{sp}(L^1(\mathbb{G}))$. Then*

(i) *$\mathbb{G}$ is y-biflat if and only if $\mathbb{G}$ is amenable;*
(ii) *$\mathbb{G}$ is y-biprojective if and only if $\mathbb{G}$ is compact.*

Proof Suppose that $\mathbb{G}$ is y-biflat. Then there is a continuous $L^1(\mathbb{G})$-bimodule morphism $\rho: L^1(\mathbb{G}) \to (L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G}))^{**}$ for which $\langle(\pi^{**}\circ\rho)(f), \widetilde{y}\rangle = \langle f, y\rangle$ for all

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$f \in L^1(\mathbb{G})$. Set

$$T = \mathrm{id}_{L^1(\mathbb{G})} \otimes 1 : L^1(\mathbb{G}) \widehat{\otimes} L^1(\mathbb{G}) \to L^1(\mathbb{G}), \quad f \otimes g \mapsto g(1) f$$

and define the operator $\rho_1 : L^1(\mathbb{G}) \to L^1(\mathbb{G})^{**}$ by $\rho_1 := T^{**} \circ \rho$. It is trivial that ρ_1 is a left $L^1(\mathbb{G})$-module morphism. On the other hand, for each $f \in L^1(\mathbb{G})$ and $g \in I_{0,y}(\mathbb{G})$, we have $\rho_1(f * g) = \rho_1(f) g(y) = 0$. Thus, $\rho_1 = 0$ on $I_{0,y}(\mathbb{G})$ since $\langle L^1(\mathbb{G}) * I_{0,y}(\mathbb{G}) \rangle$ is dense in $I_{0,y}(\mathbb{G})$; see Lemma 3.1. This shows that ρ_1 induces a left $L^1(\mathbb{G})$-module morphism $\widetilde{\rho} : L^1(\mathbb{G})/I_{0,y}(\mathbb{G}) \to L^1(\mathbb{G})^{**}$. Note that $\mathbb{C} \cong L^1(\mathbb{G})/I_{0,y}(\mathbb{G})$ is a Banach $L^1(\mathbb{G})$-bimodule with the multiplication given by $f \cdot \lambda = \lambda \cdot f = f(y)\lambda$ for all $f \in L^1(\mathbb{G})$ and $\lambda \in \mathbb{C}$. Define the right $L^1(\mathbb{G})$-module morphism $\theta : L^\infty(\mathbb{G}) \to \mathbb{C}$ by $\theta = \widetilde{\rho}^* |_{L^\infty(\mathbb{G})}$. It is easy to check that $\theta \circ y^* = \mathrm{id}_{\mathbb{C}^*}$. This shows that $\mathbb{C}$ is a flat left $L^1(\mathbb{G})$-module. It follows from [1, Theorem 2.3] that $\mathbb{G}$ is amenable. So the proof of (i) is complete. The converse follows from [17, Proposition 2.2 and Lemma 3.1]. The proof of (ii) is similar. ■

4 On a Result by Hu, Neufang, and Ruan

We commence this section with a key lemma.

Lemma 4.1 *Let $\mathbb{G}$ be an amenable locally compact quantum group. Then for each separable subspace $\mathfrak{X}$ of $L^1(\mathbb{G})$, there exists a separable closed subalgebra $\mathfrak{A}$ of $L^1(\mathbb{G})$ including $\mathfrak{X}$ and a sequence of normal states (f_n) such that for each $f \in \mathfrak{A}$,*

$$\|f * f_n - f(1) f_n\| \to 0 \quad \textit{and} \quad \|f_n * f - f(1) f_n\| \to 0. \tag{4.1}$$

Proof Suppose that (g_i) is a dense sequence of the unit ball of $\mathfrak{X}$. Since $\mathbb{G}$ is amenable, there is a net of normal states (f_α) in $L^1(\mathbb{G})$ such that

$$\|f * f_\alpha - f(1) f_\alpha\| \to 0, \quad \|f_\alpha * f - f(1) f_\alpha\| \to 0$$

for all $f \in L^1(\mathbb{G})$; see [12, Theorem 4.6]. We choose inductively a sequence of states (f_n) in $L^1(\mathbb{G})$ such that $\|f_i * f_n - f_n\| < \frac{1}{n}$, $\|f_n * f_i - f_n\| < \frac{1}{n}$ for $1 \le i < n$, and

$$\|g_i * f_n - g_i(1) f_n\| < \frac{1}{n}, \quad \|f_n * g_i - g_i(1) f_n\| < \frac{1}{n}$$

for $1 \le i < n$. Put $\mathfrak{Y} := \{g_n : n \ge 1\} \cup \{f_n : n \ge 1\}$, and define $\mathfrak{A}$ to be the closed subalgebra of $L^1(\mathbb{G})$ generated by $\mathfrak{Y}$. Then it is easily verified that

$$\|f * f_n - f(1) f_n\| \to 0, \quad \|f_n * f - f(1) f_n\| \to 0$$

for all $f \in \mathfrak{A}$, which completes the proof. ■

It is known that for a Banach algebra $\mathcal{A}$ there are two multiplications $\square$ and $\diamond$ on the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$, each extending the multiplication on $\mathcal{A}$. For $m, n \in \mathcal{A}^{**}$ and $x \in \mathcal{A}^*$, the *left Arens product* $m \square n \in \mathcal{A}^{**}$ satisfies $\langle m \square n, x \rangle = \langle m, n \cdot x \rangle$, where $n \cdot x \in \mathcal{A}^*$ is defined by $\langle n \cdot x, a \rangle = \langle n, x \cdot a \rangle$ for all $a \in \mathcal{A}$. Similarly, the *right Arens product* $m \diamond n \in \mathcal{A}^{**}$ satisfies $\langle m \diamond n, x \rangle = \langle n, x \cdot m \rangle$ where $x \cdot m \in \mathcal{A}^*$ is given by $\langle x \diamond m, a \rangle = \langle m, a \cdot x \rangle$ for all $a \in \mathcal{A}$. The Banach algebra $\mathcal{A}$ is called *Arens regular* if $\square$ and $\diamond$ coincide on $\mathcal{A}^{**}$.

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Suppose now that $\mathfrak{A}$ is a subalgebra of $L^1(\mathbb{G})$. We say that $\mathfrak{A}$ *has an invariant mean* if there is a normal state f in $\mathfrak{A}$ such that $f * g = g * f = g(1)f$ for all $g \in \mathfrak{A}$. It was shown by Hu, Neufang, and Ruan [6, Theorem 3.10] that if $\mathbb{G}$ is amenable and $L^1(\mathbb{G})$ is separable and Arens regular, then $\mathbb{G}$ is compact. We show below that this is true without the assumption that $L^1(\mathbb{G})$ is separable.

Theorem 4.2 *Let $\mathbb{G}$ be an amenable locally compact quantum group such that $L^1(\mathbb{G})$ is Arens regular. Then $\mathbb{G}$ is compact.*

Proof Let $\mathfrak{X}_1$ be an arbitrary separable subspace of $L^1(\mathbb{G})$. By Lemma 4.1, $\mathfrak{X}_1$ contained in a closed separable subalgebra $\mathfrak{A}_1$ of $L^1(\mathbb{G})$ with a sequence of normal states (f_n) in $\mathfrak{A}_1$ satisfying the condition (4.1). Suppose now that f and f' are two weak* cluster points of (f_n) in $\mathfrak{A}_1^{**}$. Then $f \square f' = f'$ and $f \diamond f' = f$. As $\mathfrak{A}_1$ is Arens regular, we conclude that $f = f'$. Thus, (f_n) has a unique weak* cluster point f_1 which implies that it is weakly Cauchy in $\mathfrak{A}_1$. Moreover, $\mathfrak{A}_1$ is weakly sequentially complete. Therefore, $f_1 \in \mathfrak{A}_1$, and it is an invariant mean for $\mathfrak{A}_1$. Put

$$\mathfrak{B}_1 = \left\{ g \in L^1(\mathbb{G}) : g * f_1 = f_1 * g = g(1) f_1 \right\},$$

and note that $\mathfrak{B}_1$ is the largest closed subalgebra of $L^1(\mathbb{G})$ that has f_1 as an invariant mean. If $\mathbb{G}$ were not compact, then $L^1(\mathbb{G}) \neq \mathfrak{B}_1$. Given $g_1 \in L^1(\mathbb{G}) \setminus \mathfrak{B}_1$ and let $\mathfrak{X}_2$ be the subspace of $L^1(\mathbb{G})$ generated by $\{g_1, f_1\}$. By the same argument as above, $\mathfrak{X}_2$ is contained in a closed separable subalgebra $\mathfrak{A}_2$ of $L^1(\mathbb{G})$ which has an invariant mean, say f_2. It is trivial that f_2 is not in $\mathfrak{B}_1$ and $f_1 * f_2 = f_2 * f_1 = f_2$. Inductively, there is a linearly independent sequence of idempotent states (f_n) in $L^1(\mathbb{G})$ such that

$$f_i * f_j = f_j * f_i = f_{\max\{i,j\}}$$

for all $i, j \geq 1$. Define $\mathfrak{A}$ to be the closed subalgebra of $L^1(\mathbb{G})$ generated by $\{f_n : n \geq 1\}$. Then $\mathfrak{A}$ is commutative and separable. Suppose that $g = \sum_{i=1}^{m} \lambda_i f_i$, where $\lambda_i \in \mathbb{C}$. Then

$$f_n * g = g * f_n = g(1) f_n$$

for all $n \geq m$. Therefore, the sequence (f_n) satisfies (4.1). Since $\mathfrak{A}$ is weakly sequentially complete and Arens regular, it has an invariant mean, say f. Thus, $f * f_n = f$ for all $n \geq 1$. Moreover, $\|f * f_n - f_n\| \to 0$. Therefore, there is an integer N such that $\|f_n - f\| < 1$ for all $n \geq N$. Since $f_n - f$ is idempotent, it follows that $f_n = f$ for all $n \geq N$ which is a contradiction. Thus, $\mathbb{G}$ is compact. ∎

Recall that a locally compact quantum group $\mathbb{G}$ is said to be *discrete* if $L^1(\mathbb{G})$ has an identity. Moreover, $\mathbb{G}$ is said to be *finite* if $L^\infty(\mathbb{G})$ is finite dimensional, or equivalently $\mathbb{G}$ is compact and discrete. Combining the previous theorem with [20, Theorem 3.3], we obtain the following result.

Corollary 4.3 *Let $\mathbb{G}$ be a co-amenable and amenable locally compact quantum group. Then $L^1(\mathbb{G})$ is Arens regular if and only if $\mathbb{G}$ is finite.*

Example 4.4 It is well known that for a locally compact group G, the Fourier algebra $A(G) = L^1(\mathbb{G}_s)$ is commutative, which implies that $\mathbb{G}_s$ is amenable. Therefore, Arens

https://doi.org/10.4153/CMB-2016-052-x Published online by Cambridge University Press

regularity of $A(G)$ implies discreteness of G. This result was originally obtained in [4, Theorem 3.2].

The result below can be obtained if we modify the proof of Proposition [5, Proposition 16] on $I_{0,1}(\mathbb{G})$.

Proposition 4.5 *Let $\mathbb{G}$ be a locally compact quantum group and let $y \in \operatorname{sp}(L^1(\mathbb{G}))$. Then $\mathbb{G}$ is finite if and only if $I_{0,y}(\mathbb{G})$ has an identity.*

Before giving the following result, let us first recall that a Banach algebra $\mathcal{A}$ is two-sided φ-amenable if there is a functional M in $\mathcal{A}^{**}$ for which $M(\varphi) = 1$ and $M(x \cdot a) = M(a \cdot x) = \varphi(a)M(x)$ for all $a \in \mathcal{A}$ and $x \in \mathcal{A}^*$.

Corollary 4.6 *Let $\mathcal{A}$ be an Arens regular Banach algebra with a bounded approximate identity and let $\mathbb{G}$ be a locally compact quantum group. Suppose that $\phi: \mathcal{A} \to L^1(\mathbb{G})$ is a continuous homomorphism with dense range. If $y \in \operatorname{sp}(L^1(\mathbb{G}))$ and $\mathcal{A}$ is two-sided $y \circ \phi$-amenable, then $\mathbb{G}$ is finite.*

Proof First note that the equality $L^1(\mathbb{G}) = \overline{\phi(\mathcal{A})}$ does not imply that the algebra $L^1(\mathbb{G})$ is Arens regular, so Theorem 4.3 does not apply directly. Suppose now that $\varphi = y \circ \phi \in \operatorname{sp}(\mathcal{A})$ and let $I_{0,\varphi}(\mathcal{A}) = \ker \varphi$. Then we show that $\overline{\phi(I_{0,\varphi}(\mathcal{A}))} = I_{0,y}(\mathbb{G})$. To prove this, let $f \in I_{0,y}(\mathbb{G})$ and choose $\varepsilon > 0$. Then there is $a \in \mathcal{A}$ such that $\|\phi(a) - f\| < \varepsilon$. Thus, $|\varphi(a)| < \varepsilon$. If we now put $c = a - \varphi(a)b$, where $b \in \mathcal{A}$ with $\varphi(b) = 1$ and $\|b\| \leq 2$, then $c \in I_{0,\varphi}(\mathcal{A})$. Moreover,

$$\|f - \phi(c)\| = \|f - \phi(a) + \varphi(a)\phi(b)\| \leq (1 + 2\|\phi\|)\varepsilon,$$

which implies that $I_{0,y}(\mathbb{G}) \subseteq \overline{\phi(I_{0,\varphi}(\mathcal{A}))}$. The reverse inclusion is clear. On the other hand, $I_{0,\varphi}(\mathcal{A})$ has a bounded approximate identity; see, for example, [8, Proposition 2.2]. Hence, [20, Theorem 4.1] yields that the ideal $I_{0,y}(\mathbb{G})$ is unital and therefore $\mathbb{G}$ is finite by Proposition 4.5. ∎

Acknowledgements The authors would like to thank the referee for his very careful reading and his invaluable comments. They acknowledge that this research was partially carried out at the IPM-Isfahan Branch.

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