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THE DOUBLE COVER OF ODD GENERAL SPIN GROUPS, SMALL REPRESENTATIONS, AND APPLICATIONS

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Abstract We construct local and global metaplectic double covers of odd general spin groups, using the cover of Matsumoto of spin groups. Following Kazhdan and Patterson, a local exceptional representation is the unique irreducible quotient of a principal series representation, induced from a certain exceptional character. The global exceptional representation is obtained as the multi-residue of an Eisenstein series: it is an automorphic representation, and it decomposes as the restricted tensor product of local exceptional representations. As in the case of the small representation of SO_{2n+1} of Bump, Friedberg, and Ginzburg, exceptional representations enjoy the vanishing of a large class of twisted Jacquet modules (locally), or Fourier coefficients (globally). Consequently they are useful in many settings, including lifting problems and Rankin–Selberg integrals. We describe one application, to a calculation of a co-period integral.

Keywords: small representations; metaplectic cover; *GSpin* groups; co-period integral

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Contents

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Let $G_n = GSpin_{2n+1}$ be the split odd general spin group of rank $n+1$. Its derived group is $G'_n = Spin_{2n+1}$, the simple split simply connected algebraic group of type B_n. The group G_n occurs as a Levi subgroup of G'_{n+1}. For a local field F of characteristic 0, let $\widetilde{G'}_{n+1}(F)$ be the metaplectic double cover of $G'_{n+1}(F)$ defined by Matsumoto [59]. We can obtain a double cover $\widetilde{G}_n(F)$ of $G_n(F)$ by restriction.

Following Banks *et al.* [10], we define a section $\mathfrak{s}$ and a 2-cocycle σ of $G'_{n+1}(F)$, representing the cohomology class in $H^2(G'_{n+1}(F), \{\pm 1\})$ of $\widetilde{G'}_{n+1}(F)$. We show that the restriction of σ to $G_n(F) \times G_n(F)$ satisfies a block-compatibility relation, with respect to standard Levi subgroups. This is a useful condition for studying parabolically induced representations.

Fix a Borel subgroup in $G_n(F)$. The preimage $\widetilde{T}_{n+1}(F)$ of the maximal torus $T_{n+1}(F)$ in the cover is a two-step nilpotent subgroup; its irreducible genuine representations are parameterized by genuine characters of its center $C_{\widetilde{T}_{n+1}(F)}$. The analogous theory for covers of GL_n was developed by Kazhdan and Patterson [47], who studied a special class of genuine characters which they called 'exceptional'. In our setting these are characters χ of $C_{\widetilde{T}_{n+1}(F)}$ satisfying $\chi(\alpha^{\vee *}(x^{\mathfrak{l}(\alpha)})) = |x|$ for all simple roots α of G_n and $x \in F^*$, where $\alpha^{\vee *}$ is a certain lift of the coroot $\alpha^\vee$ to the cover and $\mathfrak{l}(\alpha)$ is the length of α.

We use an exceptional character χ to construct a genuine principal series representation of $\widetilde{G}_n(F)$, which has a unique irreducible quotient denoted $\Theta = \Theta_{G_n,\chi}$. The quotient Θ is an exceptional representation, or a small representation in the terminology of Bump *et al.* [17]. Our purpose is to develop a theory of these representations.

Let $\mathcal{O}$ be a unipotent class of G_n; it corresponds to a partition of $2n+1$ for which an even number appears with an even multiplicity. Let $V_{\mathcal{O}}$ be the corresponding unipotent subgroup. We consider certain characters of $V_{\mathcal{O}}(F)$, called 'generic'. Roughly, a character ψ of $V_{\mathcal{O}}(F)$ is generic if it is in general position. The definitions are similar to those of Bump *et al.* [17] (see also [20, 22]) for SO_{2n+1}, and are given in § 2.3.2. The following result characterizes the sense of 'smallness' of Θ.

Theorem 1. *Assume that F is a p-adic field with an odd residual characteristic. Let $\mathcal{O}_0 = (2^n 1)$ if n is even; otherwise, $\mathcal{O}_0 = (2^{n-1}1^3)$. Let $\mathcal{O}$ be any class greater than or non-comparable with $\mathcal{O}_0$, and let ψ be a generic character of $V_{\mathcal{O}}(F)$. The twisted Jacquet module of Θ with respect to $V_{\mathcal{O}}(F)$ and ψ is zero.*

Theorem 1 is proved in § 2.3.2. To remove the restriction on the residual characteristic, we only need to know that exceptional representations of the double cover of $GL_n(F)$

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(with the [47] parameter $\mathrm{c} = 0$) do not have Whittaker models, also for $n \geqslant 3$ and even residual characteristic; this is an expected result ([19, p. 145]; see [25, Lemma 6]).[1]

The local theory has a global counterpart. Let F be a number field with a ring of adèles $\mathbb{A}$. Let $\Theta = \Theta_{G_n,\chi}$ be the exceptional representation of $\widetilde{G}_n(\mathbb{A})$ defined with respect to a global exceptional character χ. It has an automorphic realization as the multi-residue of an Eisenstein series with respect to the Borel subgroup. Also $\Theta \cong \prod_\nu' \Theta_{G_n,\chi_\nu}$ (restricted tensor product). The analogs of the Jacquet modules are Fourier coefficients, over the quotient $V_{\mathcal{O}}(F)\backslash V_{\mathcal{O}}(\mathbb{A})$, with respect to generic characters of $V_{\mathcal{O}}(\mathbb{A})$ trivial on $V_{\mathcal{O}}(F)$. See § 3.4.4 for the definitions. The local–global principle (see, e.g., [39, Proposition 1]) immediately implies the following global corollary of Theorem 1.

Theorem 2. *Let $\mathcal{O}$ be any orbit greater than or non-comparable with $\mathcal{O}_0$. Any Fourier coefficient with respect to $\mathcal{O}$ and a generic character vanishes identically on the space of Θ.*

A minimal representation is a representation supported on the minimal coadjoint orbit. If $n \geqslant 4$, Vogan [84, Theorem 2.13] proved that SO_{2n+1} (or its cover groups) does not afford such representations. Bump *et al.* [17] constructed a local and global 'small' representation $\Theta_{SO_{2n+1}}$ of SO_{2n+1}. It is a representation of a cover $\widetilde{SO}_{2n+1}(F)$; this cover was obtained by restriction of the fourfold cover of $SL_{2n+1}(F)$ of Matsumoto [59]. In the cases when $n = 2, 3$, it is in fact the minimal representation. It is small in the sense that it is supported on the orbit $\mathcal{O}_0$ [17, 18]. The use of the fourfold cover implies a minor technical restriction on the field, namely that -1 is a square.

The arguments of Vogan [84] apply also to G_n; that is, for $n \geqslant 4$ there is no minimal representation. It is reasonable to call Θ a small representation of G_n.

Our local and global results are parallel to those of [17, 18], and are obtained using similar methods. For example, because the unipotent subgroups of G_n are in bijection with those of SO_{2n+1}, manipulations on Jacquet modules are similar. One notable difference is in the restriction of the cover to Levi subgroups. In contrast with the cover of SO_{2n+1}, here, direct factors of Levi subgroups do not commute in the cover. This implies that representations of Levi subgroups cannot be studied using the usual tensor product. In this property, as well as in other details, the cover of G_n is more related to the cover of GL_n than to that of SO_{2n+1}.

Let $Q_k = M_k \ltimes U_k$ be a maximal parabolic subgroup with a Levi part M_k isomorphic to $GL_k \times G_{n-k}$. In the particular case of $k = n$, $\widetilde{GL}_n(F)$ and $\widetilde{G}_0(F)$ do commute, and we can define a tensor product. Let $\chi^{(1)}$ be an exceptional character in the sense of Kazhdan and Patterson [47], and let $\Theta_{GL_n,\chi^{(1)}}$ be the corresponding global exceptional representation of the double cover $\widetilde{GL}_n(\mathbb{A})$ of [47]. Also, let $\chi^{(2)}$ be a genuine character of $\widetilde{G}_0(\mathbb{A})$ (this cover is split). Assume that $\chi = \chi^{(1)} \otimes \chi^{(2)}$; for the precise meaning of this equality, see § 3.3. We compute the constant term of an automorphic form θ in the space of Θ along the unipotent radical U_n, and prove the following result.

[1] Since the time of writing this paper we proved this result in [44, Theorem 2.6], so Theorem 1 holds in general.

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Theorem 3. *The function $m \mapsto \theta^{U_n}(m)$ on $\widetilde{M}_n(\mathbb{A})$ belongs to the space of*

$$\Theta_{GL_n,|\det|^{-1/2}\chi^{(1)}} \otimes \Theta_{G_0,\chi^{(2)}}.$$

The definition of the constant term θ^{U_n} and the proof of the theorem occupy § 3.4.3. Note that $|\det|^{-1/2}\chi^{(1)}$ is also an exceptional character.

For SO_{2n+1} and any k, the mapping $m \mapsto \theta^{U_k}(m)$ belongs to the space of the tensor product $\Theta_{GL_k} \otimes \Theta_{SO_{2(n-k)+1}}$, for a uniquely determined exceptional representation Θ_{GL_n} ($\Theta_{SO_{2(n-k)+1}}$ is unique). This was conjectured in [17] and proved for $k = 1$, and in general proved in [45]. We mention that for the current applications the case of $k = n$ is sufficient. For the lifting results of [18] (see below) the constant term was not used.

We will study representations of M_k using a 'larger' induced representation, similar to the construction of Kable [42]; see § 2.2.4. A metaplectic tensor product for cover groups of GL_n has been studied in [24, 42, 62, 78, 81], but will not be used here. Refer to the discussion in § 2.2.5.

There are several applications to our work. Essentially, one can simply replace $\Theta_{SO_{2n+1}}$ with Θ. This has the benefit of removing the restriction on the field with respect to the fourth roots of unity.

We describe one application, whose details are given in § 4. In general, let G be a split reductive algebraic F-group, where F is a number field. Let $Q = M \ltimes U$ be a maximal parabolic subgroup of G with a Levi part M, and let τ be an irreducible unitary cuspidal globally generic automorphic representation of $M(\mathbb{A})$. Denote the central character of τ by ω_τ. Denote by $E(g; \rho, s)$ the Eisenstein series corresponding to an element ρ in the space of the representation of $G(\mathbb{A})$ induced from τ; $g \in G(\mathbb{A})$ and $s \in \mathbb{C}$. For $s_0 \in \mathbb{C}$, let $E_{s_0}(g; \rho)$ denote the residue of $E(g; \rho, s)$ at s_0. The space spanned by the residues $E_{s_0}(\cdot; \rho)$ is called the residual representation E_τ.

Periods of automorphic forms are often related to poles of L-functions and to questions of functoriality. Ginzburg *et al.* [29] described such relations in a general setup, and considered several examples. They conjectured [29, Conjecture 1.4] that the pole at $s = 1$ of a partial L-function corresponding to τ is related to the non-triviality of certain period integrals, the existence of a residual representation, and the existence of a representation τ_0 such that τ is the Langlands functorial transfer of τ_0.

Among the examples given in [29] is the case of $G = SO_{2n+1}$ and $M = GL_n$. Let A^+ be the subgroup of idèles of F whose finite components are trivial, and Archimedean components are equal, real, and positive. Assume that ω_τ is trivial on A^+. The pole of the Eisenstein series at $s = 1/2$ is determined by the presence of a pole of the partial symmetric square L-function at $s = 1$. In [45] we elaborated on this case and proved a result relating a co-period integral

$$\int_{SO_{2n+1}(F)\backslash SO_{2n+1}(\mathbb{A})} E_{1/2}(g; \rho)\theta(g)\theta'(g)\, dg$$

to the 'theta period' integral

$$\int_{GL_n(F)\backslash GL_n(\mathbb{A})^1} \rho(b)\theta^{U_n}(b)\theta'^{U_n}(b)\, db.$$

Here, θ (respectively, θ') is an automorphic form in the space of $\Theta_{SO_{2n+1},\vartheta}$ (respectively, $\Theta_{SO_{2n+1},\vartheta^{-1}}$), where ϑ is a character of order 4 of the group of fourth roots of unity (ϑ is implicit in the notation $\Theta_{SO_{2n+1}}$); $GL_n(\mathbb{A})^1$ is the kernel of $|\det|$ on $GL_n(\mathbb{A})$.

Bump and Ginzburg [19] constructed a Rankin–Selberg integral representing the partial symmetric square L-function. In particular, they showed that, if this function has a pole at $s=1$, a certain period integral does not vanish. Their period integral was related in [45], under an additional assumption on ω_τ, to the theta period above.

Using the exceptional representation Θ, this result can be put in a more general setting of G_n, and, in particular, will hold for any number field. Let τ be as above (but without the assumption that $\omega_\tau|_{A^+}=1$), and let η be a unitary Hecke character. The Eisenstein series $E(g;\rho,s)$ is now defined with respect to G_n and the parabolic subgroup Q_n. According to Hundley and Sayag [37], the series $E(g;\rho,s)$ is holomorphic at $\Re(s)>0$ except perhaps for a simple pole at $s=1/2$. The existence of this pole is determined by the presence of a pole of the partial L-function $L^S(s,\tau,Sym^2\otimes\eta)$ at $s=1$ (in [37], the twisting is with respect to η^{-1}, but their conventions are different; see Remark 4.1). Takeda [82] constructed a Rankin–Selberg integral for this function and proved that, if S is large enough and $\omega_\tau^2\eta^n\neq 1$, then $L^S(s,\tau,Sym^2\otimes\eta)$ is holomorphic at $s=1$. In particular, the series is holomorphic at $s=1/2$ unless $\omega_\tau^2\eta^n$ is trivial on A^+.

Denote the center of $G_n(\mathbb{A})$ by $C_{G_n(\mathbb{A})}$. We select global exceptional characters χ and χ' such that $\chi\cdot\chi'\cdot\eta=1$ on $C_{G_n(\mathbb{A})}$. Let θ (respectively, θ') belong to the space of $\Theta_{G_n,\chi}$ (respectively, $\Theta_{G_n,\chi'}$). Here is our result, which follows by a minor modification to [45].

Theorem 4. *Consider the co-period integral*

$$\mathcal{I}(E_{1/2}(\cdot;\rho),\theta,\theta')=\int_{C_{G_n(\mathbb{A})}G_n(F)\backslash G_n(\mathbb{A})} E_{1/2}(g;\rho)\theta(g)\theta'(g)\,dg.$$

Assume that $\omega_\tau^2\eta^n$ *is trivial on* A^+. *Then the following hold.*

(1) *There is a normalization of measures (explicitly given in the proof) such that*

$$\mathcal{I}(E_{1/2}(\cdot;\rho),\theta,\theta')=\int_K\int_{GL_n(F)\backslash GL_n(\mathbb{A})^1}\rho(bk)\theta^{U_n}(bk){\theta'}^{U_n}(bk)\,db\,dk.$$

Here, K *is the product of local maximal compact subgroups.*

(2) *The co-period* $\mathcal{I}(E_{1/2}(\cdot;\rho),\theta,\theta')$ *is non-zero for some* (ρ,θ,θ') *if and only if*

$$\int_{GL_n(F)\backslash GL_n(\mathbb{A})^1}\rho_1(m)\theta_1(b)\theta_1'(b)\,db\neq 0$$

for some cusp form ρ_1 *in the space of* τ, *and* θ_1 *(respectively,* θ_1'*) in the space of* $\Theta_{GL_n,|\det|^{-1/2}\chi^{(1)}}$ *(respectively,* $\Theta_{GL_n,|\det|^{-1/2}\chi'^{(1)}}$*).*

The proof is given in § 4. Note that, according to Theorem 3, the function θ^{U_n} belongs to the space of an exceptional representation on $\widetilde{GL}_n(\mathbb{A})$.

Theorem 4 motivates a local counterpart, which will be used as an ingredient in a proof of a conjecture of Lapid and Mao on Whittaker–Fourier coefficients [55], for even

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orthogonal groups. Let ε be Arthur's elliptic tempered parameter for SO_{2n}, and let $\mathcal{S}_\varepsilon$ be the corresponding group [3]. Assume that we have an irreducible automorphic cuspidal (ψ-)generic representation π of $SO_{2n}(\mathbb{A})$ in the A-packet associated to ε (π is expected to be unique). Further, let W and $W^\wedge$ be two global Whittaker–Fourier coefficients on the spaces of π and $\pi^\wedge$ ($\pi^\wedge$ is the contragradient representation). The conjecture of Lapid and Mao relates the product $W(e)W^\wedge(e)$ to the size of $\mathcal{S}_\varepsilon$. The exceptional representation of G_n can perhaps be also used to prove the conjecture for $GSpin_{2n}$.

The minimal representation for the group SO_7, which is a representation of $\widetilde{SO}_7$, was constructed and studied by Roskies [69], Sabourin [70], and Torasso [83]. It was used by Bump *et al.* [16] to construct a Rankin–Selberg integral for the 14-dimensional irreducible representation of the L-group of SO_7, corresponding to the third fundamental weight. Bump *et al.* [18] used $\Theta_{SO_{2n+1}}$ to construct a lift, with certain functorial properties, from genuine automorphic representations of $\widetilde{SO}_k(\mathbb{A})$ to $\widetilde{SO}_m(\mathbb{A})$, for integers k and m of different parity. Both results can be extended to the context of G_n. In particular, our results lead to a lift from genuine automorphic representations of $\widetilde{GSpin_k}(\mathbb{A})$ to $\widetilde{GSpin_m}(\mathbb{A})$.

Loke and Savin [57] constructed local and global exceptional representations for simply connected Chevalley groups. Their approach was different from that of [47]. They started with defining a global automorphic representation π, which was invariant under the action of the Weyl group. Their local and global exceptional representations were obtained by unramified twists of π_ν or π. The global representation was also realized as a multi-residue of an Eisenstein series. Their exposition is elegant and applicable to a wide range of groups.

Our results have some overlap with ongoing work of Loke and Savin;[2] we thank them for informing us about their work. They and the author were working independently. The approach and techniques are different. For example, they do not use a cocycle.

Minimal representations have been studied and used by many authors. The fundamental example is the Weil representation of $\widetilde{Sp}_n$, which was used in the theta correspondence between a pair of dual reductive groups, to lift representations from one group to another [67]. The Weil representation also played an important role in the descent method, in the construction of Fourier–Jacobi coefficients [34, 36, 41, 75]. Among the works on minimal representations are [26, 56, 71, 72]. Works on minimal representations for simply laced groups include [15, 31, 46, 48, 49, 84]. Minimal representations enjoy the vanishing of a large class of Fourier coefficients, which makes them valuable for applications involving lifts and Rankin–Selberg integrals [27, 30, 35].

Early works on the metaplectic groups include the work of Weil [85] on $\widetilde{Sp}_n$, Kubota [51, 52], who studied the r-fold cover of GL_2, and Moore [64] and Steinberg [77], who studied central extensions of simple Chevelley groups and also considered metaplectic groups. Matsumoto [59] constructed the metaplectic r-fold cover of any simple simply connected split group G over a local field. For GL_n, the metaplectic groups were constructed and studied by Kazhdan and Patterson [47]. Over a p-adic field containing $2r$ different $2r$th roots of unity and such that $2r$ is coprime to the residue characteristic,

[2] Private communication.

McNamara [61] constructed $\widetilde{G}$ for any split reductive group G, using the results of Brylinski and Deligne [14] and Finkelberg and Lysenko [23]. Sun [79] studied metaplectic covers of G, defined using covers of its derived group G', assuming that G is also connected and that G' is a simple simply connected Chevalley group.

Banks *et al.* [10] elaborated on the work of Matsumoto [59], by describing an explicit section and a 2-cocycle σ_G representing the corresponding cohomology class in $H^2(G(F), \mu_r)$ of the cover of [59], where F is a local field and μ_r is the subgroup of r rth roots of unity. They proved several compatibility results, which make their cocycle a convenient choice. For example, if H is a 'standard' subgroup of G, which means that H is a simple simply connected split group generated in G by certain data, the restriction of σ to $H(F) \times H(F)$ is σ_H. The cocycle $\sigma_{SL_{n+1}}$ for $SL_{n+1}(F)$ was used in [10] to define a cocycle on $GL_n(F)$, which (in contrast with the cocycle of [47]) is block compatible.

The group $GSpin_m$ has been the focus of study of a few recent works. Asgari [4, 5] studied its local L-functions, Asgari and Shahidi [6, 7] proved functoriality results, and Hundley and Sayag [37] extended the descent construction to $GSpin_m$.

1. Preliminaries

1.1. The groups

Let F be a field of characteristic 0. For any $r \geqslant 1$, let $\mu_r = \mu_r(F)$ be the subgroup of the rth roots of unity in F. Put $F^{*r} = (F^*)^r$. If F is any local field, let $(,)_r$ be the Hilbert symbol of order r of F, and we usually denote by ψ a fixed non-trivial additive character of F. Then γ_ψ is the normalized Weil factor associated to ψ ([85, § 14]; $\gamma_\psi(a)$ is $\gamma_F(a, \psi)$ in the notation of [68], $\gamma_\psi(\cdot)^4 = 1$). If F is a local p-adic field, its ring of integers is $\mathfrak{O}$, the maximal ideal is $\mathcal{P} = \varpi\mathfrak{O}$, and $|\varpi|^{-1} = q = |\mathfrak{O}/\mathcal{P}|$.

In the group GL_n, fix the Borel subgroup $B_{GL_n} = T_{GL_n} \ltimes N_{GL_n}$ of upper triangular invertible matrices, where T_{GL_n} is the diagonal torus. Denote by I_n the identity matrix of $GL_n(F)$.

We define the special odd orthogonal group

$$SO_{2n+1}(F) = \{g \in SL_{2n+1}(F) : {}^t g J_{2n+1} g = J_{2n+1}\},$$

where ${}^t g$ is the transpose of g, and, for any $k \geqslant 1$, $J_k \in GL_k(F)$ is the matrix with 1 on the anti-diagonal and 0 elsewhere. Fix the Borel subgroup $B_{SO_{2n+1}} = T_{SO_{2n+1}} \ltimes N_{SO_{2n+1}}$, where $B_{SO_{2n+1}} = B_{GL_{2n+1}} \cap SO_{2n+1}$ and $T_{SO_{2n+1}}$ is the torus. If $t \in T_{SO_{2n+1}}(F)$, $t = diag(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1})$, where $diag(\cdots)$ denotes a diagonal or block diagonal matrix. Denote by ϵ_i, $1 \leqslant i \leqslant n$, the ith coordinate function, $\epsilon_i(t) = t_i$.

Let $Spin_{2n+1}$ be the simple split simply connected algebraic group of type B_n. It is the algebraic double cover of SO_{2n+1}. The standard Borel subgroup $B_{Spin_{2n+1}}$ of $Spin_{2n+1}$ is the preimage of $B_{SO_{2n+1}}$, $B_{Spin_{2n+1}} = T_n \ltimes N_{Spin_{2n+1}}$. Each ϵ_i can be pulled back to T_n; this pull back will still be denoted ϵ_i. Denote the set of roots of $Spin_{2n+1}$ by $\Sigma_{Spin_{2n+1}}$ and the positive roots by $\Sigma^+_{Spin_{2n+1}}$. The set of simple roots of $Spin_{2n+1}$ is $\Delta_{Spin_{2n+1}} = \{\alpha_i : 1 \leqslant i \leqslant n\}$, where $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $\alpha_n = \epsilon_n$.

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Define $\epsilon_i^\vee$ with respect to the standard $\mathbb{Z}$-pairing $(,)$; i.e., $(\epsilon_i, \epsilon_j^\vee) = \delta_{i,j}$. The set $\Delta_n^\vee = \{\alpha_i : 1 \leqslant i \leqslant n\}$ of simple coroots is given by $\alpha_i^\vee = \epsilon_i^\vee - \epsilon_{i+1}^\vee$ for $1 \leqslant i \leqslant n-1$ and $\alpha_n^\vee = \epsilon_n^\vee$. Because $Spin_{2n+1}$ is simply connected, any $t \in T_n(F)$ can be written uniquely as $t = \prod_{i=1}^n \alpha_i^\vee(t_i)$ for $t_i \in F^*$. For a description of the Levi subgroups of $Spin_{2n+1}$, see Matić [58].

The group $GSpin_{2n+1}$ is an F-split connected reductive algebraic group, which can be defined using a based root datum as in [5, 6, 37]. It is also embedded in $Spin_{2n+3}$ as the Levi part of the parabolic subgroup corresponding to $\Delta_{Spin_{2n+3}} - \{\alpha_1\}$ (see [58]). Since we will be constructing a cover of $GSpin_{2n+1}(F)$ using a cover of $Spin_{2n+3}(F)$, it is natural for us to view $GSpin_{2n+1}(F)$ as this subgroup. Henceforth we adapt this identification. The simple roots of $GSpin_{2n+1}$ are $\{\alpha_2, \ldots, \alpha_{n+1}\}$. In the degenerate case $n = 0$, $GSpin_{2n+1} = GL_1$.

The group $Spin_{2n+1}$ is the derived group of $GSpin_{2n+1}$. Denote $G_n = GSpin_{2n+1}$ and $G'_n = Spin_{2n+1}$. Additionally, let Σ_{G_n} (respectively, $\Sigma_{G_n}^+$) denote the set of roots (respectively, positive roots) of G_n, determined according to the embedding $G_n < G'_{n+1}$. The set of simple roots of G_n is $\Delta_{G_n} = \Delta_{G'_{n+1}} - \{\alpha_1\}$. The corresponding Borel subgroup of G_n is $B_n = T_{n+1} \ltimes N_n$. For $0 \leqslant k \leqslant n$, denote by $Q_k = M_k \ltimes U_k$ the standard maximal parabolic subgroup of G_n with a Levi part M_k isomorphic to $GL_k \times G_{n-k}$. The modulus character of $Q(F)$ for a parabolic subgroup $Q < G_n$ is denoted by $\delta_{Q(F)}$. Let W_n be the Weyl group of G_n. The longest element of W_n is denoted $\mathbf{w_0}$. If F is p-adic, let $K = G_n(\mathfrak{O})$ be the hyperspecial subgroup.

We will also encounter the quasi-split group $GSpin_{2n}$. Let SO_{2n} be a quasi-split even orthogonal group, split over F or a quadratic extension of F. The group $Spin_{2n}$ is the simply connected algebraic double cover of SO_{2n}. In the split case, it is the simple simply connected algebraic group of type D_n. In the non-split case, its relative root system is of type B_{n-1}. Regarding SO_{2n} as a subgroup of GL_{2n}, define the Borel subgroup $B_{SO_{2n}} = SO_{2n} \cap B_{GL_{2n}}$. Then $B_{Spin_{2n}}$ is the preimage of $B_{SO_{2n}}$. This fixes a set of simple roots. The group $GSpin_{2n}$ can be defined as the Levi subgroup of the maximal parabolic subgroup of $Spin_{2(n+1)}$ corresponding to the subset of simple roots obtained by removing the first root [50, § 2.3]. For a definition using a based root datum, see [5, 6, 37].

In general, if G is a group, denote by C_G the center of G. If $H < G$, $C(G, H)$ is the centralizer of H in G. For any two elements $x, y \in G$, $[x, y] = xyx^{-1}y^{-1}$ denotes their commutator. Also put ${}^x y = xyx^{-1}$ and ${}^x H = \{{}^x h : h \in H\}$.

1.2. Properties of $GSpin_{2n+1}$

We collect a few structure properties of G_n that will be used throughout. The center of G_n is

$$C_{G_n(F)} = \left\{ \prod_{i=1}^{n} \alpha_i^\vee(t_1^2)\alpha_{n+1}^\vee(t_1) : t_1 \in F^* \right\}. \tag{1.1}$$

If $t = \prod_{i=1}^{n+1} \alpha_i^\vee(t_i) \in T_{n+1}(F)$,

$${}^{\mathbf{w_0}}t = \prod_{i=1}^{n+1} \alpha_i^\vee(t_i^{-1}) \cdot \prod_{i=1}^{n} \alpha_i^\vee(t_1^2)\alpha_{n+1}^\vee(t_1). \tag{1.2}$$

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To see this, one may work over the algebraic closure of F, where t is a product of elements $\alpha_i^\vee(y_i)$ with $2 \leqslant i \leqslant n+1$, which are inverted by $\mathbf{w_0}$, and an element of the center.

Let M be a Levi subgroup of G_n. As in classical groups, M is isomorphic to $GL_{k_1} \times \cdots \times GL_{k_l} \times G_{n-k}$, $k = k_1 + \cdots + k_l$, with $0 \leqslant k \leqslant n$ [5]. We define an isomorphism of $GL_k \times G_{n-k}$ with M_k. This isomorphism can be used to define an isomorphism between $GL_{k_1} \times \cdots \times GL_{k_l} \times G_{n-k}$ and M using the standard embedding of $GL_{k_1} \times \cdots \times GL_{k_l}$ in GL_k.

The derived group SL_k of GL_k is generated by the root subgroups of $\{\alpha_i : 2 \leqslant i \leqslant k\}$. To complete the description of the embedding of GL_k, let $\eta_1^\vee, \ldots, \eta_k^\vee$ be the standard cocharacters of T_{GL_k} and map $\eta_i^\vee \mapsto \epsilon_{i+1}^\vee - \epsilon_1^\vee$ for $1 \leqslant i \leqslant k$. Under this embedding, the image of a torus element $\prod_{i=1}^k \eta_i^\vee(a_i)$ of $GL_k(F)$ is

$$\prod_{i=1}^{k} \alpha_i^\vee \left(\prod_{j=i}^{k} a_j^{-1} \right). \tag{1.3}$$

Regarding G_{n-k}, the set $\{\alpha_i : k+2 \leqslant i \leqslant n+1\}$ identifies G'_{n-k}, and, if $\theta_1, \ldots, \theta_{n-k+1}$ are the characters of T_{n-k+1}, define $\theta_1^\vee \mapsto \epsilon_1^\vee$ and, for $2 \leqslant i \leqslant n-k+1$, $\theta_i^\vee \mapsto \epsilon_{k+i}^\vee$. Denote by $\beta_1^\vee, \ldots, \beta_{n-k+1}^\vee$ the simple coroots of T_{n-k+1}. We have $\beta_i^\vee = \theta_i^\vee - \theta_{i+1}^\vee$ for $1 \leqslant i \leqslant n-k$, and also $\beta_{n-k+1}^\vee = 2\theta_{n-k+1}^\vee$. When $k = 0$, $\beta_i^\vee = \alpha_i^\vee$ for all $1 \leqslant i \leqslant n+1$. The image of $\prod_{i=1}^{n-k+1} \beta_i^\vee(t_i)$ in $G_n(F)$ is

$$\begin{cases} \displaystyle\prod_{i=1}^{k} \alpha_i^\vee(t_1) \prod_{i=1}^{n-k+1} \alpha_{k+i}^\vee(t_i) & k < n, \\ \displaystyle\prod_{i=1}^{n} \alpha_i^\vee(t_1^2)\alpha_{n+1}^\vee(t_1) & k = n. \end{cases} \tag{1.4}$$

For $k = n$, the image of $\beta_1^\vee$ is $C_{G_n(F)}$.

The restriction of the projection $G'_n \to SO_{2n+1}$ to unipotent subgroups is an isomorphism. In particular, the unipotent radical U_k corresponds to the unipotent radical U'_k of the standard parabolic subgroup Q'_k of SO_{2n+1}, whose Levi part M'_k is isomorphic to $GL_k \times SO_{2(n-k)+1}$. In coordinates,

$$U'_k(F) = \left\{ \begin{pmatrix} I_k & u_1 & u_2 \\ & I_{2(n-k)+1} & u'_1 \\ & & I_k \end{pmatrix} \in SO_{2n+1}(F) \right\} \quad (u'_1 = -J_{2(n-k)+1}{}^t u_1 J_k).$$

If $b_0 \in GL_k(F)$, b is the image of b_0 in $M_k(F)$, and b' is the image of b_0 in $M'_k(F)$ ($b' = diag(b, I_{2(n-k)+1}, J_k{}^t b^{-1} J_k)$), the following diagram commutes:

$$\begin{array}{ccc} U_k & \xrightarrow{\mapsto^{b}} & U_k \\ \downarrow & & \uparrow \\ U'_k & \xrightarrow{\mapsto^{b'}} & U'_k. \end{array}$$

Here, the horizontal arrows denote conjugation.

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For calculations it will sometimes be more convenient to use the cocharacters of T_{GL_n} and the coroot of T_1 to write an element in T_{n+1}, instead of the coroots of T_{n+1}. Then we write

$$t = \prod_{i=1}^{n} \eta_i^{\vee}(a_i)\beta_1^{\vee}(t_1), \tag{1.5}$$

and the image of t in $G_n(F)$ is

$$\prod_{i=1}^{n} \alpha_i^{\vee}\left(\left(\prod_{j=i}^{n} a_j^{-1}\right) t_1^2\right) \alpha_{n+1}(t_1). \tag{1.6}$$

In these coordinates,

$$^{\mathbf{w_0}}t = \prod_{i=1}^{n} \eta_i^{\vee}(a_i^{-1})\beta_1^{\vee}\left(\left(\prod_{i=1}^{n} a_i\right)^{-1} t_1\right), \tag{1.7}$$

$$\delta_{B_n(F)}(t) = \delta_{B_{SO_{2n+1}}(F)}(diag(a_1, \ldots, a_n, 1, a_n^{-1}, \ldots, a_1^{-1})) = \prod_{i=1}^{n} |a_i|^{2(n-i)+1}. \tag{1.8}$$

We use the character ϵ_1 to define the following 'canonical' character Υ of G_n, which is the extension of $-\epsilon_1$ to G_n. The extension exists because $G_n(F) = \{\alpha_1^{\vee}(x) : x \in F^*\} \ltimes G'_n(F)$ and $\epsilon_1|_{T_{n+1}(F)\cap G'_n(F)}$ is trivial. We call Υ canonical because its restriction to GL_k is $\det$, as is evident from (1.3). For example, if t is given by (1.5), $\Upsilon(t) = t_1^{-2}\prod_{i=1}^{n} a_i$. The restriction of Υ to G_{n-k} is the corresponding character of G_{n-k}. For $n = 0$, $\Upsilon(x) = x^{-2}$.

1.3. Further definitions and notation

1.3.1. Central extensions. Let G be a group. We recall a few notions of abstract central extensions (i.e., ignoring topologies). A central extension of G by a group A is a short exact sequence of groups

$$1 \to A \xrightarrow{\iota} \widetilde{G} \xrightarrow{p} G \to 1$$

such that $\iota(A) < C_{\widetilde{G}}$. For any subset $X \subset G$, denote $\widetilde{X} = p^{-1}(X)$. Assume that we have a fixed faithful character $\vartheta : A \to \mathbb{C}^*$. Let $H < G$. A representation π of $\widetilde{H}$ is called ϑ-genuine if $\pi(\iota(a)h) = \vartheta(\iota(a))\pi(h)$ for all $a \in A$ and $h \in \widetilde{H}$. In particular, if $A = \mu_2$, such a representation is simply called genuine. Any representation π' of G can be pulled back to a non-genuine representation of $\widetilde{G}$ by composing it with p.

A section of H is a mapping $\varphi : H \to \widetilde{G}$ such that $\varphi(1) = 1$ and $p \circ \varphi = id_H$. If $H_1 < H$, we say that φ splits H_1, or H_1 is split under φ, if the restriction $\varphi|_{H_1}$ is a homomorphism. In this case, because $\varphi(H_1) \cap \iota(A) = \{1\}$, $\widetilde{H}_1 = \iota(A) \cdot \varphi(H_1)$ (an inner direct product). Now any ϑ-genuine representation of $\widetilde{H}_1$ is uniquely determined by its restriction to $\varphi(H_1)$ and conversely, any representation of $\varphi(H_1)$ can be extended uniquely to a ϑ-genuine representation of $\widetilde{H}_1$. Of course, the splitting φ might not be unique, but, once fixed, a representation of H_1 has a unique extension to a ϑ-genuine representation of $\widetilde{H}_1$.

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A 2-cocycle defined on G is a mapping $\sigma : G \times G \to A$ such that $\sigma(1, 1) = 1$ and, for all $g, g', g'' \in G$,

$$\sigma(g, g')\sigma(gg', g'') = \sigma(g, g'g'')\sigma(g', g''). \tag{1.9}$$

If $\varphi : G \to \widetilde{G}$ is a section, then the function $(g, g') \mapsto \iota^{-1}(\varphi(g)\varphi(g')\varphi(gg')^{-1})$ is a 2-cocycle. Once we fix σ, the elements of $\widetilde{G}$ can be written as pairs (g, a), where $g \in G$ and $a \in A$. In this realization, the multiplication in $\widetilde{G}$ is defined by

$$(g, a) \cdot (g', a') = (gg', \sigma(g, g')aa').$$

Remark 1.1. Note one subtlety of the notation. For clarity reasons, we write $\widetilde{X}(F)$ instead of $\widetilde{X(F)}$, but always mean $p^{-1}(X(F))$.

1.3.2. Representations. Let G be an ℓ-group (see [11, (1.1)]). Throughout, unless mentioned otherwise, representations of G will be complex, smooth, and admissible. If π is a representation of G, the representation contragradient to π is denoted $\pi^\wedge$. If π is a representation of $H < G$ and $g \in G$, ${}^g\tau$ is the representation of ${}^gH$ defined on the space of τ by ${}^g\tau(x) = \tau({}^{g^{-1}}x)$.

Parabolically induced representations will always be normalized as in [12, (1.8)]. We use *Ind* to denote regular induction; *ind* signifies compact induction. The (normalized) Jacquet functor $j_{U,\psi}$ is defined as in [12, (1.8)], where U is a unipotent subgroup and ψ is a character of U. If ψ is trivial, we write $j_U = j_{U,\psi}$.

1.3.3. The Hilbert symbol. For an integer $r > 1$, let $c = (,)_r^{-1}$. We recall that c is an antisymmetric bi-character; i.e., $c(1, 1) = 1$, $c(xy, z) = c(x, z)c(y, z)$, and $c(x, y) = c(y, x)^{-1}$. Also, $c(x, y) = 1$ for all $y \in F^*$ if and only if $x \in F^{*r}$. If $|r| = 1$ in F^*, c is trivial on $\mathfrak{O}^* \times \mathfrak{O}^*$, and $c(x, y) = 1$ for all $y \in \mathfrak{O}^*$ if and only if $x \in \mathfrak{O}^*F^{*r}$ [86, § XIII.5, Proposition 6].

1.3.4. The Weil factor. Let ψ be given, and let γ_ψ be the Weil factor (see § 1.1). We recall that $\gamma_\psi(xy) = \gamma_\psi(x)\gamma_\psi(y)(x, y)_2$, and, if $a \in F^*$ and $\psi_a(x) = \psi(ax)$, $\gamma_{\psi_a}(x) = (a, x)_2\gamma_\psi(x)$. Moreover, $\gamma_{\psi_a} = \gamma_{\psi_b}$ if and only if $ab^{-1} \in F^{*2}$. Also $\gamma_\psi(x^2) = 1$ and $\gamma_\psi(x^{-1}) = \gamma_\psi(x)$. See the appendix of Rao [68].

2. Local theory

2.1. The double cover of $G_n(F)$

2.1.1. Definition of the r-fold cover. Let F be a local field of characteristic 0, and let $r > 1$ be an integer. Assume that F contains all the rth roots of unity; i.e., $|\mu_r| = r$. Let $\widetilde{G}'_{n+1}(F)$ be the r-fold cover of $G'_{n+1}(F)$, constructed by Matsumoto [59] using $c = (,)_r^{-1}$ as the Steinberg symbol. The group $\widetilde{G}'_{n+1}(F)$ fits into an exact sequence

$$1 \to \mu_r \xrightarrow{\iota} \widetilde{G}'_{n+1}(F) \xrightarrow{p} G'_{n+1}(F) \to 1.$$

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Let $\widetilde{G}_n(F) = p^{-1}(G_n(F))$. We have an exact sequence

$$1 \to \mu_r \xrightarrow{\iota} \widetilde{G}_n(F) \xrightarrow{p} G_n(F) \to 1,$$

and $\widetilde{G}_n(F)$ is an r-fold cover of $G_n(F)$.

Although most of this work will focus on the double cover, we start with the more general setting. From § 2.1.6 onward, except for § 3.1, we will assume that $r = 2$.

Banks *et al.* [10] gave an explicit description of a section $\mathfrak{s}$ of $G'_{n+1}(F)$ and used it to construct a 2-cocycle $\sigma_{G'_{n+1}}$, representing the cohomology class of $\widetilde{G'}_{n+1}(F)$ in $H^2(G'_{n+1}(F), \mu_r)$. We obtain a section of $G_n(F)$ and a 2-cocycle by restricting $\mathfrak{s}$ and $\sigma_{G'_{n+1}}$.

We briefly describe the construction and results of [10]. For $\alpha \in \Sigma_{G'_{n+1}}$, let $\mathcal{U}_\alpha$ be the corresponding root subgroup. Fix an isomorphism $n_\alpha : F \to \mathcal{U}_\alpha$ (based on an explicit decomposition of the Chevalley algebra corresponding to $G'_{n+1}(F)$). For $x \in F^*$, put $w_\alpha(x) = n_\alpha(x)n_{-\alpha}(-x^{-1})n_\alpha(x)$. Note that $\alpha^\vee(x) = h_\alpha(x)$ in the notation of [10] ($h_\alpha(x) = w_\alpha(x)w_\alpha(1)^{-1}$). If $\alpha' \in \Sigma_{G'_{n+1}}$, let $\langle \alpha, \alpha'^\vee \rangle$ be the standard pairing between characters and cocharacters, where $\alpha'^\vee$ is the coroot corresponding to α'.

Given $\alpha \in \Sigma_{G'_{n+1}}$, let $n^*_\alpha : F \to \widetilde{G'}_{n+1}(F)$ be the canonical lift of Steinberg [76]. Using these lifts, one defines $w^*_\alpha(x)$ and $\alpha^{\vee *}(x)$: $w^*_\alpha(x) = n^*_\alpha(x)n^*_{-\alpha}(-x^{-1})n^*_\alpha(x)$ and $\alpha^{\vee *}(x) = w^*_\alpha(x)w^*_\alpha(1)$.

Let $\alpha \in \Delta_{G'_{n+1}}$. For any $x, y \in F^*$, define $c_\alpha(x, y) = c(x, y)$ if $n > 0$ and α is a long root; otherwise ($n = 0$ or $\alpha = \alpha_{n+1}$), put $c_\alpha(x, y) = c(x^2, y)$. Also set $w_\alpha = w_\alpha(-1)$ and $w^*_\alpha = w^*_\alpha(-1)$.

Remark 2.1. In the case when $n = 0$, there is only one root, α_1, which is by definition a long root. In this case, in [10], it was defined that $c_\alpha(x, y) = c(x, y)$. The present definition seems to be correct because it preserves the compatibility with restriction to 'standard' subgroups; see § 2.1.2.

Following Matsumoto [59], Banks, Levi, and Sepanski described a list of identities in the cover group describing the multiplication laws between the elements. The following partial list of identities in $\widetilde{G'}_{n+1}(F)$ [10, § 1] will be used repeatedly in computations:

$$\alpha^{\vee *}(x)\alpha^{\vee *}(y) = \iota(c_\alpha(x, y))\alpha^{\vee *}(xy), \tag{2.1}$$

$$\alpha^{\vee *}(x)\alpha'^{\vee *}(y) = \iota(c_\alpha(x, y^{\langle \alpha, \alpha'^\vee \rangle}))\alpha'^{\vee *}(y)\alpha^{\vee *}(x), \tag{2.2}$$

$$w^*_\alpha \alpha'^{\vee *}(x) w^{*-1}_\alpha = \alpha^{\vee *}(x^{-\langle \alpha, \alpha'^\vee \rangle})\alpha'^{\vee *}(x), \tag{2.3}$$

$$w^{*-1}_\alpha = w^*_\alpha \alpha^{\vee *}(-1)\iota(c_\alpha(-1, -1))^{r-1}, \tag{2.4}$$

$$w^*_{\alpha''}(x)n^*_{\alpha''}(z)w^*_{\alpha''}(x)^{-1} = n^*_{-\alpha''}(-x^{-2}z), \tag{2.5}$$

$$w^*_\alpha n^*_\alpha(x) = n^*_\alpha(-x^{-1})\alpha^{\vee *}(x^{-1})n^*_{-\alpha}(x^{-1}), \tag{2.6}$$

$$n^*_{\alpha'''}(z)\alpha^{\vee *}(y) = \alpha^{\vee *}(y)n^*_{\alpha'''}(zy^{-\langle \alpha''', \alpha'^\vee \rangle}), \tag{2.7}$$

$$w^{*-1}_\alpha n^*_{\alpha'''}(z)w^*_\alpha = n_{\mathbf{s}_\alpha \alpha'''}(d_{\alpha, \alpha'''} z). \tag{2.8}$$

Here, $\alpha, \alpha' \in \Delta_{G'_{n+1}}$, $x, y \in F^*$, $z \in F$, $\alpha'' \in \Sigma_{G'_{n+1}}$, and $\alpha''' \in \Sigma^+_{G'_{n+1}}$. In (2.8), $\alpha \neq \alpha'''$, $\mathbf{s}_\alpha$ is the reflection along α in the Weyl group of G'_{n+1}, and $d_{\alpha,\alpha'''} = \pm 1$, depending only on $G'_{n+1}(F)$.

Banks *et al.* [10] defined the following section $\mathfrak{s}$ of $G'_{n+1}(F)$. First, it is the only splitting of $N_{n+1}(F)$, and it satisfies $\mathfrak{s}(n_\alpha(x)) = n^*_\alpha(x)$ for all $\alpha \in \Sigma^+_{G'_{n+1}}$. On $T_{n+1}(F)$ it satisfies

$$\mathfrak{s}\left(\prod_{i=1}^{n+1} \alpha_i^\vee(t_i)\right) = {\alpha_{n+1}^\vee}^*(t_{n+1})\alpha_n^{\vee *}(t_n) \cdot \ldots \cdot \alpha_1^{\vee *}(t_1) \prod_{i=1}^{n+1} \iota(c_{\alpha_i}(t_i, t_i)). \tag{2.9}$$

Let $\mathfrak{W}_{n+1}$ be the set of elements $w = w_{\alpha_{i_1}} w_{\alpha_{i_2}} \cdots w_{\alpha_{i_k}}$, where $\alpha_{i_1}, \ldots, \alpha_{i_k} \in \Delta_{G'_{n+1}}$ and w is assumed to be a reduced expression. For such w, put $l(w) = k$. As sets, $\mathfrak{W}_{n+1} \cong W_{n+1}$. We use boldface to denote the elements of W_{n+1}. If $\mathbf{w} \in W_{n+1}$, its representative in $\mathfrak{W}_{n+1}$ is w.

The section $\mathfrak{s}$ is extended to $G'_{n+1}(F)$ with the following properties [10, Lemmas 2.2 and 2.3]:

$$\mathfrak{s}(w_\alpha) = w^*_\alpha, \tag{2.10}$$

$$\mathfrak{s}(ww') = \mathfrak{s}(w)\mathfrak{s}(w'), \tag{2.11}$$

$$\mathfrak{s}(utwu') = \mathfrak{s}(u)\mathfrak{s}(t)\mathfrak{s}(w)\mathfrak{s}(u'). \tag{2.12}$$

Here, $\alpha \in \Delta_{G'_{n+1}}$, $w, w' \in \mathfrak{W}_{n+1}$ satisfy $l(ww') = l(w) + l(w')$, $u, u' \in N_{G'_{n+1}}(F)$, and $t \in T_{n+1}(F)$.

Finally, the 2-cocycle $\sigma = \sigma_{G'_{n+1}}$ is defined by $\sigma(g, g') = \mathfrak{s}(g)\mathfrak{s}(g')\mathfrak{s}(gg')^{-1}$. The image of σ belongs to $\iota(\mu_r)$, but we implicitly compose it with ι^{-1}. The following formulas hold [10, Proposition 2.4]:

$$\sigma(ugu'', g'u') = \sigma(g, u''g'), \tag{2.13}$$

$$\sigma(t, w) = 1, \tag{2.14}$$

where $u, u', u'' \in N_{G'_{n+1}}(F)$, $t \in T_{n+1}(F)$, and $w \in \mathfrak{W}_{n+1}$.

We pull back the character Υ to a non-genuine character of $\widetilde{G}_n(F)$, still denoted Υ.

For any $g, g' \in G'_{n+1}(F)$, denote $[g, g']_\sigma = \sigma(g, g')\sigma(g', g)^{-1}$. Let $x, x' \in \widetilde{G}'_{n+1}(F)$ be with $p(x) = g$ and $p(x') = g'$. If $[g, g'] = 1$, then $\mathfrak{s}(gg') = \mathfrak{s}(g'g)$, and

$$[x, x'] = \mathfrak{s}(g)\mathfrak{s}(g')\mathfrak{s}(g)^{-1}\mathfrak{s}(g')^{-1} = [g, g']_\sigma. \tag{2.15}$$

In particular, x and x' commute if and only if $[g, g'] = 1$ and $[g, g']_\sigma = 1$.

The following claim describes the value of the cocycle on the torus.

Claim 2.1. *Let $t = \prod_{i=1}^{n+1} \alpha_i^\vee(t_i)$, $t' = \prod_{i=1}^{n+1} \alpha_i^\vee(t'_i)$. Then*

$$\sigma(t, t') = c(t_{n+1}^2, t'_{n+1})c(t_n, {t'}_{n+1}^{-2}) \prod_{i=1}^{n} c(t_i, t'_i) \prod_{i=1}^{n-1} c(t_i, {t'}_{i+1}^{-1}).$$

In particular, if $n = 0$, $\sigma(t, t') = c(t_1^2, t'_1)$ and, for $n = 1$, $\sigma(t, t') = c(t_2^2, t'_2)c(t_1, {t'}_2^{-2})$ $c(t_1, t'_1)$.

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Proof. The proof follows from (2.9), (2.1), (2.2), and the fact that, for $1 \leqslant i < i' \leqslant n+1$, if $i < i'-1$ then $\langle \alpha_i, \alpha_{i'}^{\vee} \rangle = 0$; if $i' \leqslant n$, $\langle \alpha_{i'-1}, \alpha_{i'}^{\vee} \rangle = -1$; and $\langle \alpha_n, \alpha_{n+1}^{\vee} \rangle = -2$. □

Note that a similar construction works for $GSpin_{2n}$. As explained in § 1.1, this group can be regarded as a subgroup of $Spin_{2(n+1)}$; then the cover $\widetilde{GSpin_{2n}}(F)$ can be obtained by restriction from $\widetilde{Spin_{2(n+1)}}(F)$. Here, $GSpin_{2n}$ will appear a priori as a subgroup of some G_m, so the cover is obtained by restricting $\widetilde{G}_m(F)$.

2.1.2. Restriction to 'standard' subgroups. Let H be a simple simply connected algebraic F-group, which is F-split. Let $\Delta \subset \Delta_{G'_{n+1}}$. Further assume that $H(F)$ is the subgroup of $G'_{n+1}(F)$ generated by the elements $\alpha^{\vee}(x)$, w_{α}, and $n_{\alpha'}(y)$, where $\alpha \in \Delta$; $x \in F^*$; $\alpha' \in \Sigma^+_{G'_{n+1}}$ is spanned by the roots in Δ; and $y \in F$. Then $H(F)$ was called a 'standard' subgroup of $G'_{n+1}(F)$ in [10]. According to [10, Lemma 2.5 and its proof], the restriction of $\mathfrak{s}$ to $H(F)$ gives the section defined by [10] on $H(F)$, and $\sigma|_{H(F)\times H(F)}$ is the 2-cocycle of [10] on $H(F)$. Moreover, if $H_1(F), \ldots, H_k(F)$ is a collection of standard subgroups that are mutually commuting and σ_i is the 2-cocycle on $H_i(F)$, by [10, Theorem 2.7],

$$\sigma(g_1 \cdots g_k, g'_1 \cdots g'_k) = \prod_{i=1}^{k} \sigma_i(g_i, g'_i).$$

This property was used in [10] to define their block-compatible cocycle for $GL_n(F)$; see § 2.1.5.

For example, if $n > 0$, according to the embedding $M_{n-1} < G_n$, the group $G'_1(F)$ is the standard subgroup generated by $\{\alpha_{n+1}^{\vee}(x), w_{\alpha_{n+1}}, n_{\alpha_{n+1}}(y)\}$. If $\sigma_{G'_1}$ is the cocycle on $G'_1(F)$,

$$\sigma_{G'_1}(\alpha_{n+1}^{\vee}(t_1), \alpha_{n+1}^{\vee}(t'_1)) = \sigma(\alpha_{n+1}^{\vee}(t_1), \alpha_{n+1}^{\vee}(t'_1)) = c_{\alpha_{n+1}}(t_1, t'_1).$$

Since $n > 0$, $c_{\alpha_{n+1}}(t_1, t'_1) = c(t_1^2, t'_1)$. Therefore, when $n = 0$, we must define $c_{\alpha_1}(x, y) = c(x^2, y)$ (see Remark 2.1).

2.1.3. Splitting of the cover for $r = 2$ and $n \leqslant 1$. We have the following minimal cases.

Claim 2.2. *Assume that $r = 2$. For $n \leqslant 1$, the cover $\widetilde{G}_n(F)$ splits under $\mathfrak{s}$. Moreover, for $n = 0$, the restriction of σ to $G_0(F) \times G_0(F)$ is trivial.*

Proof. In the case when $n = 0$, $\sigma|_{G_0(F)\times G_0(F)} = 1$ because $c_{\alpha_1}(x, y) = c(x^2, y) = 1$. Assume that $n = 1$. Since $G_1 = GL_2$, if the cocycle is non-trivial, then it is equal up to a coboundary to one of the cocycles defined by Kazhdan and Patterson [47], all of which are non-trivial on $SL_2(F)$. However, $\sigma|_{G'_1 \times G'_1} = \sigma_{G'_1}$ (by § 2.1.2) and $\sigma_{G'_1} = 1$, as can be seen using (2.1)–(2.7); note that we have only one simple root α_2, and in this case $c_{\alpha_2} = 1$. □

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2.1.4. A splitting of the hyperspecial subgroup. Assume that $|r| = 1$ in F (in particular, F is p-adic) and that $q > 3$. By Moore [64, pp. 54–56], there is a unique splitting κ of $G'_{n+1}(\mathfrak{O})$, which is in particular a splitting of $K(= G_n(\mathfrak{O}))$. Denote $K^* = \kappa(K)$. As in [47, Proposition 0.1.3], we have the following relation between κ and $\mathfrak{s}$.

Claim 2.3. *The sections $\mathfrak{s}$ and κ agree on $K \cap N_n(F)$, $K \cap T_{n+1}(F)$, and $\mathfrak{W}_n$ ($\mathfrak{W}_n \subset K$).*

Proof. Let $\alpha \in \Delta_{G'_{n+1}}$. First, assume that $\alpha \neq \alpha_{n+1}$, and let SL_2 be embedded in G'_{n+1} along α. Let σ_{SL_2} be the 2-cocycle of [10] on $SL_2(F)$. According to [10, Corollary 3.8 and Lemma 2.5], σ_{SL_2} is the cocycle of Kubota [51] on $SL_2(F)$. Kubota [52, Theorem 2] proved that the mapping $\gamma : SL_2(\mathfrak{O}) \to \mu_r$, which is 1 on $\left(\begin{smallmatrix} * & * \\ x & y \end{smallmatrix}\right)$ if $|x| \in \{0, 1\}$ and otherwise equals $c(x, y)$, satisfies $\sigma_{SL_2}(k, k') = \gamma(k)\gamma(k')\gamma(kk')^{-1}$.

Again by [10, Lemma 2.5], $\mathfrak{s}(k)\mathfrak{s}(k')\mathfrak{s}(kk')^{-1} = \sigma_{SL_2}(k, k')$, whence $\varphi(k) = \mathfrak{s}(k)/\iota(\gamma(k))$ is a splitting of $SL_2(\mathfrak{O})$. Hence φ coincides with κ on $SL_2(\mathfrak{O})$ [64, Lemma 11.1]. Whenever $\gamma(k) = 1$, $\mathfrak{s}(k) = \kappa(k)$, and in particular $\mathfrak{s}$ and κ agree on $n_\alpha(\mathfrak{O})$, $\alpha^\vee(\mathfrak{O}^*)$, and w_α.

If $\alpha = \alpha_{n+1}$ and $r > 2$, restriction to $SL_2(F)$ gives a non-trivial cover (of order $r/\gcd(2, r)$), and the proceeding discussion applies (with γ defined using $c(,)^2$ instead of $c(,)$). If $r = 2$, $\mathfrak{s}|_{SL_2(F)}$ is a homomorphism ($\sigma_{G'_1} = 1$; see the proof of Claim 2.2), and hence $\mathfrak{s}|_{SL_2(\mathfrak{O})} = \kappa|_{SL_2(\mathfrak{O})}$.

The section κ is in particular a splitting of $K \cap T_{n+1}(F)$, and the same holds for $\mathfrak{s}$ (because c is trivial on $\mathfrak{O}^* \times \mathfrak{O}^*$). Writing $t = \prod_{i=1}^{n+1} \alpha_i^\vee(t_i) \in K \cap T_{n+1}(F)$ ($t_i \in \mathfrak{O}^*$) and noting that $\mathfrak{s}(\alpha_i^\vee(t_i)) = \kappa(\alpha_i^\vee(t_i))$ for each i, since $\alpha_i^\vee(t_i)$ belongs to the copy of $SL_2(\mathfrak{O})$ along $\alpha_i \in \Delta_{G'_{n+1}}$, one deduces that $\kappa(t) = \mathfrak{s}(t)$.

Regarding $\mathfrak{W}_n$, according to (2.11) it is enough to show that $\kappa(w_\alpha) = \mathfrak{s}(w_\alpha)$ for $\alpha \in \Delta_{G_n}$, which holds because w_α belongs to a suitable copy of $SL_2(\mathfrak{O})$.

It remains to consider $\mathcal{U}_{\alpha'}$, where $\alpha' \in \Sigma^+_{G_n}$. Let $y \in \mathfrak{O}$, and take $\alpha \in \Delta_{G'_{n+1}}$, $x \in \mathfrak{O}$, and $w \in \mathfrak{W}_n$ such that $w^{-1}n_\alpha(x)w = n_{\alpha'}(y)$. The fact that κ is a splitting of K and the assertions already proved imply that $\kappa(w^{-1}n_\alpha(x)w) = \mathfrak{s}(w)^{-1}\mathfrak{s}(n_\alpha(x))\mathfrak{s}(w)$. Then (2.10), (2.11), and (2.8) give $\mathfrak{s}(w)^{-1}\mathfrak{s}(n_\alpha(x))\mathfrak{s}(w) = n^*_{\alpha'}(y)$, whence $\mathfrak{s}(n_{\alpha'}(y)) = \kappa(n_{\alpha'}(y))$. Note that we actually apply (2.8) repeatedly, according to the decomposition of w into a product of simple reflections w_{α_i} ($\alpha_i \in \Delta_{G_n}$), and after j conjugations, if we have to conjugate $n^*_{\alpha'''}(x_j)$ by $w^*_{\alpha_i}$, then $\alpha''' \in \Sigma^+_{G_n} - \Delta_{G_n}$, whence $\alpha_i \neq \alpha'''$ (this is needed for (2.8)). □

2.1.5. Block compatibility. As mentioned in the introduction, the 2-cocycle defined on $SL_{n+1}(F)$ was used in [10] to define a 2-cocycle σ_n for $GL_n(F)$. Specifically, they defined

$$\sigma_n(b, b') = c(\det b, \det b')^{-1}\sigma_{SL_{n+1}}(diag(b, \det b^{-1}), diag(b', \det b'^{-1})) \quad (b, b' \in GL_n(F)).$$

If $n = 1$, $\sigma_n = 1$. Their cocycle is block compatible [10, Theorem 3.11], in the sense that, for any $g_i, g'_i \in GL_{n_i}(F)$, $1 \leqslant i \leqslant l$,

$$\sigma_n(diag(g_1, \ldots, g_l), diag(g'_1, \ldots, g'_l)) = \prod_{i=1}^{l} \sigma_{n_i}(g_i, g'_i) \prod_{i<j} c(\det g_i, \det g'_j)^{-1}.$$

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We seek similar block compatibility. We begin with the following lemma, which encapsulates several arguments of [10] and summarizes a list of properties which, when satisfied by a pair of subgroups, implies a certain plausible block formula.

Lemma 2.4. *Assume that H_1 and H_2 are two subgroups of $G'_{n+1}(F)$ with the following properties.*

(1) *For each i and $h \in H_i$, there is a Bruhat decomposition $h = utwv$ with $u, v \in N_{G'_{n+1}}(F)$, $t \in T_{n+1}(F)$, and $w \in \mathfrak{W}_{n+1}$, and such that $u, t, w, v \in H_i$.*

(2) *The subgroups H_1 and H_2 commute.*

(3) *If $w_1 \in H_1 \cap \mathfrak{W}_{n+1}$ and $w_2 \in H_2 \cap \mathfrak{W}_{n+1}$, then $l(w_1 w_2) = l(w_1) + l(w_2)$.*

(4) *If $i \in \{1, 2\}$, $w \in H_i \cap \mathfrak{W}_{n+1}$, and $t' \in H_{3-i} \cap T_{n+1}(F)$, then $\sigma(w, t') = 1$.*

(5) *If $t_1 \in H_1 \cap T_{n+1}(F)$ and $t_2 \in H_2 \cap T_{n+1}(F)$, then $\sigma(t_2, t_1) = 1$.*

Then, for any $h_1, h'_1 \in H_1$ and $h_2, h'_2 \in H_2$,

$$\sigma(h_1 h_2, h'_1 h'_2) = \sigma(h_1, h'_1)\sigma(h_2, h'_2)\sigma(h_1, h'_2).$$

Moreover, if $h_1 = u_1 t_1 w_1 v_1$ and $h'_2 = u'_2 t'_2 w'_2 v'_2$, with $u_1, v_1, u'_2, v'_2 \in N_{G'_{n+1}}(F)$, $t_1, t'_2 \in T_{n+1}(F)$, $w_1, w'_2 \in \mathfrak{W}_{n+1}$, $u_1, t_1, w_1, v_1 \in H_1$, and $u'_2, t'_2, w'_2, v'_2 \in H_2$, then

$$\sigma(h_1, h'_2) = \sigma(t_1, t'_2).$$

Proof. Let $i \in \{1, 2\}$, $t \in H_i \cap T_{n+1}(F)$, $w \in H_i \cap \mathfrak{W}_{n+1}$, $t' \in H_{3-i} \cap T_{n+1}(F)$, and $w' \in H_{3-i} \cap \mathfrak{W}_{n+1}$. Note that, by (2) and (2.12),

$$\sigma(w, t') = \mathfrak{s}(w)\mathfrak{s}(t')\mathfrak{s}(wt')^{-1} = \mathfrak{s}(w)\mathfrak{s}(t')\mathfrak{s}(t'w)^{-1} = \mathfrak{s}(w)\mathfrak{s}(t')\mathfrak{s}(w)^{-1}\mathfrak{s}(t')^{-1},$$

whence (4) implies that $\mathfrak{s}(w)\mathfrak{s}(t')\mathfrak{s}(w)^{-1} = \mathfrak{s}(t')$. Thus

$$\begin{aligned}\sigma(tw, t') &= \mathfrak{s}(tw)\mathfrak{s}(t')\mathfrak{s}(twt')^{-1} = \mathfrak{s}(t)\mathfrak{s}(w)\mathfrak{s}(t')\mathfrak{s}(tt'w)^{-1}\\ &= \mathfrak{s}(t)\mathfrak{s}(w)\mathfrak{s}(t')\mathfrak{s}(w)^{-1}\mathfrak{s}(tt')^{-1} = \mathfrak{s}(t)\mathfrak{s}(t')\mathfrak{s}(tt')^{-1} = \sigma(t, t'),\end{aligned} \tag{2.16}$$

where we also used (2.12) and (2). Similarly,

$$\begin{aligned}\sigma(twt', w') &= \sigma(tt'w, w') = \mathfrak{s}(tt')\mathfrak{s}(w)\mathfrak{s}(w')\mathfrak{s}(tt'ww')^{-1}\\ &= \mathfrak{s}(tt')\sigma(w, w')\mathfrak{s}(tt')^{-1} = \mathfrak{s}(tt')\mathfrak{s}(tt')^{-1} = 1.\end{aligned} \tag{2.17}$$

Here we used (2.11), (2), and (3) to deduce that $\sigma(w, w') = 1$.

Now (2.14), (1.9), (2.16), and (2.17) imply that

$$\sigma(tw, t'w') = \sigma(tw, t'w')\sigma(t', w') = \sigma(tw, t')\sigma(twt', w') = \sigma(t, t').$$

Let $h \in H_i$ and $h' \in H_{3-i}$. Write $h = utwv$ and $h' = u't'w'v'$, with $u, v \in H_i \cap N_{G'_{n+1}}(F)$ and $u', v' \in H_{3-i} \cap N_{G'_{n+1}}(F)$ (this is possible by (1)). Then (2.13) and (2) give

$$\sigma(h, h') = \sigma(twv, u't'w') = \sigma(twu', vt'w') = \sigma(u'tw, t'w'v) = \sigma(tw, t'w') = \sigma(t, t').$$

Hence for $i = 1$ ($h \in H_1$), we deduce the second assertion; i.e., $\sigma(h_1, h'_2) = \sigma(t_1, t'_2)$. When $i = 2$ ($h \in H_2$, $h' \in H_1$), $\sigma(h, h') = 1$ because of (5). Then, exactly as in [10, top of p. 157], a repeated application of (1.9) implies that $\sigma(h_1 h_2, h'_1 h'_2) = \sigma(h_1, h'_1)\sigma(h_2, h'_2)\sigma(h_1, h'_2)$. □

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Consider the group $M_k(F)$, which is a product of $H_2 = GL_k(F)$ and $H_1 = G_{n-k}(F)$. These subgroups clearly satisfy conditions (1)–(3) of Lemma 2.4. The following claim checks the validity of the other conditions.

Claim 2.5. *Let* $a = \prod_{i=1}^{k} \eta_i^\vee(a_i) \in T_{GL_k}(F)$ *and* $t = \prod_{i=1}^{n-k+1} \beta_i^\vee(t_i) \in T_{n-k+1}(F)$, *and consider their images in* $G_n(F)$, *given by (1.3) and (1.4). Then*

$$\sigma(a,t) = 1,$$
$$\sigma(t,a) = \begin{cases} c(t_1, \det a^{-1}) & k < n, \\ c(t_1^2, \det a^{-1}) & k = n. \end{cases}$$

In particular, condition (5) of Lemma 2.4 holds. Further, let $w_1 \in H_1 \cap \mathfrak{W}_{n+1}$ *and* $w_2 \in H_2 \cap \mathfrak{W}_{n+1}$. *Then* $\sigma(w_1, a) = 1$ *and, if* $r = 2$, $\sigma(w_2, t) = 1$, *whence condition (4) also holds.*

Proof. The computations of $\sigma(a,t)$ and $\sigma(t,a)$ are immediate from (1.3), (1.4), and Claim 2.1. The equality $\sigma(w_1, a) = 1$ follows from the fact that, for any $k + 2 \leqslant i \leqslant n + 1$ and $1 \leqslant j \leqslant k$, $\langle \alpha_i, \alpha_j^\vee \rangle = 0$, and then (2.3) implies that $w_{\alpha_i}^*$ commutes with $\alpha_j^{\vee *}(x)$ for any $x \in F^*$.

Regarding $\sigma(w_2, t)$, we show that, if $r = 2$,

$$\mathfrak{s}(w_2)\mathfrak{s}(t)\mathfrak{s}(w_2)^{-1} = \mathfrak{s}(t). \tag{2.18}$$

Then the result follows from (2.12) (note that $w_2 t = t w_2$). It is enough to establish this for $w = w_{\alpha_i}^*$ with $2 \leqslant i \leqslant k$.

First, assume that $k < n$. Using (2.3) and then (2.1) and (2.2),

$$\begin{aligned} w\alpha_{i+1}^{\vee}{}^*(t_1)\alpha_i^{\vee *}(t_1)\alpha_{i-1}^{\vee}{}^*(t_1)w^{-1} &= \alpha_i^{\vee *}(t_1)\alpha_{i+1}^{\vee}{}^*(t_1)\alpha_i^{\vee *}(t_1^{-2})\alpha_i^{\vee *}(t_1)\alpha_i^{\vee *}(t_1)\alpha_{i-1}^{\vee}{}^*(t_1) \\ &= \alpha_{i+1}^{\vee}{}^*(t_1)\alpha_i^{\vee *}(t_1)\alpha_{i-1}^{\vee}{}^*(t_1)\iota(c_{\alpha_i}(t_1, t_1^{-1})^2 c_{\alpha_i}(t_1^{-2}, t_1)) \\ &= \alpha_{i+1}^{\vee}{}^*(t_1)\alpha_i^{\vee *}(t_1)\alpha_{i-1}^{\vee}{}^*(t_1). \end{aligned}$$

This computation implies (2.18). For $k = n$, the computation is similar (one distinguishes between the cases $i < n$ and $i = n$). □

Remark 2.2. For a general r, the cocycle does not seem to satisfy a convenient formula on $M_k(F) \times M_k(F)$. One can remedy this by defining another isomorphism of $GL_k \times G_{n-k}$ with M_k. Let SL_k be generated by the simple roots $\{\alpha_i : 1 \leqslant i \leqslant k-1\}$, and let G'_{n-k} be generated by $\{\alpha_i : k+2 \leqslant i \leqslant n+1\}$. If $\{\eta_i^\vee\}$ are the cocharacters of T_{GL_k} and $\{\theta_i\}$ are the characters of T_{n-k+1}, map $\eta_i^\vee \mapsto \epsilon_i^\vee - \epsilon_{k+1}^\vee$ for $1 \leqslant i \leqslant k$ and $\theta_i^\vee \mapsto \epsilon_{k+i}^\vee$ for $1 \leqslant i \leqslant n-k+1$. This is a twist of the embeddings described in § 1.2, by a Weyl element of G'_{n+1}.

The evaluation of σ on $T_{GL_k}(F) \times T_{GL_k}(F)$ and $T_{n-k+1}(F) \times T_{n-k+1}(F)$ provides evidence that the cocycle indeed satisfies certain block-compatibility properties. Claim 2.1, along with (1.3)–(1.4), shows that, for any $0 \leqslant k \leqslant n$,

$$\sigma\left(\prod_{i=1}^{k} \eta_i^\vee(a_i), \prod_{i=1}^{k} \eta_i^\vee(a'_i)\right) = c(\det a, \det a') \prod_{1 \leqslant i < j \leqslant k} c(a_i, a'_j)^{-1},$$

$$\sigma\left(\prod_{i=1}^{n-k+1}\beta_i^\vee(t_i),\ \prod_{i=1}^{n-k+1}\beta_i^\vee(t_i')\right)=c(t_{n-k+1}^2,t'_{n-k+1})c(t_{n-k},{t'}_{n-k+1}^{-2})\\ \times\prod_{i=1}^{n-k}c(t_i,t_i')\prod_{i=1}^{n-k-1}c(t_i,{t'}_{i+1}^{-1}).$$

We see that $\sigma(a,a')=c(\det a,\det a')\sigma_k(a,a')$ for $a,a'\in T_{GL_k}(F)$ and for $t,t'\in T_{n-k+1}(F)$, $\sigma(t,t')=\sigma_{G'_{n-k+1}}(t,t')$.

Henceforth until the end of § 2, $r=2$. Lemma 2.4 and Claim 2.5 imply that, for $b,b'\in GL_k(F)$ and $h,h'\in G_{n-k}(F)$,

$$\sigma(bh,b'h')=\sigma(b,b')\sigma(h,h')c(\Upsilon(h),\det b').$$

The subgroup $GL_k(F)$ (of $M_k(F)$) is contained in the subgroup $SL_{k+1}(F)<G'_{n+1}(F)$ generated by $\{\alpha_i:1\leqslant i\leqslant k\}$. If we regard $SL_{k+1}(F)$ as a group of matrices in the standard way, i.e., identify $n_{\epsilon_i-\epsilon_j}(x)$ $(1\leqslant i<j\leqslant k+1,\ x\in F)$ with the matrix having 1 on the diagonal, x on the (i,j)th place, and 0 elsewhere, b takes the form $diag(\det b^{-1},b)$. Since $SL_{k+1}(F)$ is a standard subgroup (in the sense of [10]), the restriction of σ to $SL_{k+1}(F)\times SL_{k+1}(F)$ is just $\sigma_{SL_{k+1}}$ of [10, § 3]. We define a 2-cocycle σ_{GL_k} of $GL_k(F)$ via

$$\sigma_{GL_k}(b,b')=\sigma_{SL_{k+1}}(diag(\det b^{-1},b),diag(\det b'^{-1},b')).$$

Here, with a minor abuse of notation, we regard the arguments b and b' on the right-hand side as matrices. This cocycle is related to σ_k of [10] by

$$\sigma_{GL_k}(b,b')=c(\det b,\det b')\sigma_k(b',b).$$

Since $\sigma(b,b')=\sigma_{SL_{k+1}}(b,b')$, we get

$$\sigma(b,b')=\sigma_{GL_k}(b,b').$$

Regarding $G_{n-k}(F)$, as a subgroup of $M_k(F)$ its embedding in $G_n(F)$ is not contained in a standard subgroup of type G'_{n-k+1} (in contrast with the embedding described by Remark 2.2). However, it is still true that

$$\sigma(h,h')=\sigma_{G'_{n-k+1}}(h,h'). \tag{2.19}$$

To see this, first note that G'_{n-k} (as a subgroup of G_{n-k}) is generated by the roots $\{\alpha_i:k+2\leqslant i\leqslant n+1\}$, and is therefore a standard subgroup, whence

$$\sigma|_{G'_{n-k}(F)\times G'_{n-k}(F)}=\sigma_{G'_{n-k+1}}|_{G'_{n-k}(F)\times G'_{n-k}(F)}.$$

Moreover, according to the computation on $T_{n-k+1}(F)\times T_{n-k+1}(F)$ above,

$$\sigma|_{T_{n-k+1}(F)\times T_{n-k+1}(F)}=\sigma_{G'_{n-k+1}}|_{T_{n-k+1}(F)\times T_{n-k+1}(F)}.$$

According to Sun [79, Proposition 1], the restrictions of a cocycle (defined using a bilinear Steinberg symbol) to the derived group and to the torus determine it uniquely. Thus we conclude that (2.19) holds.

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Therefore,

$$\sigma(bh, b'h') = \sigma_{GL_k}(b, b')\sigma_{G'_{n-k+1}}(h, h')c(\Upsilon(h), \det b'). \tag{2.20}$$

Let $Q < G_n$ be a standard parabolic subgroup with a Levi part M isomorphic to $GL_{k_1} \times \cdots \times GL_{k_l} \times G_{n-k}$, where $k = k_1 + \cdots + k_l$. Our standard embedding $M < G_n$ was defined in § 1.2. According to (2.20), for $(b_1, \ldots, b_l, h), (b'_1, \ldots, b'_l, h') \in M(F)$ $(h, h' \in G_{n-k}(F))$,

$$\begin{aligned}&\sigma((b_1, \ldots, b_l, h), (b'_1, \ldots, b'_l, h'))\\&\quad = \left(\prod_{i=1}^{l} \sigma_{GL_{k_i}}(b_i, b'_i)c(\Upsilon(h), \det b'_i)\right)\left(\prod_{i>j}^{l} c(\det b_i, \det b'_j)\right)\sigma_{G'_{n-k+1}}(h, h').\end{aligned} \tag{2.21}$$

We mention one particularly convenient Levi subgroup, M_n. Then $c(\Upsilon(h), \cdot) = 1$ for all $h \in G_0(F)$ (see § 1.2). Therefore the subgroups $\widetilde{GL}_n(F)$ and $\widetilde{G}_0(F)$ of $\widetilde{M}_n(F)$ are commuting, and $\widetilde{M}_n(F)$ is simply their direct product with amalgamated μ_2. Also note that (since $r = 2$) the cover $\widetilde{G}_0(F)$ is abelian and splits, and $\sigma|_{G_0(F)\times G_0(F)}$ is trivial (Claim 2.2). Then

$$\sigma\left(\prod_{i=1}^{n} \eta_i^\vee(a_i)\beta_1^\vee(t_1), \prod_{i=1}^{n} \eta_i^\vee(a'_i)\beta_1^\vee(t'_1)\right) = c(\det a, \det a') \prod_{1\leqslant i<j\leqslant k} c(a_i, a'_j)^{-1}. \tag{2.22}$$

We end this section with a remark about a possible generalization of these block-compatibility results. Lemma 2.4 holds in the generality of the construction of [10]. That is, one can replace $G'_{n+1}(F)$ with an arbitrary simple simply connected split group G' over a local field. If $H < G'$ is a Levi subgroup and H_1 and H_2 are two direct factors of H, the first three conditions of the lemma hold, but the other two depend on the relations between the roots and coroots of H_1 and H_2. These might be 'too close', so there is no block compatibility.

Now consider a (proper) Levi subgroup $G < G'$, and assume that we are interested in its cover obtained by restricting the cover of G'. In this case we can apply the procedure described in Remark 2.2 to a maximal Levi subgroup H of G: if $\Delta_{G'}$ is the set of simple roots of G', $\Delta_G \subset \Delta_{G'} - \{\alpha\}$ for some $\alpha \in \Delta_{G'}$, then one can use α to define a subgroup of G' (not of G) isomorphic to H, on which block compatibility is expected to hold. The downside is that now we must switch back and forth between two isomorphic copies of H, but they are twists of each other (by a Weyl element of G').

Proving a general statement about the block compatibility of the cocycle should be possible along the lines described here. We hope to return to this problem in the future.

2.1.6. Subgroups of the torus. For $n > 1$, $\widetilde{T}_{n+1}(F)$ is not abelian. Its irreducible genuine representations are parameterized by genuine characters of its center, as follows from an analog of the Stone–von Neumann theorem [47, 61]. In this section we compute $C_{\widetilde{T}_{n+1}(F)}$ and several other subgroups, which will be used in § 2.2 to study these representations.

The following subgroup of $T_{n+1}(F)$ will play an important role in constructing irreducible representations of $\widetilde{T}_{n+1}(F)$:

$$T_{n+1}(F)^2 = \left\{ \prod_{i=1}^{n+1} \alpha_i^\vee(t_i) : \forall 1 \leqslant i \leqslant n, t_i \in F^{*2}, t_{n+1} \in F^* \right\}.$$

Equality (1.6) implies that $T_{n+1}(F)^2 = \{\prod_{i=1}^n \eta_i^\vee(a_i^2)\beta_1^\vee(t_1) : a_i, t_1 \in F^*\}$.

Claim 2.6. *The subgroup $T_{n+1}(F)^2$ splits under $\mathfrak{s}$. In particular, any character of $T_{n+1}(F)^2$ can be extended to a genuine character of $\widetilde{T}_{n+1}(F)^2$ (uniquely, using the splitting $\mathfrak{s}$).*

Proof. This is evident from (2.9), (2.1), and (2.2). The point is that $\alpha_i^{\vee *}(x)$ and $\alpha_j^{\vee *}(y)$ commute whenever both $x, y \in F^{*2}$ or $n+1 \in \{i, j\}$. □

Next we describe the center of $\widetilde{G}_n(F)$.

Claim 2.7. $C_{\widetilde{G}_n(F)} = \widetilde{C}_{G_n(F)}$ *and splits under* $\mathfrak{s}$.

Proof. Let $t = \prod_{i=1}^n \alpha_i^\vee(t_1^2)\alpha_{n+1}^\vee(t_1) \in C_{G_n(F)}$ (see (1.1)). Then $[t, g] = 1$ for all $g \in G_n(F)$, and we must show that $[t, g]_\sigma = 1$ ($[,]_\sigma$ was defined in § 2.1.1). Since $[,]_\sigma$ is bi-multiplicative, it suffices to consider $g = t', u, w$, where $t' \in T_{n+1}(F)$, $u \in N_n(F)$, and $w = w_{\alpha_i}$ with $2 \leqslant i \leqslant n+1$. Now $[t, t']_\sigma = 1$ because of (2.2), and $[t, u]_\sigma = 1$ by (2.13). Using (2.15), it is enough to show that $\mathfrak{s}(w)\mathfrak{s}(t)\mathfrak{s}(w)^{-1} = \mathfrak{s}(t)$. This follows from (2.1)–(2.3) and the observation in the proof of Claim 2.6. Now, the splitting follows from Claim 2.6. □

Recall that, by [47, Proposition 0.1.1],

$$C_{\widetilde{GL}_n(F)} = p^{-1}\left(\left\{ \prod_{i=1}^n \eta_i^\vee(d) : d \in (F^*)^{2/\gcd(2,n+1)} \right\}\right). \tag{2.23}$$

Also, set $T_{GL_n}(F)^2 = \{\prod_{i=1}^n \eta_i^\vee(a_i^2) : a_i \in F^*\}$. By [47, p. 57], $C_{\widetilde{T}_{GL_n}(F)} = \widetilde{T}_{GL_n}(F)^2 C_{\widetilde{GL}_n(F)}$.

Note that $C_{\widetilde{G}_n(F)} < \widetilde{T}_{n+1}(F)^2$, in contrast with the case of $\widetilde{GL}_n(F)$, where $C_{\widetilde{GL}_n(F)}$ is contained in $\widetilde{T}_{GL_n}(F)^2$ only when n is even. This causes technical difficulties when trying to adapt definitions of metaplectic tensor product from GL_n [42, 62, 81] to G_n; see §§ 2.2.4–2.2.5 below.

We compute the center of the torus.

Claim 2.8. *We have $C_{\widetilde{T}_{n+1}(F)} = \widetilde{T}_{n+1}(F)^2 C_{\widetilde{GL}_n(F)}$. More specifically,*

$$C_{\widetilde{T}_{n+1}(F)} = p^{-1}\left(\left\{ \prod_{i=1}^n \eta_i^\vee(a_i^2 d)\beta_1^\vee(t_1) : a_i \in F^*, d \in (F^*)^{2/\gcd(2,n+1)}, t_1 \in F^* \right\}\right). \tag{2.24}$$

Proof. In general, if $b = \prod_{i=1}^{n} \eta_i^\vee(b_i)\beta_1^\vee(t_1)$ and $b' = \prod_{i=1}^{n} \eta_i^\vee(b'_i)\beta_1^\vee(t'_1)$, equality (2.22) implies that

$$[b, b']_\sigma = \prod_{i=1}^{n} c(b_i, b'_i) c\left(\prod_{i=1}^{n} b_i, \prod_{i=1}^{n} b'_i\right). \tag{2.25}$$

Set $d = \prod_{i=1}^{n} b_i^{-1}$. On the one hand, for any $1 \leqslant j \leqslant n$ and $z \in F^*$, $[b, \eta_j^\vee(z)]_\sigma = c(b_j d^{-1}, z)$. Then, if $b \in C_{\widetilde{T}_{n+1}(F)}$, $b_j d^{-1} \in F^{*2}$, whence $b_j = a_j^2 d$ for some $a_j \in F^*$. It follows that $d^{n-1} \in F^{*2}$, or equivalently $d^{n+1} \in F^{*2}$ and $d \in (F^*)^{2/\gcd(2,n+1)}$. Hence b is of the required form. On the other hand, if $d \in (F^*)^{2/\gcd(2,n+1)}$,

$$\left[\prod_{i=1}^{n} \eta_i^\vee(a_i^2 d)\beta_1^\vee(t_1), \eta_j^\vee(z)\right]_\sigma = c(d^{n+1}, z) c(a_j^2, z) \prod_{i=1}^{n} c(a_i^2, z).$$

Because $c(d^{n+1}, z) = c(a_i^2, z) = 1$, we obtain $[\prod_{i=1}^{n} \eta_i^\vee(a_i^2 d)\beta_1^\vee(t_1), \eta_j^\vee(z)]_\sigma = 1$ for all $1 \leqslant j \leqslant n$. Since $[\prod_{i=1}^{n} \eta_i^\vee(a_i^2 d)\beta_1^\vee(t_1), \eta_{n+1}^\vee(z)]_\sigma = 1$ clearly holds and $[,]_\sigma$ is bi-multiplicative, any element on the right-hand side of (2.24) belongs to the center of $\widetilde{T}_{n+1}(F)$.

Now $C_{\widetilde{T}_{n+1}(F)} = \widetilde{T}_{n+1}(F)^2 C_{\widetilde{GL}_n(F)}$ follows immediately from (2.23). □

Remark 2.3. If n is even, $C_{\widetilde{GL}_n(F)} < \widetilde{T}_{n+1}(F)^2$, whence $C_{\widetilde{T}_{n+1}(F)} = \widetilde{T}_{n+1}(F)^2$.

Remark 2.4. In the case of the cover of $SO_{2n+1}(F)$ obtained by restricting the fourfold cover of $SL_{2n+1}(F)$ of Matsumoto [59], we have [18, § 6]

$$C_{\widetilde{T}_{SO_{2n+1}}(F)} = p^{-1}(\{diag(a_1^2, \ldots, a_n^2, 1, a_n^{-2}, \ldots, a_1^{-2}) : a_i \in F^*\}).$$

We describe certain maximal abelian subgroups of $\widetilde{T}_{n+1}(F)$. A representation of $C_{\widetilde{T}_{n+1}(F)}$ can always be extended to such a subgroup, and then induced to a representation of $\widetilde{T}_{n+1}(F)$.

Claim 2.9. *Assume that* $|2| = 1$ *and that* $q > 3$ *in* F. *The subgroup* $C(\widetilde{T}_{n+1}(F), \widetilde{T}_{n+1}(F) \cap K^*)$ *(i.e., the centralizer of* $\widetilde{T}_{n+1}(F) \cap K^*$ *in* $\widetilde{T}_{n+1}(F)$*; see § 1.1) is a maximal abelian subgroup of* $\widetilde{T}_{n+1}(F)$ *containing* $\widetilde{T}_{n+1}(F) \cap K^*$. *We have*

$$C(\widetilde{T}_{n+1}(F), \widetilde{T}_{n+1}(F) \cap K^*) = C_{\widetilde{T}_{n+1}(F)} \cdot (\widetilde{T}_{n+1}(F) \cap K^*). \tag{2.26}$$

Proof. Equality (2.26) follows if we show that $C(\widetilde{T}_{n+1}(F), \widetilde{T}_{n+1}(F) \cap K^*)$ is equal to

$$p^{-1}\left(\left\{\prod_{i=1}^{n} \eta_i^\vee(a_i d)\beta_1^\vee(t_1) : a_i \in \mathfrak{O}^* F^{*2}, d \in \mathfrak{O}^*(F^*)^{2/\gcd(2,n+1)}, t_1 \in F^*\right\}\right). \tag{2.27}$$

The proof of this is similar to the proof of Claim 2.8, and is therefore omitted. □

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It can be useful for applications to have a convenient choice of a maximal abelian subgroup, over any field. We follow the construction of Bump and Ginzburg [19, p. 141]. Let

$$T_{GL_n}(F)^{\mathrm{m}} = \left\{\prod_{i=1}^{n} \eta_i^{\vee}(a_i) : a_1, \ldots, a_n \in F^*, a_{n-2i}^{-1}a_{n-2i-1} \in F^{*2}, 0 \leqslant i < \lfloor n/2 \rfloor\right\}.$$

(This is T_F^{e} of [19].) Then $\widetilde{T}_{GL_n}(F)^{\mathrm{m}}$ is a maximal abelian subgroup. The advantage of this subgroup is that it always contains $\widetilde{C}_{GL_n(F)}$, regardless of the parity of n. Similarly, we have the following.

Claim 2.10. *Let*

$$T_{n+1}(F)^{\mathrm{m}} = \left\{\prod_{i=1}^{n} \eta_i^{\vee}(a_i)\beta_1^{\vee}(t_1) : a_1, \ldots, a_n \in F^*, a_{n-2i}^{-1}a_{n-2i-1} \in F^{*2}, 0 \leqslant i < \lfloor n/2 \rfloor, t_1 \in F^*\right\}.$$

Then $\widetilde{T}_{n+1}(F)^{\mathrm{m}}$ is a maximal abelian subgroup of $\widetilde{T}_{n+1}(F)$ containing $\widetilde{C}_{GL_n(F)}$.

Proof. If b belongs to an abelian subgroup of $\widetilde{T}_{n+1}(F)$ containing $\widetilde{T}_{n+1}(F)^{\mathrm{m}}$, write $p(b) = \prod_{i=1}^{n} \eta_i^{\vee}(b_i)\beta_1^{\vee}(t_1)$. Then, for all $0 \leqslant j < \lfloor n/2 \rfloor$ and $x, y \in F^*$, by (2.25),

$$[b, \eta_j^{\vee}(x)\eta_{j+1}^{\vee}(xy^2)]_{\sigma} = c(b_j, x)c(b_j, xy^2) = c(b_jb_{j+1}, x).$$

Because $c(b_jb_{j+1}, x)$ must be equal to 1, we obtain $b_j^{-1}b_{j+1} \in F^{*2}$, whence $b \in \widetilde{T}_{n+1}(F)^{\mathrm{m}}$.

To see that $\widetilde{T}_{n+1}(F)^{\mathrm{m}}$ is an abelian subgroup, let $t = \prod_{i=1}^{n} \eta_i^{\vee}(a_i)\beta_1^{\vee}(t_1)$, $t' = \prod_{i=1}^{n} \eta_i^{\vee}(a'_i)\beta_1^{\vee}(t'_1)$ belong to $T_{n+1}(F)^{\mathrm{m}}$. If n is even, $\prod_{i=1}^{n} a_i \in F^{*2}$ and (2.25) implies that $[t, t']_{\sigma} = \prod_{i=1}^{n} c(a_i, a'_i)$. This equals 1 because, if $a_{j+1}^{-1}a_j, {a'_{j+1}}^{-1}a'_j \in F^{*2}$,

$$c(a_j, a'_j)c(a_{j+1}, a'_{j+1}) = c(a_{j+1}a_{j+1}^{-1}a_j, a'_j)c(a_{j+1}, a'_{j+1}) = c(a_{j+1}, a'_ja'_{j+1}) = 1.$$

If n is odd, $\prod_{i=2}^{n} a_i \in F^{*2}$, and we proceed similarly. □

2.2. Principal series representations

2.2.1. Representations of the torus. We describe the representations of $\widetilde{T}_{n+1}(F)$. Since $r = 2$, $\widetilde{T}_{n+1}(F)$ is abelian for $n \leqslant 1$, as may be seen either by a direct verification using (2.2) or from Claim 2.2. For $n > 1$, it is a two-step nilpotent group.

We follow the exposition of McNamara [61] (§§ 13.5–13.7; we only use arguments which do not impose restrictions on the field). See also Kazhdan and Patterson [47, §§ 0.3, I.1–I.2] and Bump *et al.* [17, § 2].

Assume that $n > 1$. Since $\widetilde{T}_{n+1}(F)$ is a two-step nilpotent group, the genuine irreducible representations of $\widetilde{T}_{n+1}(F)$ are parameterized by the genuine characters of $C_{\widetilde{T}_{n+1}(F)}$. Let χ be a genuine character of $C_{\widetilde{T}_{n+1}(F)}$, and choose a maximal abelian subgroup $X < \widetilde{T}_{n+1}(F)$. Of course, $C_{\widetilde{T}_{n+1}(F)} < X$. We can extend χ to a character of X, and then induce it to

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$\widetilde{T}_{n+1}(F)$. Denote $\rho(\chi) = ind_X^{\widetilde{T}_{n+1}(F)}(\chi)$. This is a genuine irreducible representation, which is independent of the actual choices of X and the extension [61, Theorem 3]. If $n \leqslant 1$, we start with a genuine character χ of $C_{\widetilde{T}_{n+1}(F)} = \widetilde{T}_{n+1}(F)$ and put $\rho(\chi) = \chi$.

Extend $\rho(\chi)$ to a representation of $\widetilde{B}_n(F)$, trivially on $\mathfrak{s}(N_n(F))$, and induce to a representation $Ind_{\widetilde{B}_n(F)}^{\widetilde{G}_n(F)}(\rho(\chi))$, whose space we denote by $V(\chi)$. The elements $f \in V(\chi)$ are smooth complex-valued functions f on $\widetilde{G}_n(F) \times \widetilde{T}_{n+1}(F)$ such that $f(tug, 1) = \delta_{B_n(F)}^{1/2}(t) f(g, t)$ and $t \mapsto f(g, t)$ belongs to the space of $\rho(\chi)$ ($t \in \widetilde{T}_{n+1}(F)$, $u \in \mathfrak{s}(N_n(F))$, and $g \in \widetilde{G}_n(F)$). For brevity, we will write $f(g) = f(g, 1)$.

The Weyl group W_n acts on the representations χ and $\rho(\chi)$. If $\mathbf{w} \in W_n$ and ξ is a representation of $C_{\widetilde{T}_{n+1}(F)}$ or $\widetilde{T}_{n+1}(F)$, ${}^{\mathbf{w}}\xi$ denotes the representation on the space of ξ given by ${}^{\mathbf{w}}\xi(t) = \xi({}^{\mathfrak{s}(w)^{-1}}t)$, where $w \in \mathfrak{W}_n$ is the representative of $\mathbf{w}$. We have ${}^{\mathbf{w}}\rho(\chi) = \rho({}^{\mathbf{w}}\chi)$.

Let χ be a genuine character of $C_{\widetilde{T}_{n+1}(F)}$. Then $\rho(\chi)^{\wedge} = \rho(\chi^{\wedge}) = \rho(\chi^{-1})$ (recall from § 1.3.2 that $\rho(\chi)^{\wedge}$ is the contragradient representation).

A genuine character χ of $C_{\widetilde{T}_{n+1}(F)}$ is called regular if ${}^{\mathbf{w}}\chi \neq \chi$ for all $1 \neq \mathbf{w} \in W_n$. For such characters, $j_{N_n}(\rho(\chi))$ is semisimple [61, Proposition 5] and is the direct sum of $\rho({}^{\mathbf{w}}\chi)$, where $\mathbf{w}$ varies over W_n. If χ' is another genuine character, the dimension of

$$Hom_{\widetilde{G}_n}(Ind_{\widetilde{B}_n(F)}^{\widetilde{G}_n(F)}(\rho(\chi)), Ind_{\widetilde{B}_n(F)}^{\widetilde{G}_n(F)}(\rho(\chi')))$$

is zero unless $\chi' = {}^{\mathbf{w}}\chi$ for some $\mathbf{w}$, in which case the dimension is 1. These statements follow from Bernstein and Zelevinsky [12] (see [61]).

Let χ be a genuine character of $C_{\widetilde{T}_{n+1}(F)}$. There are unique $m_1, \ldots, m_{n+1} \in \mathbb{R}$ such that, for all $t \in C_{\widetilde{T}_{n+1}(F)}$, if $p(t) = \prod_{i=1}^{n+1} \alpha_i^{\vee}(t_i)$, $|\chi(t)| = \prod_{i=1}^{n+1} |t_i|^{m_i}$. Define $\mathrm{Re}\chi = \sum_{i=2}^{n+1} m_i \alpha_i$. We say that χ belongs to the positive Weyl chamber if $\langle \mathrm{Re}\chi, \alpha^{\vee} \rangle > 0$ for all $\alpha \in \Delta_{G_n}$.

For a genuine character χ of $C_{\widetilde{T}_{n+1}(F)}$, $\chi|_{\widetilde{T}_{n+1}(F)^2}$ is a genuine character, and, by Claim 2.6, it is obtained as the unique extension (via $\mathfrak{s}$) to $\widetilde{T}_{n+1}(F)^2$ of a character χ_0 of $T_{n+1}(F)^2$. According to Claim 2.8, one can further extend to $C_{\widetilde{T}_{n+1}(F)}$ and obtain χ, according to $\chi|_{C_{\widetilde{GL}_n(F)}}$. We will see that the 'important' properties of χ are shared by χ_0. Motivated by this observation, let $\Pi(\chi)$ be the set of genuine characters χ' of $C_{\widetilde{T}_{n+1}(F)}$ which agree with χ on $\widetilde{T}_{n+1}(F)^2$. By Claim 2.8, $|\Pi(\chi)| = 1$ or $[F^* : F^{*2}]$, depending on the parity of n.

2.2.2. Unramified representations. In this section, assume that $|2| = 1$ and that $q > 3$ in F. Then K is split under κ, and $K^* = \kappa(K)$ (§ 2.1.4). An irreducible genuine representation π of $\widetilde{G}_n(F)$ is called unramified if it has a non-zero vector fixed by K^*. Since Claim 2.3 implies that, for any $w \in \mathfrak{W}_n \subset K$, $\mathfrak{s}(w) = \kappa(w) \in K^*$, ${}^{\mathbf{w}}\pi$ is also unramified for all $\mathbf{w} \in W_n$.

Let χ be a genuine character of $C_{\widetilde{T}_{n+1}(F)}$. The subgroup $\widetilde{T}_{n+1}(F) \cap K^*$ is abelian (Claim 2.9). Let X be a maximal abelian subgroup of $\widetilde{T}_{n+1}(F)$ which contains $\widetilde{T}_{n+1}(F) \cap K^*$. According to Claim 2.9 and (2.26), $X = C(\widetilde{T}_{n+1}(F), \widetilde{T}_{n+1}(F) \cap K^*)$. Assume that χ can be extended to a character of X such that this extension is trivial on $\widetilde{T}_{n+1}(F) \cap K^*$.

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According to [61, Lemma 2], the subspace of $V(\chi)$ fixed by K^* is one dimensional. In particular, $Ind_{\widetilde{B}_n(F)}^{\widetilde{G}_n(F)}(\rho(\chi))$ is unramified, and we also call χ an unramified character. A function $f \in V(\chi)$ is unramified if it is fixed by K^*, and normalized if in addition $f(1) = 1$. We will use the following observation from [47, Lemma I.1.3], [61, Lemma 2]: let f be unramified. If $t \in \widetilde{T}_{n+1}(F)$ does not belong to X, $f(t) = 0$. This is because, for $k \in \widetilde{T}_{n+1}(F) \cap K^*$, $tk = [t, k]kt$, $\chi(k) = 1$, and one can choose k such that $[k, t] \neq 1$ (by the maximality of X); therefore

$$f(t) = f(tk, 1) = [k, t]\delta_{B_n(F)}^{1/2}(t)\chi(k)f(1, t) = [k, t]f(t). \tag{2.28}$$

According to (2.26), any genuine character of $C_{\widetilde{T}_{n+1}(F)}$, which is trivial on $C_{\widetilde{T}_{n+1}(F)} \cap K^*$, can be extended uniquely to a character of X which is trivial on $\widetilde{T}_{n+1} \cap K^*$. Therefore, any such character is unramified.

Let $\alpha \in \Sigma_{G_n}^+$. If α is a long root and $n > 1$, put $\mathfrak{l}(\alpha) = 2$; otherwise, $\mathfrak{l}(\alpha) = 1$. Set $a_\alpha = \alpha^{\vee*}(\varpi^{\mathfrak{l}(\alpha)})$. For any character χ of $C_{\widetilde{T}_{n+1}(F)}$ (not necessarily unramified), $\chi(a_\alpha)$ is defined.

2.2.3. Intertwining operators. Let χ be a genuine character of $C_{\widetilde{T}_{n+1}(F)}$, and let $\rho(\chi)$ be the corresponding representation of $\widetilde{T}_{n+1}(F)$ (see § 2.2.1). Let $\mathbf{w} \in W_n$. Put $N_n^w = {}^wN_n \cap N_n$. Let $M(w, \chi) : V(\chi) \to V({}^{\mathbf{w}}\chi)$ be the intertwining operator defined by

$$M(w, \chi)f(g) = \int_{N_n^w(F)\backslash N_n(F)} f(\mathfrak{s}(w)^{-1}\mathfrak{s}(u)g)\,du \quad (g \in \widetilde{G}_n(F)).$$

If $|2| = 1$ and $q > 3$ in F, $\mathfrak{s}(w) = \kappa(w) \in K^*$ (Claim 2.3), and we follow Casselman [21] in the normalization of the measure du. The following claim adapted from [47, §§ I.2 and I.6] provides the basic properties of the intertwining operators.

Claim 2.11. *The integral defining $M(w, \chi)$ is absolutely convergent if $\langle Re\chi, \alpha^\vee\rangle > 0$ for all $\alpha \in \Sigma_{G_n}^+$ such that $\mathbf{w}\alpha < 0$. The integral has a meromorphic continuation by which it is defined for all χ. If $\mathbf{w}, \mathbf{w}' \in W_n$, $ww' \in \mathfrak{W}_n$, and $l(ww') = l(w) + l(w')$, then $M(ww', \chi) = M(w, {}^{\mathbf{w}'}\chi)M(w', \chi)$. Finally, if χ is unramified and $f \in V(\chi)$ is the unramified normalized element, $M(w, \chi)f = c(\mathbf{w}, \chi)f$, where*

$$c(\mathbf{w}, \chi) = \prod_{\{\alpha \in \Sigma_{G_n}^+ : \mathbf{w}\alpha < 0\}} \frac{1 - q^{-1}\chi(a_\alpha)}{1 - \chi(a_\alpha)}.$$

Remark 2.5. The constant $c(\mathbf{w}, \chi)$ is given by the Gindikin–Karpelevich formula [21, § 3]. This formula was proved in the context of $\widetilde{GL}_n$ by [47, § I.2], and by McNamara [60, 61] for any split reductive algebraic group over a p-adic field, as long as $|\mu_{2r}| = 2r$ and q is coprime to $2r$.

Proof. We assume that $n > 1$, since otherwise the cover splits and the statement is well known. These results were proved by Kazhdan and Patterson [47, §§ I.2, I.6] for $\widetilde{GL}_n(F)$, over any local field. We provide the calculation of the constant $c(\mathbf{w}, \chi)$. Over a p-adic field, the meromorphic continuation can also be proved using the continuation principle of Bernstein; see Banks [9] (see also [47, p. 67] and [65]).

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Assume that χ is unramified, and take f as in the claim. It is enough to consider $\mathbf{w}_\alpha$ for $\alpha \in \Delta_{G_n}$. Let $2 \leqslant l \leqslant n+1$, and put $\alpha = \alpha_l$. The volume of $\mathfrak{O}$ with respect to the additive measure of F is 1. Hence

$$c(\mathbf{w}_\alpha, \chi) = M(w_\alpha, \chi) f(1) = 1 + \int_{\mathcal{U}_\alpha - K} f(w_\alpha^{*-1}\mathfrak{s}(u))\, du.$$

Write $u = n_\alpha(x)$ for some $x \in F$ with $x \notin \mathfrak{O}$; then $\mathfrak{s}(u) = n_\alpha^*(x)$. Using (2.2)–(2.7), we see that

$$w_\alpha^{*-1} n_\alpha^*(x) = \alpha^{\vee *}(x^{-1}) n_\alpha^*(-x) \alpha^{\vee *}(-1) w_\alpha^* n_\alpha^*(-x^{-1}) w_\alpha^{*-1}.$$

By virtue of Claim 2.3, we have $w_\alpha^* = \mathfrak{s}(w_\alpha) \in K^*$, $n_\alpha^*(-x^{-1}) = \mathfrak{s}(n_\alpha(-x^{-1})) \in K^*$ ($|x| > 1$), and $\alpha^{\vee *}(-1) = \mathfrak{s}(\alpha^\vee(-1)) \in K^*$. Therefore

$$f(w_\alpha^{*-1}\mathfrak{s}(u)) = f(\alpha^{\vee *}(x^{-1})) = \delta_{B_n(F)}^{1/2}(\alpha^\vee(x^{-1})) f(1, \alpha^{\vee *}(x^{-1})) = |x| f(1, \alpha^{\vee *}(x^{-1})).$$

Assume that $l \leqslant n$. If $f(1, \alpha^{\vee *}(x^{-1})) \neq 0$, by the observation in § 2.2.2 (see (2.28)), we must have $\alpha^{\vee *}(x^{-1}) \in C(\widetilde{T}_{n+1}(F), \widetilde{T}_{n+1}(F) \cap K^*)$. Assume that this holds, and set $l' = l+1$ if $l \leqslant n-1$; otherwise, $l' = n-1$ (recall that $n > 1$). Then equality (2.2) applied to $\alpha^{\vee *}(x^{-1})$ and $\alpha_{l'}^{\vee *}(z)$ implies that $c(x^{-1}, z) = 1$ for all $z \in \mathfrak{O}^*$, whence $x \in \mathfrak{O}^* F^{*2}$. We have shown that the integrand vanishes unless $x \in \mathfrak{O}^* F^{*2}$, in which case $f(1, \alpha^{\vee *}(x^{-1})) = \chi(\alpha^{\vee *}(x^{-1}))$. When $l = n+1$, this last equality holds for all $x \in F^*$.

Thus, for all $2 \leqslant l \leqslant n+1$,

$$\int_{\mathcal{U}_\alpha(F) - K} f(w_\alpha^{*-1}\mathfrak{s}(u))\, du = \sum_{v=1}^{\infty} \chi(a_\alpha)^v = (1 - q^{-1}) \frac{\chi(a_\alpha)}{1 - \chi(a_\alpha)}.$$

(The change $du \mapsto d^*x$ contributed a factor canceled by $\delta_{B_n(F)}^{1/2}$.) Therefore $c(\mathbf{w}_\alpha, \chi) = (1 - q^{-1}\chi(a_\alpha))/(1 - \chi(a_\alpha))$, completing the proof. □

2.2.4. Structure of certain induced representations. In this section, F is p-adic. For any $H < G_n$, set $H(F)^\star = \{h \in H(F) : \Upsilon(h) \in F^{*2}\}$. In particular, if $H = GL_k$, $H(F)^\star = \{h \in GL_k(F) : \det h \in F^{*2}\}$, and $G_0(F)^\star = G_0(F)$. The subgroup $H(F)^\star$ is normal in $H(F)$, and $H(F)^\star \backslash H(F)$ is abelian. The index of $H(F)^\star$ in $H(F)$ is either 1 or $[F^* : F^{*2}]$. These properties hold also for $\widetilde{H}(F)^\star (= p^{-1}(H(F)^\star))$ and $\widetilde{H}(F)$.

The subgroup $H(F)^\star$ is open in $H(F)$, and $\widetilde{H}(F)^\star$ is open in $\widetilde{H}(F)$ ([64, pp. 54–56]; see also [47, Proposition 0.1.2]).

If ξ is a genuine irreducible representation of $\widetilde{H}(F)$, let $\xi^\star = \xi|_{\widetilde{H}(F)^\star}$. The representation $\xi^\star$ is a finite sum of (at most $[F^* : F^{*2}]$) genuine irreducible representations [11, 2.9].

We will need the following results of Kable [42, in Propositions 3.1 and 3.2].

Proposition 2.12. *Let H be a Levi subgroup of GL_n, and let τ be a genuine irreducible representation of $\widetilde{H}(F)$. If n is odd, $\tau^\star$ is irreducible, and $ind_{\widetilde{H}(F)^\star}^{\widetilde{H}(F)}(\tau^\star) = \oplus_\omega \omega \cdot \tau$, where ω ranges over the (finite set of) characters of $\widetilde{H}(F)^\star \backslash \widetilde{H}(F)$. Furthermore, $Hom_{\widetilde{H}(F)}(\omega\tau, \tau) = 0$ unless $\omega = 1$. If n is even, $\tau^\star = \oplus_a {}^a\sigma$, where σ is any irreducible summand of $\tau^\star$ and a ranges over $\widetilde{H}(F)^\star \backslash \widetilde{H}(F)$. Furthermore, ${}^a\sigma \ncong \sigma$ for all $a \notin \widetilde{H}(F)^\star$.*

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Let H_1 be a Levi subgroup of GL_k, and let H_2 be a Levi subgroup of G_{n-k}. Then $H = H_1 \times H_2$ is a Levi subgroup of G_n. We assume that the cocycle defined on $G_n(F)$ satisfies the block-compatibility criterion (2.21).

Let τ be a genuine irreducible representation of $\widetilde{H}_1(F)$, and let π be a genuine irreducible representation of $\widetilde{H}_2(F)$. The following discussion describes a replacement for the usual tensor '$\tau \otimes \pi$', which is not defined when the groups are not commuting.

According to (2.21), the subgroups $\widetilde{H}_1(F)^\star$ and $\widetilde{H}_2(F)^\star$ are commuting in $\widetilde{G}_n(F)$, and hence

$$p^{-1}(H_1(F)^\star \times H_2(F)^\star) \cong \{(\zeta, \zeta) : \zeta \in \mu_2\} \backslash (\widetilde{H}_1(F)^\star \times \widetilde{H}_2(F)^\star).$$

Here on the right-hand side we have an outer direct product. In other words, $p^{-1}(H_1(F)^\star \times H_2(F)^\star)$ is the direct product of $\widetilde{H}_1(F)^\star$ and $\widetilde{H}_2(F)^\star$ with amalgamated μ_2. The representation $\tau^\star \otimes \pi^\star$ is defined as the usual tensor product, and it can be regarded as a genuine irreducible representation of $p^{-1}(H_1(F)^\star \times H_2(F)^\star)$. The representations $\tau^\star \otimes \pi$ and $\tau \otimes \pi^\star$ are defined similarly, because the same arguments apply to the pairs $(\widetilde{H}_1(F)^\star, \widetilde{H}_2(F))$ and $(\widetilde{H}_1(F), \widetilde{H}_2(F)^\star)$. We have the following semisimple representation:

$$I(\tau, \pi)^\star = ind_{p^{-1}(H_1(F)^\star \times H_2(F)^\star)}^{\widetilde{H}(F)}(\tau^\star \otimes \pi^\star).$$

In the case of $H_2 = G_0$, $H_2(F)^\star = H_2(F)$; hence $(\widetilde{H}_1(F), \widetilde{H}_2(F))$ do commute, and

$$I(\tau, \pi)^\star = ind_{p^{-1}(H_1(F)^\star \times H_2(F))}^{\widetilde{H}(F)}(\tau^\star \otimes \pi) = ind_{\widetilde{H}_1(F)^\star}^{\widetilde{H}_1(F)}(\tau^\star) \otimes \pi.$$

Remark 2.6. In this case, the tensor $\tau \otimes \pi$ is defined, and of course will be preferred over $I(\tau, \pi)^\star$. See Claim 2.21 in § 2.3.1 below.

Lemma 2.13. *Assume that* $0 < k < n$. *The representation* $I(\tau, \pi)^\star$ *is a direct sum of* $[F^* : F^{*2}]$ *copies of*

$$ind_{p^{-1}(H_1(F)^\star \times H_2(F))}^{\widetilde{H}(F)}(\tau^\star \otimes \pi).$$

Similarly, it is a direct sum of $[F^* : F^{*2}]$ *copies of*

$$ind_{p^{-1}(H_1(F) \times H_2(F)^\star)}^{\widetilde{H}(F)}(\tau \otimes \pi^\star).$$

Proof. The arguments are similar to those of Kable [42, Theorem 3.1]. Since $p^{-1}(H_1(F)^\star \times H_2(F)^\star)$ is a normal subgroup of $\widetilde{H}(F)$ with a finite index, and the quotient of $p^{-1}(H_1(F)^\star \times H_2(F))$ by $p^{-1}(H_1(F)^\star \times H_2(F)^\star)$ is abelian,

$$ind_{p^{-1}(H_1(F)^\star \times H_2(F)^\star)}^{\widetilde{H}(F)}(\tau^\star \otimes \pi^\star) = \bigoplus_{\omega} ind_{p^{-1}(H_1(F)^\star \times H_2(F))}^{\widetilde{H}(F)}(\tau^\star \otimes \omega\pi), \tag{2.29}$$

where the summation ranges over the characters of $\widetilde{H}_2(F)^\star \backslash \widetilde{H}_2(F)$ (these are non-genuine characters). Any such character ω takes the form $\omega(h) = \omega_a(h) = c(\Upsilon(h), \det a)$ for some $a \in \widetilde{H}_1(F)$. By the block compatibility of the cocycle (and (2.15)), $ha = c(\Upsilon(h), \det a)ah$

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$(a \in \widetilde{H}_1(F),\ h \in \widetilde{H}_2(F))$, whence $\tau^\star \otimes \omega_a \pi = {}^a({}^{a^{-1}}(\tau^\star) \otimes \pi)$. Thus the right-hand side of (2.29) is equal to a direct sum of $[\widetilde{H}_1(F) : \widetilde{H}_1(F)^\star] = [F^* : F^{*2}]$ copies of

$$ind_{p^{-1}(H_1(F)^\star \times H_2(F))}^{\widetilde{H}(F)}(\tau^\star \otimes \pi). \tag{2.30}$$

One can similarly consider $p^{-1}(H_1(F) \times H_2(F)^\star)$, and repeating the steps above obtain

$$\begin{aligned} ind_{p^{-1}(H_1(F)^\star \times H_2(F)^\star)}^{\widetilde{H}(F)}(\tau^\star \otimes \pi^\star) &= \bigoplus_{h \in \widetilde{H}_2(F)^\star \backslash \widetilde{H}_2(F)} ind_{p^{-1}(H_1(F) \times H_2(F)^\star)}^{\widetilde{H}(F)}(\omega_h \tau \otimes \pi^\star) \\ &= [F^* : F^{*2}] ind_{p^{-1}(H_1(F) \times H_2(F)^\star)}^{\widetilde{H}(F)}(\tau \otimes \pi^\star). \end{aligned}$$

Here, $\omega_h(a) = c(\Upsilon(h), \det a)$. □

The aforementioned results of Kable [42, Propositions 3.1, 3.2] do not apply to the group $G_n(F)$, mainly because its center does not play a role similar to that of the center of $GL_n(F)$. Namely, $C_{G_n(F)} < G_n(F)^\star$ for all n (see § 1.2). However, if H_2 is a Levi subgroup of a proper parabolic subgroup of G_n, it is possible to extend Proposition 2.12 to H_2. We will only need the following result on T_{n+1}.

Claim 2.14. *Let π be a genuine irreducible representation of $\widetilde{T}_{n+1}(F)$, and assume that n is even (including zero). Then $\pi^\star = \oplus_h {}^h\rho$, where ρ is any irreducible summand of $\pi^\star$ and h ranges over $\widetilde{T}_{n+1}(F)^\star \backslash \widetilde{T}_{n+1}(F)$.*

Proof. Equality (2.25) implies that $\widetilde{C}_{GL_n(F)}$ is contained in the center of $\widetilde{T}_{n+1}(F)^\star$ and, furthermore, if $z \in \widetilde{C}_{GL_n(F)}$ with $p(z) = \prod_{i=1}^n \eta_i^\vee(d)$ and $t \in \widetilde{T}_{n+1}(F)$ with $p(t) = \prod_{i=1}^n \eta_i^\vee(t_i)\beta_1^\vee(t_{n+1})$, $tzt^{-1} = c(\prod_{i=1}^n t_i, d)z$ (see (2.15) and (2.25)). Given these observations, the arguments of Kable [42, Proposition 3.2] readily apply to our case. □

Recall that for a genuine irreducible representation ξ of $\widetilde{T}_{n+1}(F)$ we have a corresponding genuine character χ_ξ of $C_{\widetilde{T}_{n+1}(F)}$, and $\xi = \rho(\chi_\xi)$. The same applies to representations of $\widetilde{T}_{GL_n}(F)$, and we use the same notation $\rho(\cdots)$.

Let χ_1' and χ_2' be genuine characters of $\widetilde{T}_{GL_k}(F)^2$ and $\widetilde{T}_{n-k+1}(F)^2$. Since

$$\widetilde{T}_{n+1}(F)^2 \cong \{(\zeta, \zeta) : \zeta \in \mu_2\} \backslash (\widetilde{T}_{GL_k}(F)^2 \times \widetilde{T}_{n-k+1}(F)^2), \tag{2.31}$$

the tensor representation $\chi_1' \otimes \chi_2'$ of $\widetilde{T}_{n+1}(F)^2$ is defined: it is a genuine character. For genuine characters χ_1 and χ_2 of $C_{\widetilde{T}_{GL_k}(F)}$ and $C_{\widetilde{T}_{n-k+1}(F)}$, put

$$\chi_1 \odot \chi_2 = \chi_1|_{\widetilde{T}_{GL_k}(F)^2} \otimes \chi_2|_{\widetilde{T}_{n-k+1}(F)^2}.$$

The last claim enables us to deduce the following result.

Lemma 2.15. *Assume that $0 < k \leqslant n$, and that τ and π are irreducible genuine representations of $\widetilde{T}_{GL_k}(F)$ and $\widetilde{T}_{n-k+1}(F)$ (respectively). Write $\tau = \rho(\chi_\tau)$ and $\pi = \rho(\chi_\pi)$, where χ_τ and χ_π are genuine characters of $C_{\widetilde{T}_{GL_k}(F)}$ and $C_{\widetilde{T}_{n-k+1}(F)}$ (respectively).*

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Also, let χ *be a genuine character of* $C_{\widetilde{T}_{n+1}(F)}$ *which agrees with* $\chi_\tau \odot \chi_\pi$ *on* $\widetilde{T}_{n+1}(F)^2$*. Then*

$$I(\tau,\pi)^\star = [F^* : F^{*2}]^r \bigoplus_{\chi' \in \Pi(\chi)} \rho(\chi'),$$

where $r = 2$ *if* n *and* k *are even and* $k < n$*,* $r = 0$ *if* n *is odd and* $k = n$*, and otherwise* $r = 0$*; the finite set* $\Pi(\chi)$ *was defined in* § *2.2.1.*

Proof. We follow the arguments of Kable [42, Theorem 3.1]. Applying Lemma 2.13 to $I(\tau,\pi)^\star$, we write it as a direct sum of $[F^* : F^{*2}]$ copies of

$$ind_{p^{-1}(T_{GL_k}(F)^\star \times T_{n-k+1}(F))}^{\widetilde{T}_{n+1}(F)}(\tau^\star \otimes \pi). \tag{2.32}$$

Now we consider the different cases.

(1) n is even, k is odd: by Proposition 2.12, the representation $\tau^\star$ is irreducible. Hence $\tau^\star \otimes \pi$ is also irreducible. As explained in the proof of Lemma 2.13, for any $a \in \widetilde{T}_{GL_k}(F)$, ${}^a(\tau^\star \otimes \pi) = \tau^\star \otimes \omega_a \pi$. Because π and $\omega_a\pi$ are both irreducible genuine representations of $\widetilde{T}_{n-k+1}(F)$, they are isomorphic if and only if their restrictions to $C_{\widetilde{T}_{n-k+1}(F)}$ are identical. Since $n-k$ is odd, $C_{\widetilde{GL}_{n-k}(F)} < C_{\widetilde{T}_{n-k+1}(F)}$. The restriction $\omega_a|_{C_{\widetilde{GL}_{n-k}(F)}}$ is trivial if and only if $a \in \widetilde{T}_{GL_k}(F)^\star$, whence

$$Hom_{\widetilde{T}_{n-k+1}(F)}(\omega_a\pi, \pi) = 0, \quad \forall a \notin \widetilde{T}_{GL_k}(F)^\star.$$

This means that $\tau^\star \otimes \pi$ satisfies Mackey's criterion, and therefore by Mackey's theory (2.32) is an irreducible representation of $\widetilde{T}_{n+1}(F)$. We claim that it is isomorphic to $\rho(\chi)$, which implies that $I(\tau,\pi)^\star = [F^* : F^{*2}]\rho(\chi)$ (since n is even, $\Pi(\chi) = \{\chi\}$).
Indeed, n is even; hence $C_{\widetilde{T}_{n+1}(F)} = \widetilde{T}_{n+1}(F)^2$, and it suffices to show that both representations agree on $\widetilde{T}_{n+1}(F)^2$. This holds because of (2.31) and $\widetilde{T}_{GL_k}(F)^2 < \widetilde{T}_{GL_k}(F)^\star$.

(2) n and k are even: write $\tau^\star = \oplus_a {}^a\sigma$ as in Proposition 2.12. Then (2.32) is the direct sum

$$\bigoplus_{a \in \widetilde{T}_{GL_k}(F)^\star \backslash \widetilde{T}_{GL_k}(F)} ind_{p^{-1}(T_{GL_k}(F)^\star \times T_{n-k+1}(F))}^{\widetilde{T}_{n+1}(F)}(\sigma \otimes \omega_a\pi). \tag{2.33}$$

If ${}^b(\sigma \otimes \omega_a\pi) \cong \sigma \otimes \omega_a\pi$ for some $b \in \widetilde{T}_{GL_k}(F)$, then ${}^b\sigma \cong \sigma$, which by Proposition 2.12 only happens for $b \in \widetilde{T}_{GL_k}(F)^\star$. Thus $\sigma \otimes \omega_a\pi$ satisfies Mackey's criterion, and each summand is irreducible. Since $n-k$ is even, $\omega_a|_{C_{\widetilde{T}_{n-k+1}(F)}} = 1$, whence $\omega_a\pi \cong \pi$, and all summands are equal. Thus if $k < n$ we get $I(\tau,\pi)^\star = [F^* : F^{*2}]^2\rho(\chi)$. If $k = n$, $I(\tau,\pi)^\star$ is equal to (2.32); hence $I(\tau,\pi)^\star = [F^* : F^{*2}]\rho(\chi)$.

(3) n is odd, k is even: as in case (2) consider (2.33). Because k is even, as above, each summand of (2.33) is irreducible. Since both n and $n-k$ are odd, any $z \in C_{\widetilde{GL}_n(F)}$ can be written as $z = z_1 z_2$, with $z_1 \in \widetilde{C}_{GL_k(F)}$ and $z_2 \in C_{\widetilde{GL}_{n-k}(F)}$, and,

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furthermore, $z \in C_{\widetilde{T}_{n+1}(F)}$ and $z_2 \in C_{\widetilde{T}_{n-k+1}(F)}$. Additionally, because k is even, $\widetilde{C}_{GL_k(F)}$ is contained in the center of $\widetilde{T}_{GL_k}(F)^{\star}$, and hence $\sigma|_{\widetilde{C}_{GL_k(F)}}$ is a character. Using these observations we see that the restrictions of $\sigma \otimes \omega_a \pi$ to $C_{\widetilde{GL}_n(F)}$ as a varies are different. Therefore the summands in (2.33) are inequivalent.

Each summand takes the form $\rho(\chi')$, where χ' is some genuine character of $C_{\widetilde{T}_{n+1}(F)}$, but χ' must agree with χ on $\widetilde{T}_{n+1}(F)^2$ because $\sigma|_{\widetilde{T}_{GL_k}(F)^2} = \tau|_{\widetilde{T}_{GL_k}(F)^2}$ ($\widetilde{T}_{GL_k}(F)^2 < C_{\widetilde{T}_{GL_k}(F)}$) and $\omega_a|_{\widetilde{T}_{n-k+1}(F)^2} = 1$. There are $[F^* : F^{*2}]$ such characters χ', and, because there are exactly $[F^* : F^{*2}]$ non-isomorphic summands in (2.33), all possible χ' appear. It follows that $I(\tau,\pi)^{\star} = [F^* : F^{*2}] \oplus_{\chi' \in \Pi(\chi)} \rho(\chi')$.

(4) n and k are odd: assume that $k < n$. Using Lemma 2.13,

$$\begin{aligned} I(\tau,\pi)^{\star} &= [F^* : F^{*2}] ind_{p^{-1}(T_{GL_k}(F) \times T_{n-k+1}(F)^{\star})}^{\widetilde{T}_{n+1}(F)} (\tau \otimes \pi^{\star}) \\ &= [F^* : F^{*2}] \bigoplus_{h \in \widetilde{T}_{n-k+1}(F)^{\star} \backslash \widetilde{T}_{n-k+1}(F)} ind_{p^{-1}(T_{GL_k}(F) \times T_{n-k+1}(F)^{\star})}^{\widetilde{T}_{n+1}(F)} (\omega_h \tau \otimes \rho). \end{aligned}$$

Here, $\omega_h(a) = c(\Upsilon(h), \det a)$ and $\pi^{\star} = \oplus_h {}^h\rho$, as in Claim 2.14. Since k is odd, by Proposition 2.12 we have $Hom_{\widetilde{T}_{GL_k}(F)}(\omega_{h'}\omega_h \tau, \omega_h \tau) = 0$ as long as $h' \notin \widetilde{T}_{n-k+1}(F)^{\star}$, and it follows that each of the last summands is irreducible. For $h \neq h'$, the summands are non-isomorphic: this follows as in case (3) (the opposite case with respect to k and $n-k$), because $\widetilde{C}_{GL_{n-k}(F)}$ is contained in the center of $\widetilde{T}_{n-k+1}(F)^{\star}$ (see (2.25)) and for $z \in C_{\widetilde{GL}_n(F)}$, $z = z_1 z_2$, where $z_1 \in C_{\widetilde{GL}_k(F)}$ and $z_2 \in \widetilde{C}_{GL_{n-k}(F)}$. Thus $I(\tau,\pi)^{\star} = [F^* : F^{*2}] \oplus_{\chi' \in \Pi(\chi)} \rho(\chi')$. Finally ,if $k = n$, by Proposition 2.12,

$$I(\tau,\pi)^{\star} = ind_{\widetilde{T}_{GL_n}(F)^{\star}}^{\widetilde{T}_{GL_n}(F)} (\tau^{\star}) \otimes \pi = \oplus_{\omega} (\omega\tau \otimes \pi) = \oplus_{\chi' \in \Pi(\chi)} \rho(\chi'). \qquad \square$$

Remark 2.7. One could define $I(\tau,\pi)^{\star} = ind_{p^{-1}(H_1(F) \times H_2(F)^{\star})}^{\widetilde{H}(F)} (\tau \otimes \pi^{\star})$. With this definition, the case $k = n$ is simpler: $I(\tau,\pi)^{\star} = \tau \otimes \pi$. Lemma 2.13 implies that, for $k < n$,

$$ind_{p^{-1}(H_1(F) \times H_2(F)^{\star})}^{\widetilde{H}(F)} (\tau \otimes \pi^{\star}) \cong ind_{p^{-1}(H_1(F)^{\star} \times H_2(F))}^{\widetilde{H}(F)} (\tau^{\star} \otimes \pi).$$

Then in Lemma 2.15 we obtain smaller multiplicities. However, the presentation in §§ 2.3.1–2.3.2 below seems to be simpler with the definition above.

2.2.5. Discussion on the metaplectic tensor product. Irreducible representations of Levi subgroups of classical groups are usually described in terms of the tensor product. For metaplectic groups the direct factors of Levi subgroups do not necessarily commute, and hence the tensor construction cannot be extended in a straightforward manner.

The metaplectic tensor product in the context of GL_n has been studied by various authors [24, 42, 62, 78, 81]. Kable [42] used a representation similar to $I(\tau,\pi)^{\star}$ (see § 2.2.4) to define a metaplectic tensor product, between genuine indecomposable representations of Levi subgroups of the double cover of $GL_n(F)$. He studied the induced space $I(\tau_1, \tau_2)^{\star}$,

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where τ_i are representations of $\widetilde{L}_i(F)$ and $L_i < GL_{n_i}$ are Levi subgroups. To any genuine character ω of $C_{\widetilde{GL}_n(F)}$, Kable defined $\tau_1 \otimes_\omega \tau_2$ as an indecomposable summand of $I(\tau_1, \tau_2)^\star$, on which $C_{\widetilde{GL}_n(F)}$ acts by ω. He proved that this construction satisfies many of the useful properties of the tensor product, e.g., associativity, compatibility with taking contragradients, and certain compatibility with Jacquet functors.

Essential for his approach was the Mackey theory for the restriction $\tau_i^\star = \tau_i|_{\widetilde{L}_i(F)^\star}$, which he developed for the double cover [42, Propositions 3.1, 3.2]. The main obstacle in trying to extend his construction to our setting is the lack of an analog of these results for $G_n(F)$.

Mezo [62] considered r-fold covers. Denote $GL_m(F)^r = \{b \in GL_m(F) : \det b \in F^{*r}\}$. Mezo defined a tensor representation for $L = GL_{n_1} \times \cdots \times GL_{n_k}$ using restrictions from $GL_{n_i}(F)$ to $GL_{n_i}(F)^r$ and a process resembling the construction in § 2.2.1. Takeda [81] constructed a global metaplectic tensor product, whose local components agree with those of Mezo [62]. He proved several properties for the global (and local) tensor, e.g., compatibility with induction and automorphicity.

For our purposes it will be sufficient to consider the space $I(\tau, \pi)^\star$, on which we can compute certain Jacquet functors simply enough. The lack of a universal property for the metaplectic tensor product places a heavy burden on details. At present, we do not attempt to extend the results of [62, 81] to our context.

2.3. Exceptional (small) representations

2.3.1. Definition and basic properties. We construct exceptional representations, adapting the results of Bump *et al.* [17] in the context of SO_{2n+1}. These representations were called ‘small’ in [17].

Let χ be a genuine character of $C_{\widetilde{T}_{n+1}(F)}$. We say that χ is exceptional if, for all $\alpha \in \Delta_{G_n}$, $\chi(\alpha^{\vee *}(x^{\mathfrak{l}(\alpha)})) = |x|$ for all $x \in F^*$. In particular, such a character is regular.

Let χ_0 be a character of $T_{n+1}(F)^2$ such that χ is obtained from χ_0 by extension, as explained in § 2.2.1. Since $\mathfrak{s}(\alpha^\vee(x^{\mathfrak{l}(\alpha)})) = \alpha^{\vee *}(x^{\mathfrak{l}(\alpha)})$, χ_0 satisfies $\chi_0(\alpha^\vee(x^{\mathfrak{l}(\alpha)})) = |x|$ for all $\alpha \in \Delta_{G_n}$ and $x \in F^*$. We see that χ is exceptional if and only if χ_0 satisfies this property. Refer to § 2.3.3 below for a detailed construction of an exceptional character.

In this section, except in Proposition 2.16, the field is p-adic. Recall that $\mathbf{w_0}$ denotes the longest element of W_n, and $w_0 \in \mathfrak{W}_n$ is the representative. Additionally, for brevity, when we refer to a Jacquet functor applied to a genuine representation along some unipotent subgroup U, we drop $\mathfrak{s}$ from the notation; for example, we write $j_U(\cdots)$ instead of $j_{\mathfrak{s}(U)}(\cdots)$.

As in [17, Theorem 2.2], we have the ‘periodicity theorem’ (see also [47, Theorem I.2.9]).

Proposition 2.16. *Let χ be an exceptional character. The representation $\Theta_{G_n,\chi}$ on the space $M(w_0, \chi)V(\chi)$ is irreducible: it is the unique irreducible subrepresentation of $V({}^{\mathbf{w_0}}\chi)$ and the unique irreducible quotient of $V(\chi)$. Furthermore, if F is p-adic, $j_{N_n}(\Theta_{G_n,\chi}) = \rho({}^{\mathbf{w_0}}\chi)$.*

Proof. Consider the p-adic case first. The facts that $V({}^{\mathbf{w_0}}\chi)$ has a unique irreducible subrepresentation and $V(\chi)$ has a unique irreducible quotient follow from the Langlands

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quotient theorem proved for metaplectic groups by Ban and Jantzen [8]. This is because χ belongs to the positive Weyl chamber. Now it is left to compute the Jacquet module and establish the irreducibility of $\Theta_{G_n,\chi}$.

One argues exactly as in [17], starting with computation of $j_{N_n}(\Theta_{G_n,\chi})$, which implies that $\Theta_{G_n,\chi}$ is irreducible. The calculation follows from the properties of the Jacquet functor described in § 2.2.1, Claim 2.11, the observation that when $|2| = 1$ and $q > 3$, $1 - q^{-1}({}^{\mathbf{w}_\alpha}\chi(a_\alpha)) = 0$ for all $\alpha \in \Delta_{G_n}$, and a computation on $G_1(F)$ and $\widetilde{SL}_2(F)$ in the remaining cases ($\widetilde{G}_1(F)$ is split).

If F is Archimedean, the proposition follows from the Langlands quotient theorem [54], whose proof by Borel and Wallach [13] is applicable to cover groups (see [8]). Note that the results of [8] do not include the characterization of the Langlands quotient in terms of the intertwining operators. □

The representation $\Theta_{G_n,\chi}$ is the exceptional representation corresponding to the exceptional character χ.

We describe the contragradient exceptional representation.

Claim 2.17. *We have $\Theta^{\wedge}_{G_n,\chi} = \Theta_{G_n,{}^{\mathbf{w_0}}\chi^{\wedge}}$. Furthermore, the restriction of ${}^{\mathbf{w_0}}\chi^{\wedge}$ to $\widetilde{T}_{n+1}(F)^2$ agrees with the restriction of χ multiplied by some non-genuine character λ of $\widetilde{T}_{n+1}(F)^2$.*

Proof. Put $\chi' = {}^{\mathbf{w_0}}\chi^{\wedge}$. Proposition 2.16 implies that $\Theta^{\wedge}_{G_n,\chi}$ is an irreducible quotient of $V(\chi')$. Let χ_0 (respectively, χ'_0) be a character of $T_{n+1}(F)^2$ such that $\chi|_{\widetilde{T}_{n+1}(F)^2}$ (respectively, $\chi'|_{\widetilde{T}_{n+1}(F)^2}$) is the extension of χ_0 (respectively, χ'_0). If $t \in T_{n+1}(F)^2$, equality (1.7) implies that $\chi'_0 = \chi_0(t \cdot \beta_1^{\vee}(\Upsilon(t)))$. Denote $\lambda(t) = \chi_0(\beta_1^{\vee}(\Upsilon(t)))$. Since $\beta_1^{\vee}(x) \in T_{n+1}(F)^2$ for all $x \in F^*$, λ is a character of $T_{n+1}(F)^2$, and $\chi'_0 = \lambda \cdot \chi_0$. It follows that $\chi'|_{\widetilde{T}_{n+1}(F)^2} = \lambda \cdot (\chi|_{\widetilde{T}_{n+1}(F)^2})$. In particular, χ' is an exceptional character. Then, by Proposition 2.16, $\Theta_{G_n,\chi'}$ is the unique irreducible quotient of $V(\chi')$. Hence $\Theta^{\wedge}_{G_n,\chi} = \Theta_{G_n,\chi'}$. □

Remark 2.8. In contrast with $\Theta_{SO_{2n+1}}$ of [17], the representation $\Theta_{G_n,\chi}$ is not self-dual.

The next proposition describes an exceptional representation in terms of exceptional representations of GL_k and G_{n-k}. We briefly recall the construction of exceptional representations of Kazhdan and Patterson [47, § I.1 and Theorem I.2.9]. Let χ_1 be a genuine character of $C_{\widetilde{T}_{GL_k}(F)} = \widetilde{T}_{GL_k}(F)^2 C_{\widetilde{GL}_k(F)}$. Call χ_1 exceptional if $\chi_1(\mathfrak{s}(\eta_i^{\vee}(x^2)\eta_{i+1}^{\vee}(x^{-2}))) = |x|$ for all $1 \leqslant i < k$ and $x \in F^*$. In this case the exceptional representation Θ_{GL_k,χ_1} is the unique irreducible quotient of $Ind_{\widetilde{B}_{GL_k}(F)}^{\widetilde{GL}_k(F)}(\rho(\chi_1))$. It is also the unique irreducible subrepresentation of $Ind_{\widetilde{B}_{GL_k}(F)}^{\widetilde{GL}_k(F)}(\rho({}^{\mathbf{w}'_0}\chi_1))$, where $\mathbf{w}'_0$ is the longest element of the Weyl group of GL_k. Note that, for $1 \leqslant i < n$, $\mathfrak{s}(\eta_i^{\vee}(x^2)\eta_{i+1}^{\vee}(x^{-2})) = {\alpha_{i+1}^{\vee}}^*(x^{\mathfrak{l}(\alpha_{i+1})})$.

Let $\mathbf{w}''_0$ be the longest element of W_{n-k} (the Weyl group of G_{n-k}). Also recall the representation $\chi_1 \odot \chi_2$ defined in § 2.2.4.

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The following claim relates an exceptional character of $C_{\widetilde{T}_{n+1}(F)}$ to a pair of such characters of $C_{\widetilde{T}_{GL_k}(F)}$ and $C_{\widetilde{T}_{n-k+1}(F)}$.

Claim 2.18. *Let χ be an exceptional character, and let $0 \leqslant k \leqslant n$. There are exceptional characters χ_1 and χ_2 of $C_{\widetilde{T}_{GL_k}(F)}$ and $C_{\widetilde{T}_{n-k+1}(F)}$ such that*

$$ {}^{\mathbf{w_0}}\chi|_{\widetilde{T}_{n+1}(F)^2} = {}^{\mathbf{w}'_0}\chi_1 \odot {}^{\mathbf{w}''_0}\chi_2. \tag{2.34}$$

Proof. Since $\widetilde{T}_{n+1}(F)^2$ is split under $\mathfrak{s}$, one can regard χ as a character of $T_{n+1}(F)^2$ satisfying $\chi(\alpha^\vee(x^{\mathfrak{l}(\alpha)})) = |x|$ for all $\alpha \in \Delta_{G_n}$ and $x \in F^*$. It is enough to show that there are characters χ_1 and χ_2 of $T_{GL_k}(F)^2$ and $T_{n-k+1}(F)^2$ with the following properties.

(1) $\chi_1(\eta_i^\vee(x^2)\eta_{i+1}^\vee(x^{-2})) = |x|$ for all $1 \leqslant i < k$ and $x \in F^*$.

(2) $\chi_2(\alpha_i^\vee(x^{\mathfrak{l}(\alpha_i)})) = |x|$ for all $k+2 \leqslant i \leqslant n+1$ and $x \in F^*$.

(3) ${}^{\mathbf{w_0}}\chi|_{T_{n+1}(F)^2} = {}^{\mathbf{w}'_0}\chi_1 \otimes {}^{\mathbf{w}''_0}\chi_2$.

Indeed, given such characters, one can extend them to genuine characters of $\widetilde{T}_{GL_k}(F)^2$ and $\widetilde{T}_{n-k+1}(F)^2$, and then extend them again, not necessarily uniquely (depending on the parity of k and n), to exceptional characters of the corresponding centers.

If $a = \prod_{i=1}^{k} \eta_i^\vee(a_i^2)$ and $t = \prod_{i=1}^{n-k} \beta_i^\vee(t_i^2)\beta_{n-k+1}^\vee(t_{n-k+1})$, using (1.2)–(1.4) and (1.7), we get

$$ {}^{\mathbf{w_0}}(at) = \begin{cases} \displaystyle\prod_{i=1}^{k} \eta_i^\vee(a_i^{-2}) \prod_{i=1}^{n-k} \beta_i^\vee(t_i^{-2}t_1^4 \det a^{-2})\beta_{n-k+1}^\vee(t_{n-k+1}^{-1}t_1^2 \det a^{-1}) & k < n, \\ \displaystyle\prod_{i=1}^{n} \eta_i^\vee(a_i^{-2})\beta_1^\vee(\det a^{-1} t_1) & k = n. \end{cases} $$

Set $\eta(a) = \chi(\prod_{i=1}^{n-k} \beta_i^\vee(\det a^{-2})\beta_{n-k+1}^\vee(\det a^{-1}))$, $\chi_1 = \eta \cdot {}^{\mathbf{w}'_0}(\chi|_{T_{GL_k}(F)^2})^{-1}$, and $\chi_2 = \chi|_{T_{n-k+1}(F)^2}$. Clearly these characters satisfy the above properties. □

In general, if $0 < k \leqslant n$ and χ_1 and χ_2 are exceptional characters of $C_{\widetilde{T}_{GL_k}(F)}$ and $C_{\widetilde{T}_{n-k+1}(F)}$, we have the representation $I(\Theta_{GL_k,\chi_1}, \Theta_{G_{n-k},\chi_2})^\star$ defined in § 2.2.4:

$$ I(\Theta_{GL_k,\chi_1}, \Theta_{G_{n-k},\chi_2})^\star = ind_{p^{-1}(GL_k(F)^\star \times G_{n-k}(F)^\star)}^{\widetilde{M}_k(F)}(\Theta^\star_{GL_k,\chi_1} \otimes \Theta^\star_{G_{n-k},\chi_2}). $$

The following two results are analogs of [17, Theorem 2.3 and Proposition 2.4].

Proposition 2.19. *Let χ be an exceptional character, and let $0 < k \leqslant n$. Then, for any exceptional characters χ_1 and χ_2 of $C_{\widetilde{T}_{GL_k}(F)}$ and $C_{\widetilde{T}_{n-k+1}(F)}$ satisfying (2.34),*

$$ j_{U_k}(\Theta_{G_n,\chi}) \subset I(\Theta_{GL_k,\chi_1}, \Theta_{G_{n-k},\chi_2})^\star. $$

Proof. We start with computing the Jacquet module along $N_{GL_k} \times N_{n-k} < M_k$ of both representations. The double coset space

$$ (GL_k(F)^\star G_{n-k}(F)^\star)\backslash M_k(F)/(T_{n+1}(F)N_{GL_k}(F)N_{n-k}(F)) $$

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has one element. According to the geometric lemma of Bernstein and Zelevinsky [12, Theorem 5.2],

$$\begin{aligned} & j_{N_{GL_k}N_{n-k}}(I(\Theta_{GL_k,\chi_1},\Theta_{G_{n-k},\chi_2})^{\star}) \\ & \quad = ind_{p^{-1}(T_{GL_k}(F)^{\star}\times T_{n-k+1}(F)^{\star})}^{\widetilde{T}_{n+1}(F)}(j^{\star}_{N_{GL_k}}(\Theta^{\star}_{GL_k,\chi_1})\otimes j^{\star}_{N_{n-k}}(\Theta^{\star}_{G_{n-k},\chi_2})). \end{aligned} \tag{2.35}$$

Here, $j^{\star}_{N_{GL_k}}$ is the Jacquet functor taking representations of $\widetilde{GL}_k(F)^{\star}$ to representations of $\widetilde{T}_{GL_k}(F)^{\star}$, and $j^{\star}_{N_{n-k}}$ is defined similarly. We have

$$j^{\star}_{N_{GL_k}}(\Theta^{\star}_{GL_k,\chi_1}) = (j_{N_{GL_k}}(\Theta_{GL_k,\chi_1}))^{\star} = \rho({}^{\mathbf{w}_0'}\chi_1)^{\star},$$

where for the first equality we used the fact that $N_{GL_k}(F) < \widetilde{GL}_k(F)^{\star}$; the second equality follows from [47, Theorem I.2.9]. Similarly, using Proposition 2.16, we get $j^{\star}_{N_{n-k}}(\Theta^{\star}_{G_{n-k},\chi_2}) = \rho({}^{\mathbf{w}_0''}\chi_2)^{\star}$. Applying Lemma 2.15 to the right-hand side of (2.35) implies that

$$j_{N_{GL_k}N_{n-k}}(I(\Theta_{GL_k,\chi_1},\Theta_{G_{n-k},\chi_2})^{\star}) = m\bigoplus_{\chi'}\rho(\chi'),$$

where $m > 0$ is an integer (depending only on k and n), and the summation is over all genuine characters χ' of $C_{\widetilde{T}_{n+1}(F)}$ such that

$$\chi'|_{\widetilde{T}_{n+1}(F)^2} = {}^{\mathbf{w}_0'}\chi_1 \odot {}^{\mathbf{w}_0''}\chi_2 = {}^{\mathbf{w}_0}\chi|_{\widetilde{T}_{n+1}(F)^2}$$

(the second equality is (2.34)). Thus ${}^{\mathbf{w}_0}\chi$ appears in the summation at least once.

Since $j_{N_{GL_k}N_{n-k}}(I(\Theta_{GL_k,\chi_1},\Theta_{G_{n-k},\chi_2})^{\star})$ is semisimple and the Jacquet functor is exact, $j_{N_{GL_k}N_{n-k}}(\mathcal{V}_0)$ is semisimple for any $\mathcal{V}_0 \subset I(\Theta_{GL_k,\chi_1},\Theta_{G_{n-k},\chi_2})^{\star}$. Thus there is an irreducible $\mathcal{V} \subset I(\Theta_{GL_k,\chi_1},\Theta_{G_{n-k},\chi_2})^{\star}$ such that $\rho({}^{\mathbf{w}_0}\chi)$ is a quotient of $j_{N_{GL_k}N_{n-k}}(\mathcal{V})$. Set

$$I^{\widetilde{M}_k}(\rho({}^{\mathbf{w}_0}\chi)) = Ind_{p^{-1}(T_{n+1}(F)N_{GL_k}(F)N_{n-k}(F))}^{\widetilde{M}_k(F)}(\rho({}^{\mathbf{w}_0}\chi)).$$

Then Frobenius reciprocity shows that

$$Hom_{\widetilde{M}_k(F)}(\mathcal{V}, I^{\widetilde{M}_k}(\rho({}^{\mathbf{w}_0}\chi))) = Hom_{\widetilde{T}_{n+1}(F)}(j_{N_{GL_k}N_{n-k}}(\mathcal{V}), \rho({}^{\mathbf{w}_0}\chi)) \neq 0, \tag{2.36}$$

whence $\mathcal{V} \subset I^{\widetilde{M}_k}(\rho({}^{\mathbf{w}_0}\chi))$. According to the Langlands quotient theorem [8], the representation $I^{\widetilde{M}_k}(\rho({}^{\mathbf{w}_0}\chi))$ has a unique irreducible subrepresentation, which is $\mathcal{V}$.

Let us turn to $j_{U_k}(\Theta_{G_n,\chi})$. Proposition 2.16 implies that

$$j_{N_{GL_k}N_{n-k}}(j_{U_k}(\Theta_{G_n,\chi})) = j_{N_n}(\Theta_{G_n,\chi}) = \rho({}^{\mathbf{w}_0}\chi).$$

The representation $j_{U_k}(\Theta_{G_n,\chi})$, as a Jacquet module of a quotient of $V(\chi)$ with respect to a non-minimal unipotent radical, does not have a cuspidal constituent. Hence, since $j_{N_{GL_k}N_{n-k}}(j_{U_k}(\Theta_{G_n,\chi}))$ is irreducible, so is $j_{U_k}(\Theta_{G_n,\chi})$. Now (2.36) with $j_{U_k}(\Theta_{G_n,\chi})$ instead of $\mathcal{V}$ shows $j_{U_k}(\Theta_{G_n,\chi}) \subset I^{\widetilde{M}_k}(\rho({}^{\mathbf{w}_0}\chi))$. Thus $j_{U_k}(\Theta_{G_n,\chi}) = \mathcal{V} \subset I(\Theta_{GL_k,\chi_1},\Theta_{G_{n-k},\chi_2})^{\star}$. □

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Corollary 2.20. *Let χ be an exceptional character, and let $0 < k \leqslant n$. For any exceptional characters χ_1, χ_3 of $C_{\widetilde{T}_{GL_k}(F)}$ and χ_2, χ_4 of $C_{\widetilde{T}_{n-k+1}(F)}$ such that (χ_1, χ_2) satisfy (2.34) with respect to χ, and $({}^{\mathbf{w}_0'}\chi_3^\wedge, {}^{\mathbf{w}_0''}\chi_4^\wedge)$ satisfy (2.34) with respect to ${}^{\mathbf{w}_0}\chi^\wedge$,*

$$Hom_{\widetilde{G}_n(F)}(\Theta_{G_n,\chi}, Ind_{\widetilde{Q}_k(F)}^{\widetilde{G}_n(F)}(I(\Theta_{GL_k,\chi_1}, \Theta_{G_{n-k},\chi_2})^\star)) \neq 0,$$

$$Hom_{\widetilde{G}_n(F)}(Ind_{\widetilde{Q}_k(F)}^{\widetilde{G}_n(F)}(I(\Theta_{GL_k,\chi_3}, \Theta_{G_{n-k},\chi_4})^\star), \Theta_{G_n,\chi}) \neq 0.$$

Proof. The first assertion follows from Proposition 2.19 using the Frobenius reciprocity. Dualizing,

$$Hom_{\widetilde{G}_n(F)}(Ind_{\widetilde{Q}_k(F)}^{\widetilde{G}_n(F)}((I(\Theta_{GL_k,\chi_1}, \Theta_{G_{n-k},\chi_2})^\star)^\wedge), \Theta_{G_n,\chi}^\wedge) \neq 0.$$

In general, if ξ is a genuine representation of $\widetilde{H}(F)$, where $H < G_n$, $(\xi^\star)^\wedge = (\xi^\wedge)^\star$ (because $\widetilde{H}(F)^\star$ contains an open neighborhood of the identity of $\widetilde{H}(F)$). Applying Claim 2.17 and its analog for GL_n, we get

$$(I(\Theta_{GL_k,\chi_1}, \Theta_{G_{n-k},\chi_2})^\star)^\wedge = I(\Theta_{GL_k, {}^{\mathbf{w}_0'}\chi_1^\wedge}, \Theta_{G_{n-k}, {}^{\mathbf{w}_0''}\chi_2^\wedge})^\star.$$

Now replace χ with ${}^{\mathbf{w}_0}\chi^\wedge$, and set $\chi_1 = {}^{\mathbf{w}_0'}\chi_3^\wedge$ and $\chi_2 = {}^{\mathbf{w}_0''}\chi_4^\wedge$. We see that

$$Hom_{\widetilde{G}_n(F)}(Ind_{\widetilde{Q}_k(F)}^{\widetilde{G}_n(F)}(I(\Theta_{GL_k,\chi_3}, \Theta_{G_{n-k},\chi_4})^\star), \Theta_{G_n,\chi}) \neq 0. \qquad \square$$

In the case when $k = n$, we can strengthen the results of Proposition 2.19 and Corollary 2.20 and obtain a result more similar to that of [17].

Claim 2.21. *There are unique exceptional characters $\chi_1, \chi_2, \chi_3, \chi_4$ such that*

$$j_{U_n}(\Theta_{G_n,\chi}) = \Theta_{GL_n,\chi_1} \otimes \Theta_{G_0,\chi_2},$$

$$Hom_{\widetilde{G}_n(F)}(\Theta_{G_n,\chi}, Ind_{\widetilde{Q}_n(F)}^{\widetilde{G}_n(F)}(\Theta_{GL_n,\chi_1} \otimes \Theta_{G_0,\chi_2})) \neq 0,$$

$$Hom_{\widetilde{G}_n(F)}(Ind_{\widetilde{Q}_n(F)}^{\widetilde{G}_n(F)}(\Theta_{GL_n,\chi_3} \otimes \Theta_{G_0,\chi_4}), \Theta_{G_n,\chi}) \neq 0.$$

Proof. In this case, $C_{\widetilde{T}_{n+1}(F)}$ is the product of $C_{\widetilde{T}_{GL_n}(F)}$ and $\widetilde{T}_1(F)$ with amalgamated μ_2. Therefore, there are unique characters χ_1 and χ_2 such that ${}^{\mathbf{w}_0}\chi = {}^{\mathbf{w}_0'}\chi_1 \otimes \chi_2$. Then the first assertion follows as in [17, Theorem 2.3], by calculating the Jacquet modules and using the fact that the representation induced from $\rho({}^{\mathbf{w}_0}\chi)$ to $\widetilde{M}_n(F)$ has a unique irreducible subrepresentation. The other assertions follow from the Frobenius reciprocity. $\square$

Due to global reasons (see Proposition 3.4 below), it will be necessary to compute the constant $c(\mathbf{w_0}, \chi)$ defined in Claim 2.11. We have the following claim.

Claim 2.22. *Let χ be an exceptional character, which is unramified. Then*

$$c(\mathbf{w_0}, \chi) = \prod_{2 \leqslant i < j \leqslant n+1} \frac{(1 - q^{-1-j+i})(1 - q^{-1+j+i-2(n+2)})}{(1 - q^{-j+i})(1 - q^{j+i-2(n+2)})} \prod_{2 \leqslant i \leqslant n+1} \frac{(1 - q^{-1-n-2+i})}{(1 - q^{-n-2+i})}.$$

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Proof. We compute $\chi(a_\alpha)$ for an arbitrary $\alpha \in \Sigma^+_{G_n}$. We will show that

$$\chi(a_\alpha) = \begin{cases} q^{-j+i} & \alpha = \epsilon_i - \epsilon_j, \quad 2 \leqslant i < j \leqslant n+1, \\ q^{j+i-2(n+2)} & \alpha = \epsilon_i + \epsilon_j, \quad 2 \leqslant i < j \leqslant n+1, \\ q^{-n-2+i} & \alpha = \epsilon_i, \quad 2 \leqslant i \leqslant n+1. \end{cases}$$

The formula for $c(\mathbf{w_0}, \chi)$ clearly follows from this.

In general, if $\alpha^\vee(x) = \prod_{l=1}^m \alpha_{i_l}^\vee(x_l)$, where $\alpha_{i_l} \in \Delta_{G'_{n+1}}$, i.e., $1 \leqslant i_l \leqslant n+1$ for all $1 \leqslant l \leqslant m$, and for each l either $x_l \in F^{*2}$ or $i_l = n+1$, then $\prod_{l=1}^m \alpha_{i_l}^\vee(x_l) \in T_{n+1}(F)^2$. Then by applying $\mathfrak{s}$ we obtain $\alpha^{\vee *}(x) = \prod_{l=1}^m \alpha_{i_l}^{\vee *}(x_l)$, whence $\chi(\alpha^{\vee *}(x)) = \prod_{l=1}^m \chi(\alpha_{i_l}^{\vee *}(x_l))$.

Start with the computation of $\chi(a_\alpha)$ for $\alpha = \epsilon_i - \epsilon_j$, $2 \leqslant i < j \leqslant n+1$. Since $\alpha^\vee = \prod_{l=i}^{j-1} \alpha_l^\vee$ and $\mathfrak{l}(\alpha) = 2$, $\alpha^\vee(\varpi^2) = \prod_{l=i}^{j-1} \alpha_l^\vee(\varpi^2)$. The definition of χ implies that $\chi({\alpha_l}^{\vee *}(\varpi^2)) = q^{-1}$, giving the result.

Next, consider $\alpha = \epsilon_i$, $2 \leqslant i \leqslant n+1$. We have $\mathfrak{l}(\alpha) = 1$ and $\alpha^\vee(\varpi) = \prod_{l=i}^{n} \alpha_l^\vee(\varpi^2)\alpha_{n+1}^\vee(\varpi)$ (if $n = 1$, there is no product), whence $\chi(a_\alpha) = q^{-n-2+i}$.

For $\alpha = \epsilon_i + \epsilon_j$ $(2 \leqslant i < j \leqslant n+1)$, $\alpha^\vee(\varpi^2) = (\epsilon_i - \epsilon_j)^\vee(\varpi^2)\epsilon_j^\vee(\varpi^2)$. Applying $\mathfrak{s}$ gives $\alpha^{\vee *}(\varpi^2) = (\epsilon_i - \epsilon_j)^{\vee *}(\varpi^2)\epsilon_j^{\vee *}(\varpi^2)$. Now the result follows from the previous calculations. □

2.3.2. Vanishing results. As described in the introduction, the main motivation for studying exceptional representations is their applications. The remarkable (local) property of these representations is the vanishing of many twisted Jacquet modules.

In this section, F is a non-Archimedean field of odd residual characteristic. Fix a non-trivial additive character ψ of F. For a column $b \in F^l$, define the 'length' of b, $\ell(b)$, with respect to the symmetric bilinear form corresponding to J_l, $\ell(b) = {}^t b J_l b$. If $U < N_n$, $u \in U(F)$, and $\alpha \in \Sigma^+_{G_n}$, denote by u_α the projection of u on $\mathcal{U}_\alpha$.

We recall the notion of unipotent classes and their corresponding unipotent subgroups and characters. Since the unipotent subgroups of G_n are in bijection with those of SO_{2n+1}, we can use the description of Bump *et al.* [17, § 4]. For a general reference, see [20, 22]. See also Ginzburg [27].

A partition of k is an m-tuple $r_1 \geqslant \cdots \geqslant r_m > 0$ such that $r_1 + \cdots + r_m = k$. There is a natural partial order on the set of partitions of k: $(r_1, \ldots, r_m) \geqslant (r'_1, \ldots, r'_{m'})$ if $\sum_{i=1}^l r_i \geqslant \sum_{i=1}^l r'_i$ for all $1 \leqslant l \leqslant \min(m, m')$. We denote $(r_1, \ldots, r_m) \gtrsim (r'_1, \ldots, r'_{m'})$ if $(r_1, \ldots, r_m)$ is greater than or non-comparable with $(r'_1, \ldots, r'_{m'})$.

A unipotent class $\mathcal{O} = (r_1, \ldots, r_m)$ of G_n corresponds to a partition of $2n+1$ in which any even number appears with even multiplicity. Write the integers occurring in the multiset $\bigcup_{i=1}^m \{r_i - 2j + 1\}_{j=1}^{r_i}$ in a decreasing order, and let $l_1 \geqslant \cdots \geqslant l_n \geqslant 0$ be the first n numbers (these are the n largest ones). Define the one-parameter subgroup $h_\mathcal{O}$ by the image of

$$h_\mathcal{O} = \prod_{i=1}^n (\eta_i^\vee)^{l_i}.$$

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The group $h_{\mathcal{O}}$ acts on N_n by conjugation. For any $\alpha \in \Sigma^+_{G_n}$ there exists $j_\alpha \geqslant 0$ such that $h_{\mathcal{O}}(t)n_\alpha(x)h_{\mathcal{O}}(t)^{-1} = n_\alpha(t^{j_\alpha}x)$ for all x and t. Let $V_{\mathcal{O}} < N_n$ be the unipotent subgroup generated by those n_α for which $j_\alpha \geqslant 2$ (i.e., $V_{\mathcal{O}}(F)$ is generated by $\{n_\alpha(x) : j_\alpha \geqslant 2, x \in F\}$). Let $V^{\mathrm{a}}_{\mathcal{O}}$ be the quotient of $V_{\mathcal{O}}$ by its derived group. Recall (§ 1.1) that $C(G_n, h_{\mathcal{O}})$ denotes the centralizer of $h_{\mathcal{O}}$ in G_n. The centralizer $C(G_n, h_{\mathcal{O}})$ acts on $V^{\mathrm{a}}_{\mathcal{O}}$ by conjugation. The stabilizer of $x \in V^{\mathrm{a}}_{\mathcal{O}}$ under this action is denoted St_x, and its connected component by St^0_x.

Any character of $V_{\mathcal{O}}(F)$ is the pull back of a character of $V^{\mathrm{a}}_{\mathcal{O}}(F)$, and, because $V^{\mathrm{a}}_{\mathcal{O}}(F)$ is abelian, such a character can be identified with a point in $V^{\mathrm{a}}_{\mathcal{O}}(F)$. If $x \in V^{\mathrm{a}}_{\mathcal{O}}(F)$, let ψ_x be the corresponding character.

Let $\overline{F}$ be the algebraic closure of F. The action of $C(G_n(\overline{F}), h_{\mathcal{O}}(\overline{F}^*))$ on $V^{\mathrm{a}}_{\mathcal{O}}(\overline{F})$ has an open orbit. For $\varepsilon \in V^{\mathrm{a}}_{\mathcal{O}}(\overline{F})$ in this orbit, $St_\varepsilon(\overline{F})^0$ is a reductive group. Let $b \in V^{\mathrm{a}}_{\mathcal{O}}(F)$. The character ψ_b is called generic if b belongs to the open orbit.

Remark 2.9. In [17], the notion of a generic character was restricted to allow only F-split stabilizers. For example, if $St_\varepsilon(\overline{F})^0 = SO_{2l}(\overline{F})$, then ψ_b is generic if $St_b(F)^0$ is the F-split group $SO_{2l}(F)$. A result of [18, Proposition 3] indicates that this notion can be relaxed by considering quasi-split stabilizers.

Recall that $\mathcal{O}_0 = (2^n1)$ if n is even; otherwise, $\mathcal{O}_0 = (2^{n-1}1^3)$. Let χ be an exceptional character of $C_{\widetilde{T}_{n+1}(F)}$. In this section, we prove Theorem 1. Namely, for any $\mathcal{O} \gtrsim \mathcal{O}_0$ and generic character ψ_b of $V_{\mathcal{O}}(F)$,

$$j_{V_{\mathcal{O}},\psi_b}(\Theta_{G_n,\chi}) = 0. \tag{2.37}$$

(As in § 2.3.1, we omitted $\mathfrak{s}$.) Let $\mathcal{O}_1 = (31^{2n-2})$. If a unipotent class $\mathcal{O}$ satisfies $\mathcal{O} \gtrsim \mathcal{O}_0$, then $\mathcal{O} \geqslant \mathcal{O}_1$. This was used by Bump *et al.* [17, Theorem 4.2] to reduce the (global) vanishing results to a statement on $\mathcal{O}_1$ [17, Proposition 4.3]. We follow (a local version of) their arguments (see [18, proof of Proposition 6]). In Lemmas 2.23–2.25, we prove $j_{V_{\mathcal{O}_1},\psi_b}(\Theta_{G_n,\chi}) = 0$. The proof of (2.37) will follow from this using properties of Jacquet modules.

In the case of $\mathcal{O}_1$, the corresponding unipotent subgroup $V_{\mathcal{O}_1}$ is U_1. If $u \in U_1(F)$, let $r(u) \in F^{2n-1}$ be the row vector given by

$$r(u) = (u_{\epsilon_2-\epsilon_3}, \dots, u_{\epsilon_2-\epsilon_{n+1}}, u_{\epsilon_2}, u_{\epsilon_2+\epsilon_{n+1}}, \dots, u_{\epsilon_2+\epsilon_3}).$$

Since U_1 is abelian, for any column $b \in F^{2n-1}$ we have a character of $U_1(F)$ given by $\psi_b(u) = \psi(r(u)b)$. Also $C(G_n, h_{\mathcal{O}_1}) = M_1$. The points belonging to the open orbit are those $b \in U_1(\overline{F})$ with $\ell(b) \neq 0$. Indeed, there are only three orbits: $\{b : \ell(b) \neq 0\}$, $\{b : \ell(b) = 0\}$, and $\{b = 0\}$. If $\ell(b) = 0$, $St_b(\overline{F})^0$ is not reductive, because it contains a unipotent radical of G_{n-1}, which is normal in $St_b(\overline{F})^0$.

Let $b \in U_1(F)$. Then ψ_b is generic if and only if $\ell(b) \neq 0$. For a generic ψ_b, $St_b(F)^0 \cong GSpin_{2(n-1)}(F)$, a quasi-split group (in [17, 18], this was $SO_{2(n-1)}(F)$), where $GSpin_{2(n-1)} < G_{n-1} < M_1$. To see this, start with the root subgroups contained in the stabilizer; these are found using the identification of U_1 with a unipotent radical U'_1 of

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SO_{2n+1} (see § 1.2). To find the coroots, write $t \in T_{n+1}(F)$ in the form (1.5), and then consider the action of $\prod_{i=1}^{n} \eta_i^\vee(a_i)$ on $U_1'(F)$ ($\beta_1^\vee(t_1) \in C_{G_n(F)}$).

Lemma 2.23. *Assume that $n = 1$, and let φ be a linear functional on the space of $\Theta_{G_1,\chi}$. If $\varphi(n^*_{\alpha_2}(x)v) = \psi(x)\varphi(v)$ for all $x \in F$, then $\varphi = 0$.*

Proof. The subgroup generated by $n_{\alpha_2}(x)$ is $U_1(F) = N_1(F)$. The functional φ is the usual Whittaker functional with respect to N_1 and ψ. We will show that $V({}^{\mathbf{w_0}}\chi)$ contains an irreducible genuine representation, which does not afford such a functional. By Proposition 2.16, this representation must be $\Theta_{G_1,\chi}$, which gives the result.

In this case, χ is a character of $\widetilde{T}_2(F)$ (see § 2.2.1). By Claim 2.2, the cover splits; hence χ can be considered as the extension of a character χ' of $T_2(F)$. The character χ' must satisfy $\chi'(\alpha_2^\vee(t_2)) = |t_2|$ for all $t_2 \in F^*$. Since $w_0 = w_{\alpha_2}$, Equalities (1.6)–(1.8) imply that

$$\delta_{B_1(F)}^{1/2}\,{}^{\mathbf{w_0}}\chi'(\alpha_1^\vee(t_1)\alpha_2^\vee(t_2)) = |t_1|^{1/2}\chi'(\alpha_1^\vee(t_1)) = \eta(\Upsilon(\alpha_1^\vee(t_1)\alpha_2^\vee(t_2))),$$

where η is some character of F^*. Hence the non-genuine function $f(g) = \eta(\Upsilon(g))$ belongs to $V({}^{\mathbf{w_0}}\chi)$, and, by its definition, it spans a one-dimensional subspace which is also a $G_1(F)$-module. Because the cover splits, f can be extended to a genuine function; hence this is also a genuine $\widetilde{G}_1(F)$-module. The action of $n^*_{\alpha_2}(x)$ on this subspace is trivial, completing the proof. □

In the case when $n = 2$, $U_1(F)$ is generated by $n_{\alpha_2}(x)$, $n_{\alpha_2+\alpha_3}(y)$, and $n_{\alpha_2+2\alpha_3}(z)$.

Lemma 2.24. *Assume that $n = 2$, and let φ be a linear functional on the space of $\Theta_{G_2,\chi}$. If $b_1, b_2, b_3 \in F$ satisfy $\ell({}^t(b_1, b_2, b_3)) \neq 0$, and, for all $x, y, z \in F$,*

$$\varphi(n^*_{\alpha_2}(x)n^*_{\alpha_2+\alpha_3}(y)n^*_{\alpha_2+2\alpha_3}(z)v) = \psi(b_1x + b_2y + b_3z)\varphi(v),$$

then $\varphi = 0$.

Proof. By Claim 2.8, $C_{\widetilde{T}_3(F)} = \widetilde{T}_3(F)^2$, and hence we can regard χ as the extension to $\widetilde{T}_3(F)^2$ of a character χ_0 of $T_3(F)^2$. If $t = \eta_1^\vee(a_1^2)\eta_2^\vee(a_2^2)\beta_1^\vee(t_1)$,

$$\delta_{B_2(F)}^{1/2}\,{}^{\mathbf{w_0}}\chi_0(t) = |a_1|^{-1}|a_2|^{-2}|t_1|^2\chi_0(\alpha_1^\vee(a_1^{-2}a_2^{-2}t_1^2)) = |a_1|\eta^{-1}(a_1^{-2}a_2^{-2}t_1^2) = |a_1|\eta(\Upsilon(t)),$$

for a suitable character η of F^*.

Let $M_1(F)^{\boxplus} = GL_1(F)^\star \times G_1(F)^\star$ and $Q_1(F)^{\boxplus} = M_1(F)^{\boxplus} \ltimes U_1(F)$. According to (2.21), and because the covers of $GL_1(F)$ and $G_1(F)$ are split, the cover of $M_1(F)^{\boxplus}$ is split. Therefore $\delta_{Q_1(F)}^{1/6}\eta\Upsilon$ can be pulled back to a genuine representation of $\widetilde{Q}_1(F)^{\boxplus}$, whence $Ind_{\widetilde{Q}_1(F)^{\boxplus}}^{\widetilde{G}_2(F)}(\delta_{Q_1(F)}^{1/6}\eta\Upsilon)$ is a genuine representation (this induction is not normalized). Because $[\widetilde{Q}_1(F) : \widetilde{Q}_1(F)^{\boxplus}]$ is finite, and for t as above $\delta_{Q_1(F)}^{1/6}(t) = |a_1|$, we get

$$Ind_{\widetilde{Q}_1(F)^{\boxplus}}^{\widetilde{G}_2(F)}(\delta_{Q_1(F)}^{1/6}\eta\Upsilon) \subset Ind_{\widetilde{B}_2(F)}^{\widetilde{G}_2(F)}(\rho({}^{\mathbf{w_0}}\chi)).$$

This inclusion and Proposition 2.16 imply that $Ind_{\widetilde{Q}_1(F)^{\boxplus}}^{\widetilde{G}_2(F)}(\delta_{Q_1(F)}^{1/6}\eta\Upsilon)$ has a unique irreducible genuine subrepresentation, which must be $\Theta_{G_2,\chi}$. We proceed to show that $[F^* : F^{*2}]$ non-isomorphic irreducible representations can be embedded in $\Theta_{G_2,\chi}$ over $\widetilde{G}_2(F)^{\star}$. These will be certain twists of the Weil representation. According to Bernstein and Zelevinsky [11, (2.9)], $\Theta^{\star}_{G_2,\chi}$ equals the sum of these representations ($[\widetilde{G}_2(F) : \widetilde{G}_2(F)^{\star}] = [F^* : F^{*2}]$). Thus it will suffice to show that the prescribed functional must vanish on each of them.

We use the exceptional isomorphism $G_2' \cong Sp_2$. Let Sp_2 be the subgroup of $GL_4(F)$ preserving the antisymmetric form on F^4 given by $(x, y) \mapsto {}^t x \begin{pmatrix} & J_2 \\ -J_2 & \end{pmatrix} y$ (the matrix J_2 was defined in § 1.1). Fix the Borel subgroup $B_{Sp_2} = T_{Sp_2} \ltimes N_{Sp_2}$ of upper triangular matrices in Sp_2. Let γ_1, γ_2 be the coordinate functions; i.e., if $y = diag(y_1, y_2, y_2^{-1}, y_1^{-1}) \in T_{Sp_2}(F)$, $\gamma_i(y) = y_i$. Denote by ϱ_1, ϱ_2 the simple roots of Sp_2. The mapping from G_2' to Sp_2 is defined by $\alpha_2 \mapsto \varrho_2$ and $\alpha_3 \mapsto \varrho_1$.

Let ω_ψ be the Weil representation of the metaplectic double cover $\widetilde{Sp}_2(F)$ of $Sp_2(F)$, realized on the space $\mathcal{S}(F^2)$ of Schwartz–Bruhat functions on the row space F^2. It satisfies the following formulas (see [66]):

$$\omega_\psi((diag(a, a^*), \zeta))\phi(\xi) = \zeta\gamma_\psi(\det a)|\det a|^{\frac{1}{2}}\phi(\xi a),$$
$$\omega_\psi\left(\left(\begin{pmatrix} I_2 & u \\ & I_2 \end{pmatrix}, \zeta\right)\right)\phi(\xi) = \zeta\psi\left(\frac{1}{2}\xi J_2 {}^t u {}^t\xi\right)\phi(\xi). \tag{2.38}$$

Here, $a \in GL_2(F)$, $\zeta \in \mu_2$, $\phi \in \mathcal{S}(F^2)$, $\xi \in F^2$, and γ_ψ is the normalized Weil factor associated to ψ (see §§1.1 and 1.3.4). Note that $\gamma_\psi|_{F^{*2}} = 1$. We have $\omega_\psi = \omega_\psi^{even} \oplus \omega_\psi^{odd}$, a decomposition into even and odd functions; each space is irreducible. For $c \in F^*$, denote $\psi_c(x) = \psi(cx)$. If $c \neq d$ modulo F^{*2}, $\omega_{\psi_c}^{even}$ and $\omega_{\psi_d}^{even}$ are non-isomorphic.

The cover $\widetilde{G}_2'(F)$ is non-trivial, and, because $\widetilde{Sp}_2(F)$ is unique, the isomorphism $G_2' \cong Sp_2$ extends to the cover groups, and we can regard ω_ψ as a representation of $\widetilde{G}_2'(F)$. We have $C_{\widetilde{G}_2(F)^{\star}} = C_{\widetilde{G}_2(F)}$, $C_{\widetilde{G}_2(F)} \cap \widetilde{G}_2'(F) = p^{-1}(\alpha_3^\vee(\mp 1))$, and $\widetilde{G}_2(F)^{\star} = C_{\widetilde{G}_2(F)^{\star}}\widetilde{G}_2'(F)$ (see Claim 2.7). Since $\alpha_3^\vee(-1)$ is mapped to the matrix $-I_4 \in Sp_2(F)$ which acts by -1 on ω_ψ^{odd} and by 1 on ω_ψ^{even}, and $\Upsilon(\alpha_3^\vee(-1)) = 1$, we can extend ω_ψ^{even} to an irreducible genuine representation $\eta\Upsilon \cdot \omega_\psi^{even}$ of $\widetilde{G}_2(F)^{\star}$.

For an even function $\phi \in \mathcal{S}(F^2)$, define

$$E(\phi)(g) = \begin{cases} \eta\Upsilon \cdot \omega_\psi^{even}(g)\phi(0) & g \in \widetilde{G}_2(F)^{\star}, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\phi \mapsto E(\phi)$ defines an embedding of $\eta\Upsilon \cdot \omega_\psi^{even}$ in $\Theta^{\star}_{G_2,\chi}$. It is enough to show that $E(\phi) \in Ind_{\widetilde{Q}_1(F)^{\boxplus}}^{\widetilde{G}_2(F)}(\delta_{Q_1(F)}^{1/6}\eta\Upsilon)$. Indeed, because $Ind_{\widetilde{Q}_1(F)^{\boxplus}}^{\widetilde{G}_2(F)}(\delta_{Q_1(F)}^{1/6}\eta\Upsilon)$ is of finite length and $\widetilde{G}_2(F)^{\star}\backslash\widetilde{G}_2(F)$ is a finite group, any $\widetilde{G}_2(F)^{\star}$-submodule of $Ind_{\widetilde{Q}_1(F)^{\boxplus}}^{\widetilde{G}_2(F)}(\delta_{Q_1(F)}^{1/6}\eta\Upsilon)$ is contained in $\Theta^{\star}_{G_2,\chi}$.

Clearly $E(\phi)$ is a smooth genuine function on $\widetilde{G}_2(F)$ with support in $\widetilde{G}_2(F)^{\star}$.

For $m \in M_1(F)^{\boxplus}$, we can write $m = cm_1h'$ with

$$c = \alpha_1^{\vee}(a_1^{-2}t_1^2)\alpha_2^{\vee}(a_1^{-2}t_1^2)\alpha_3^{\vee}(a_1^{-1}t_1) \in C_{G_2(F)}, \quad m_1 = \alpha_2^{\vee}(a_1^2)\alpha_3^{\vee}(a_1t_1^{-1}t_2), \quad h' \in G_1'(F).$$

We must show that

$$E(\phi)(mg) = |a_1|\eta\Upsilon(a_1^{-2}t_1^2)E(\phi)(g), \quad \forall g \in \widetilde{G}_2(F).$$

Note that we can regard m as an element of $\widetilde{M}_1(F)^{\boxplus}$ because the cover splits. Since $M_1(F)^{\boxplus} < G_2(F)^{\star}$, it is enough to assume that $g \in \widetilde{G}_2(F)^{\star}$.

Since h' is generated by the elements $n_{\alpha_3}(x)$ $(x \in F)$ and w_{α_3}, whose images in $Sp_2(F)$ belong to $\{diag(a, a^*) : a \in SL_2(F)\}$, $\omega_{\psi}^{even}((h',1))\phi(0) = \phi(0)$. Also the image of m_1 in $Sp_2(F)$ is

$$\varrho_1^{\vee}(a_1t_1^{-1}t_2)\varrho_2^{\vee}(a_1^2) = diag(a_1t_1^{-1}t_2, a_1t_1t_2^{-1}, (a_1t_1t_2^{-1})^{-1}, (a_1t_1^{-1}t_2)^{-1}).$$

Thus $\eta\Upsilon \cdot \omega_{\psi}^{even}((m,1))\phi(0) = |a_1|\eta\Upsilon(a_1^{-2}t_1^2)\phi(0)$, as required.

Regarding the invariance under $\mathfrak{s}(U_1(F))$, let $V(F) < Sp_2(F)$ be the unipotent subgroup

$$\left\{ v(x,y,z) = \begin{pmatrix} 1 & 0 & y & z \\ & 1 & x & y \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} : x, y, z \in F \right\}.$$

The image of $n_{\alpha_2}(x)n_{\alpha_2+\alpha_3}(y)n_{\alpha_2+2\alpha_3}(z) \in U_1(F)$ in $Sp_2(F)$ is $v(x,y,z)$, which acts trivially at 0 by (2.38).

We conclude that E is an embedding. Of course, this applies to any ψ_c. It remains to show that $\eta\Upsilon \cdot \omega_{\psi}^{even}$ does not afford the functional appearing in the proposition. Indeed, the subgroup $U_1(F)$ is in bijection with $V(F)$ and, exactly as in [17, Proposition 2.5], as long as $2b_1b_3 + b_2^2 \neq 0$, any such functional vanishes on the space of ω_{ψ}^{even}. □

Remark 2.10. The proof follows the arguments of [17, Proposition 2.5]. There, the analog of $G_2(F)^{\star}$ is the kernel of the spinor norm map, $SO_5(F)'$.

The following lemma combines [17, Theorem 2.6] and [18, Proposition 3].

Lemma 2.25. *Assume that* $n \geqslant 1$. *For any generic character* ψ_b *of* $V_{\mathcal{O}_1}(F)$, $j_{V_{\mathcal{O}_1},\psi_b}(\Theta_{G_n,\chi}) = 0$.

Remark 2.11. If $\ell(b) \in F^{*2}$, it is possible to conjugate ψ_b into the character $u \mapsto \psi(u_{\epsilon_2})$. Then the statement of Lemma 2.25 is similar to that of [17, Theorem 2.6]. The general case was stated in [18, Proposition 3].

Proof. Exactly as in [17], the proof is derived using a sequence of claims, all of which follow from the results already proved in §§ 2.3.1–2.3.2. The arguments of [17] referring to unipotent subgroups can be repeated without a change. The difference is that here, instead of using a tensor product representation to describe $j_{U_k}(\Theta_{G_n,\chi})$, we use the

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induced representation of Proposition 2.19 (or Claim 2.21 when $k = n$). We briefly present the arguments of [17] and focus on the necessary changes. The restriction of the cover $\widetilde{G}_n(F)$ to $GSpin_{2(n-1)}(F)$ gives a cover of the latter. The key observation is that $j_{U_1,\psi_b}(\Theta_{G_n,\chi})$ is either a supercuspidal representation of $\widetilde{GSpin_{2(n-1)}}(F)$ or zero.

Claim 2.26. *Assume that* $2 \leqslant k \leqslant n$. *Let* $c_1, \ldots, c_{k-1} \in F$ *and* $d_1, d_2, d_3 \in F$ *be given with* $\ell({}^t(d_1, d_2, d_3)) \neq 0$. *Define a character* ψ *of* $N_{GL_k}(F)U_k(F)$ *by*

$$\psi(u) = \begin{cases} \psi\left(\sum_{i=1}^{k-1} c_i u_{\alpha_{i+1}} + d_1 u_{\epsilon_{k+1}-\epsilon_{n+1}} + d_2 u_{\epsilon_{k+1}} + d_3 u_{\epsilon_{k+1}+\epsilon_{n+1}}\right) & k < n, \\ \psi\left(\sum_{i=1}^{k-1} c_i u_{\alpha_{i+1}} + u_{\epsilon_{n+1}}\right) & k = n. \end{cases}$$

Then $j_{N_{GL_k}U_k,\psi}(\Theta_{G_n,\chi}) = 0$. *Here* N_{GL_k} *is embedded in the* GL_k *part of* M_k.

Proof. We use induction on n; the base case is $n = 2$. For any $1 \leqslant l < k$, let G_{n-l} be embedded in M_l, and let $Q'_{k-l} = M'_{k-l} \ltimes U'_{k-l} < G_{n-l}$ be the standard parabolic subgroup with a Levi part M'_{k-l} isomorphic to $GL_{k-l} \times G_{n-k}$. Let $N'_{GL_{k-l}}$ be the embedding of $N_{GL_{k-l}}$ in the GL_{k-l} part of M'_{k-l}. As a unipotent subgroup of SO_{2n+1}, $N_{GL_k}U_k$ takes the form

$$\begin{pmatrix} z_1 & u_1 & u_2 & u_3 & u_4 \\ & z_2 & v_1 & v_2 & * \\ & & I_{2(n-k)+1} & * & * \\ & & & * & * \\ & & & & * \end{pmatrix},$$

where $z_1 \in N_{GL_l}$, $z_2 \in N'_{GL_{k-l}}$, $u_1, \ldots, u_4$ are the coordinates of U_l, and v_1, v_2 are the coordinates of U'_{k-l}.

Assume that $c_l = 0$ for some $1 \leqslant l < k$. Then

$$j_{N_{GL_k}U_k,\psi} = j_{N_{GL_l},\psi_1} j_{N'_{GL_{k-l}}U'_{k-l},\psi_2} j_{U_l},$$

where ψ_1 and ψ_2 are obtained from ψ by restriction.

By the induction hypothesis, or by Lemma 2.23, if $n - l = 1$ (and then $k = n$),

$$j_{N'_{GL_{k-l}}U'_{k-l},\psi_2}(\Theta_{G_{n-l},\chi_2}) = 0 \tag{2.39}$$

for any exceptional character χ_2 of $C_{\widetilde{T}_{n-l+1}(F)}$.

Write as in Proposition 2.19, $j_{U_l}(\Theta_{G_n,\chi}) \subset I(\Theta_{GL_l,\chi_1}, \Theta_{G_{n-l},\chi_2})^{\star}$. It is enough to prove that

$$j_{N'_{GL_{k-l}}U'_{k-l},\psi_2}(I(\Theta_{GL_l,\chi_1}, \Theta_{G_{n-l},\chi_2})^{\star}) = 0.$$

Moreover, according to Lemma 2.13, it suffices to prove that $j_{N'_{GL_{k-l}}U'_{k-l},\psi_2}$ vanishes on

$$ind_{p^{-1}(GL_l(F)^{\star}\times G_{n-l}(F))}^{\widetilde{M}_l(F)}(\Theta^{\star}_{GL_l,\chi_1} \otimes \Theta_{G_{n-l},\chi_2}). \tag{2.40}$$

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Let $St_{\psi_2}(F)$ be the normalizer of $N'_{GL_{k-l}}(F)U'_{k-l}(F)$ and the stabilizer of ψ_2 in $G_{n-l}(F)$. The double coset space

$$(GL_l(F)^{\star}G_{n-l}(F))\backslash M_l(F)/(GL_l(F)St_{\psi_2}(F))$$

has one element. Using the geometric lemma of Bernstein and Zelevinsky [12, Theorem 5.2], the application of $j_{N'_{GL_{k-l}}U'_{k-l},\psi_2}$ to (2.40) is seen to be equal to

$$ind_{p^{-1}(GL_l(F)^{\star}\times St_{\psi_2}(F))}^{p^{-1}(GL_l(F)\times St_{\psi_2}(F))}(\Theta^{\star}_{GL_l,\chi_1}\otimes j_{N'_{GL_{k-l}}U'_{k-l},\psi_2}(\Theta_{G_{n-l},\chi_2})),$$

which vanishes by (2.39).

Thus we may assume that $c_1 = \cdots = c_{k-1} = 1$. By virtue of Corollary 2.20 and Lemma 2.13, it is enough to show that

$$j_{N_{GL_k}U_k,\psi}(Ind_{\widetilde{Q}_1(F)}^{\widetilde{G}_n(F)}(ind_{p^{-1}(GL_1(F)^{\star}\times G_{n-1}(F))}^{\widetilde{M}_1(F)}(\Theta^{\star}_{GL_1,\chi_3}\otimes\Theta_{G_{n-1},\chi_4}))) = 0.$$

Let χ' be an exceptional character of $C_{\widetilde{T}_n(F)}$. If $n = 2$, let $r = 1$; otherwise, take $2 \leqslant r \leqslant n-1$. Let ψ' be a non-trivial character of $N_{GL_r}(F)U_r(F)$, where $N_{GL_r}U_r$ is a subgroup of G_{n-1} embedded in the standard maximal parabolic subgroup with a Levi part $GL_r \times G_{n-1-r}$. If $n > 2$, assume that ψ' is defined as in the statement of the lemma, with respect to $c'_1,\ldots,c'_{r-1},d'_1,d'_2,d'_3 \in F$, where $\ell({}^t(d'_1,d'_2,d'_3)) \neq 0$. According to the induction hypothesis and Lemma 2.23,

$$j_{N_{GL_r}U_r,\psi'}(\Theta_{G_{n-1},\chi'}) = 0.$$

As above, [12, Theorem 5.2] implies that

$$j_{N_{GL_r}U_r,\psi'}(ind_{p^{-1}(GL_1(F)^{\star}\times G_{n-1}(F))}^{\widetilde{M}_1(F)}(\Theta^{\star}_{GL_1,\chi_3}\otimes\Theta_{G_{n-1},\chi_4})) = 0. \tag{2.41}$$

Now, the filtration argument of [17] applies to our case as well. Specifically, the stabilizer in $M_k(F)$ of ψ contains $N_{GL_k}(F)GSpin_{2(n-k)}(F)$, where $GSpin_{2(n-k)}(F)$ is the stabilizer of ψ in $G_{n-k}(F)$ (GL_k and G_{n-k} are regarded as subgroups of M_k). The space of the representation induced from $\widetilde{Q}_1(F)$ to $\widetilde{G}_n(F)$ has a filtration according to the double cosets $Q_1(F)\backslash G_n(F)/(N_{GL_k}(F)GSpin_{2(n-k)}(F))$. Let w be a representative of some double coset. Then either w conjugates one of the root subgroups of $N_{GL_k}U_k$ on which ψ is non-trivial, into U_1, in which case $j_{N_{GL_k}U_k,\psi}$ clearly vanishes, or $\psi|_{{}^w(N_{GL_k}U_k)\cap(N_{GL_k}U_k)} = \psi'$ with ψ' as above, and then the Jacquet module vanishes by (2.41). □

For $n = 1, 2$ the lemma follows immediately from Lemmas 2.23 and 2.24. Henceforth, assume that $n > 2$.

Claim 2.27. *The representation $j_{U_1,\psi_b}(\Theta_{G_n,\chi})$, if non-zero, is a supercuspidal representation of $\widetilde{GSpin}_{2(n-1)}(F)$.*

Proof. Assume that $\ell(b) \in F^{*2}$. Then we may assume that $b = e_n(\in F^{2n-1})$; i.e., $\psi_b(u) = \psi(u_{\epsilon_2})$. Exactly as in [17, Lemma 2.8], using an 'exchange of roots', the claim can be reduced to showing the vanishing of $\tau = j_{U_{l+1},\psi_2}(\Theta_{G_n,\chi})$, where $0 < l < n$ and ψ_2

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is the character of $U_{l+1}(F)$ given by $\psi_2(u) = \psi(u_{\epsilon_{l+1}})$. Let $Y_{l+1} < GL_{l+1}$ be the mirabolic subgroup, the subgroup of matrices with the last row equal to $(0, \ldots, 0, 1)$. Then τ is a genuine representation of $\widetilde{Y}_{l+1}(F)$, and we must show that $\tau = 0$. Suppose that this does not hold. According to Bernstein and Zelevinsky [11, (5.14)], there are $c_1, \ldots, c_l \in F$ such that the character $\psi_1(u) = \psi(\sum_{i=1}^{l} c_i u_{\alpha_{i+1}})$ of $N_{GL_{l+1}}(F)$ satisfies $j_{N_{GL_{l+1}},\psi_1}(\tau) \neq 0$. But this means that $j_{N_{GL_{l+1}}U_{l+1},\psi}(\Theta_{G_n,\chi}) \neq 0$, where $\psi = \psi_1 \cdot \psi_2$, contradicting Claim 2.26. If $\ell(b) \in F^*{-}F^{*2}$, the argument is similar, with a minor change to ψ_2. □

Claim 2.28. *For any genuine supercuspidal representation τ of $\widetilde{GSpin}_{2(n-1)}(F)$ and exceptional characters χ_1 and χ_2 of $C_{\widetilde{T}_{GL_n}}(F)$ and $\widetilde{T}_1(F)$,*

$$Hom_{\widetilde{GSpin}_{2(n-1)}(F)}(j_{U_1,\psi_b}(Ind_{\widetilde{Q}_n(F)}^{\widetilde{G}_n(F)}(\Theta_{GL_n,\chi_1} \otimes \Theta_{G_0,\chi_2})), \tau) = 0. \tag{2.42}$$

Proof. Since in this particular case the tensor representation is defined, we are in the same situation as in [17, Lemma 2.9], except that we have the quasi-split $GSpin_{2(n-1)}(F)$ instead of split $SO_{2(n-1)}(F)$. The space (2.42) can be analyzed using the Bruhat theory of Silberger [73, Theorems 1.9.4 and 1.9.5]. One considers the double coset space $Q_n(F)\backslash G_n(F)/(U_1(F)GSpin_{2(n-1)}(F))$, and then uses the theory of derivations of Bernstein and Zelevinsky [11]. Since τ is supercuspidal, this space is of dimension at most 1. Because $n \geqslant 3$ and we are assuming that the residue characteristic of F is odd, Θ_{GL_n,χ_1} does not have a Whittaker model ([47, § I.3]; see also [19, p. 145]). In this case, the dimension of (2.42) is zero. The arguments follow closely those of [43, § 4] for quasi-split $SO_{2(n-1)}(F)$, Soudry [74, § 8], and Ginzburg *et al.* [32, Theorem 6.2(c)]. □

As in [17], the proposition follows. Set $\tau = j_{U_1,\psi_b}(\Theta_{G_n,\chi})$; then Claims 2.21, 2.27 and 2.28 applied to τ imply that $Hom_{\widetilde{GSpin}_{2(n-1)}(F)}(\tau, \tau) = 0$, a contradiction. □

Now we are ready to prove the theorem.

Proof of Theorem 1. Let $\Theta = \Theta_{G_n,\chi}$, let $\mathcal{O} = (r_1, \ldots, r_m)$ be such that $\mathcal{O} \gtrsim \mathcal{O}_0$, and let ψ_b be a generic character of $V_{\mathcal{O}}(F)$. The proof is a local analog of [17, Theorem 4.2(i)]. We briefly reproduce the arguments. Let e (respectively, o) be the number of even (respectively, odd) numbers r_i greater than 1. By the definition of a unipotent class, e is even, whence $m - e$ is odd.

We only show the case when $o > 0$. Put $r = n - o - (m + e - 1)/2$. The unipotent subgroup $V_{\mathcal{O}}(F)$ corresponds to the following unipotent subgroup of $SO_{2n+1}(F)$ (see § 1.2):

$$\left\{ \begin{pmatrix} \ddots & & & & & \alpha & \iddots \\ & I_o & 0 & x & y & z & \alpha' \\ & & I_e & 0 & v & y' & \\ & & & I_{m-e} & 0 & x' & \\ & & & & I_e & 0 & \\ & & & & & I_o & \\ & & & & & & \ddots \end{pmatrix} \right\}.$$

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Here, α is an $r\times o$ matrix, and, if h is an $i\times j$ matrix, $h' = -J_j {}^t h J_i$. The middle $(2o+e+m)\times(2o+e+m)$ block, regarded as a unipotent subgroup of $G_{n-r} < M_r$, corresponds to the unipotent class $(3^o 2^e 1^{m-o-e})$, and we denote it by $V_{(3^o 2^e 1^{m-o-e})}$. We have

$$C(G_{n-r}(F), h_{(3^o2^e1^{m-o-e})}(F)) = GL_o(F)\times GL_e(F)\times G_{(m-e-1)/2}(F).$$

According to the description of Carter [20, p. 398], if ε belongs to the open orbit (see the discussion at the beginning of this section), $St_\varepsilon(\overline{F})^0$ is a reductive group of Lie type

$$\begin{cases} B_{(o-1)/2}\times C_{e/2}\times D_{(m-o-e)/2} & o \text{ is odd,}\\ D_{o/2}\times C_{e/2}\times D_{(m-o-e-1)/2} & o \text{ is even.}\end{cases}$$

Since the coordinates of α and z vanish in $V_{\mathcal{O}}^{\mathrm{a}}$, ψ_b is trivial on the coordinates of α' and z. According to the definition, the restriction of ψ_b to $V_{(3^o2^e1^{m-o-e})}(F)$ is a generic character. The restriction of ψ_b to the coordinates of x (respectively, v) is characterized by an $o\times(m-e)$ (respectively, $e\times e$) matrix $\mathbf{x}$ (respectively, $\mathbf{v}$). We may assume that, if $(\mathbf{x},\mathbf{v})\in V_{\mathcal{O}}^{\mathrm{a}}(F)$ is the point defined by $\mathbf{x}$ and $\mathbf{v}$, $St_{(\mathbf{x},\mathbf{v})}(\overline{F})^0$ is already of the prescribed Lie type. It follows that ψ_b is trivial on y, and, using a conjugation by a suitable $g\in GL_k(F)$ (a product of a unipotent matrix and a Weyl element), we may also assume that $\ell(\mathbf{x}_1)\neq 0$.

Using another conjugation, we can move the $(r+1)$th row (which contains $\mathbf{x}_1$) to the first row. Again by the transitivity of the Jacquet functor, it is enough to prove that $j_{U,\psi'_b}(\Theta)=0$, for the subgroup $U(F)<U_1(F)$ of matrices u whose first row is

$$\begin{pmatrix}1 & 0_{r+o-1+e} & x_1 & y_1 & z_{1,1} & \alpha'_1 & z_{1,2}\end{pmatrix},$$

where, for a matrix h, h_i is the ith row of h; $(h_{i,1},h_{i,2})=h_i$; $h_{i,2}\in F$; and $\psi'_b(u)=\psi(\mathbf{x}_1({}^t x_1))$.

If $j_{U,\psi'_b}(\Theta)\neq 0$, there exists some character ψ_c of $U_1(F)$ which agrees with ψ'_b on $U(F)$, such that $j_{U_1,\psi_c}(\Theta)\neq 0$. However, any such character takes the form $\psi_c(u)=\psi(r(u)c)$, where ${}^t c$ is the row $(c_1,\ldots,c_{r+o-1+e},\mathbf{x},0_{r+o-1+e})\in F^{2n-1}$. Since $\ell(c)\neq 0$, this contradicts Lemma 2.25. □

We also have the following vanishing result.

Proposition 2.29. *For all $3\leqslant k\leqslant n$, $j_{N_{GL_k}U_k,\psi}(\Theta_{G_n,\chi})=0$, where $\psi(u)=\psi(\sum_{i=1}^{k-1}u_{\alpha_{i+1}})$.*

Proof. Since $j_{N_{GL_k}U_k,\psi}=j_{N_{GL_k},\psi}\,j_{U_k}$, using Proposition 2.19 and Lemma 2.13 we see that it is enough to prove that

$$j_{N_{GL_k},\psi}(ind_{p^{-1}(GL_k(F)\times G_{n-k}(F)^\star)}^{\widetilde{M}_k(F)}(\Theta_{GL_k,\chi_1}\otimes\Theta^\star_{G_{n-k},\chi_2}))=0. \tag{2.43}$$

As above, we use [12, Theorem 5.2]. Let $St_\psi(F)$ be the normalizer of $N_{GL_k}(F)$ and the stabilizer of ψ in $GL_k(F)$. The space

$$(GL_k(F)G_{n-k}(F)^\star)\backslash M_k(F)/(St_\psi(F)G_{n-k}(F))$$

has one element. Since $\psi|_{N_{GL_k}}$ is the standard Whittaker character and Θ_{GL_k,χ_1} is not generic [47, § I.3], $j_{N_{GL_k},\psi}(\Theta_{GL_k,\chi_1})=0$. Hence (2.43) follows. □

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2.3.3. Explicit construction of exceptional characters. Let η be a character of F^*. Start with defining a character χ_0 of $T_{n+1}(F)^2$, depending on η. For $t = \prod_{i=1}^{n} \eta_i^\vee(a_i^2)\beta_1^\vee(t_1)$, let

$$\chi_0(t) = \prod_{i=1}^{n} |a_i|^{n-i+1}\eta(\Upsilon(t)).$$

Using (1.6), we see that $\chi_0(\alpha^\vee(x^{\mathfrak{l}(\alpha)})) = |x|$ for all $\alpha \in \Delta_{G_n}$ and $x \in F^*$. Extend χ_0 to a genuine character χ of $\widetilde{T}_{n+1}(F)^2$ using $\mathfrak{s}$: $\chi(\zeta\mathfrak{s}(t)) = \zeta\chi_0(t)$ ($\zeta \in \mu_2$). If n is even, χ is exceptional.

Let ψ be a non-trivial additive character of F. If $z = \prod_{i=1}^{n} \eta_i^\vee(d)$ and $z' = \prod_{i=1}^{n} \eta_i^\vee(d')$, where $d, d' \in (F^*)^{2/\gcd(2,n+1)}$, using (2.9), (2.1), and (2.2), we get $\sigma(z, z') = c(d, d')^{\lceil n/2\rceil}$. Therefore, $\zeta\mathfrak{s}(z) \mapsto \zeta\gamma_\psi^{\lceil n/2\rceil}(d)$ is a genuine character of $C_{\widetilde{GL}_n(F)}$ (if n is even, $\gamma_\psi(d) = 1$).

Since $[t, z]_\sigma = 1$, we can define the following exceptional character of $C_{\widetilde{T}_{n+1}(F)}$:

$$\chi(\zeta\mathfrak{s}(t)\mathfrak{s}(z)) = \zeta\chi_0(t)|d|^{n(n+1)/4}\eta(d)^n\gamma_\psi^{\lceil n/2\rceil}(d). \tag{2.44}$$

In fact, every exceptional character can be written in this form, for suitable ψ and η.

We mention that, in the context of exceptional representations of GL_n, the character η is the 'determinantal character' of Bump and Ginzburg [19, p. 143].

Let $\widetilde{T}_{n+1}(F)^{\mathrm{m}}$ be the maximal abelian subgroup given in Claim 2.10. The following formula defines an extension χ' of χ to $\widetilde{T}_{n+1}(F)^{\mathrm{m}}$. For $t = \prod_{i=1}^{n} \eta_i^\vee(a_i)\beta_1^\vee(t_1) \in T_{n+1}(F)^{\mathrm{m}}$, let

$$\chi'(\zeta\mathfrak{s}(t)) = \zeta\prod_{i=1}^{n} |a_i|^{(n-i+1)/2}\eta(\Upsilon(t)) \prod_{i=0}^{\lceil n/2\rceil-1} \gamma_\psi(a_{n-2i}).$$

Now consider the unramified case. Assume that all data are unramified. This means that $|2| = 1, q > 3$ in F; ψ is unramified ($\psi|_{\mathfrak{O}} = 1$, $\psi|_{\mathcal{P}^{-1}} \neq 1$); and η is unramified ($\eta|_{\mathfrak{O}^*} = 1$). Then $\gamma_\psi|_{\mathfrak{O}^*} = 1$ (see, e.g., [80, Lemma 3.4]). Therefore χ is trivial on $C_{\widetilde{T}_{n+1}(F)} \cap K^*$. In this case, χ is an unramified character (see § 2.2.2).

3. Global theory

3.1. The r-fold cover of $G_n(\mathbb{A})$

Let F be a number field containing all r rth roots of unity, with a ring of adèles $\mathbb{A}$. The global r-fold cover $\widetilde{G}_n(\mathbb{A})$ of $G_n(\mathbb{A})$ is defined similarly to the definition of $\widetilde{GL}_n(\mathbb{A})$ of [47] (§ 0.2; see also [24, 81, 82]). At each place ν we have the exact sequence of § 2.1.1,

$$1 \to \mu_r(F_\nu) \xrightarrow{\iota_\nu} \widetilde{G}_n(F_\nu) \xrightarrow{p_\nu} G_n(F_\nu) \to 1.$$

Denote by c the inverse of the global Hilbert symbol; c is the product of the local functions c_ν. At each ν, let $K_\nu < G_n(F_\nu)$ be either $G_n(\mathfrak{O}_\nu)$ if F_ν is p-adic, or a maximal compact subgroup in the Archimedean case, and put $K = \prod_\nu K_\nu$. For simplicity, fix an identification of $\mu_r(F_\nu)$ with $\mu_r = \mu_r(F)$ (which is usually identified with a subgroup

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of $\mathbb{C}^*$). Let S_∞ be the set of Archimedean places of F. Fix a finite set S_0 containing S_∞ and the places ν such that $|r|_\nu < 1$ or $q_\nu \leqslant 3$. For any finite set $S \supset S_0$, denote

$$G_n(\mathbb{A})_S = \prod_{\nu \in S} G_n(F_\nu) \prod_{\nu \notin S} K_\nu, \quad G_n(\mathbb{A})_S^* = \prod_{\nu \in S} \widetilde{G}_n(F_\nu) \prod_{\nu \notin S} K_\nu^*.$$

For $\nu \in S$, $\widetilde{G}_n(F_\nu) < G_n(\mathbb{A})_S^*$ (in particular, $\iota_\nu(\mu_r) < G_n(\mathbb{A})_S^*$). Note that K_ν^* was defined given the assumption on S (see § 2.1.4). Let $\widehat{\mu_r}$ be the subgroup of $G_n(\mathbb{A})_S^*$ generated by the elements $\iota_\nu(\zeta)\iota_{\nu'}(\zeta)^{-1}$, where $\zeta \in \mu_r$ and $\nu, \nu' \in S$. Define $\widehat{G}_n(\mathbb{A})_S = \widehat{\mu_r} \backslash G_n(\mathbb{A})_S^*$. Then

$$G_n(\mathbb{A}) = \lim_{S \longrightarrow} G_n(\mathbb{A})_S, \quad \widetilde{G}_n(\mathbb{A}) = \lim_{S \longrightarrow} \widehat{G}_n(\mathbb{A})_S.$$

We have an exact sequence

$$1 \to \mu_r \xrightarrow{\iota} \widetilde{G}_n(\mathbb{A}) \xrightarrow{p} G_n(\mathbb{A}) \to 1.$$

Regarding the topologies on these groups, refer to [47, § 0.2].

Let $\prod'_\nu G_n(F_\nu)$ be the restricted direct product with respect to the subgroups $\{K_\nu\}_\nu$. Then $G_n(\mathbb{A}) = \prod'_\nu G_n(F_\nu)$. Similarly, define $\prod'_\nu \widetilde{G}_n(F_\nu)$ with respect to $\{K_\nu^*\}_\nu$. Set $\mu_r^\times = \{(i_\nu(\zeta_\nu))_\nu \in \prod'_\nu \widetilde{G}_n(F_\nu) : \prod_\nu \zeta_\nu = 1\}$. Then $\widetilde{G}_n(\mathbb{A}) = \mu_r^\times \backslash \prod'_\nu \widetilde{G}_n(F_\nu)$.

In § 2.1.4, we defined κ_ν for $\nu \notin S_0$, extend κ_ν to a smooth section of $G_n(F_\nu)$. Define sections κ_ν also for $\nu \in S_0$ arbitrarily. Set $\kappa = \prod_\nu \kappa_\nu$. This is a section of $G_n(\mathbb{A})$, well defined because $\kappa_\nu(K_\nu) = K_\nu^*$ for $\nu \notin S_0$. The corresponding 2-cocycle is defined by $\gamma(g, g') = \kappa(g)\kappa(g')\kappa(gg')^{-1}$, well defined since, for $\nu \notin S_0$, κ_ν is a splitting of K_ν.

Consider $\mathfrak{s} = \prod_\nu \mathfrak{s}_\nu$, where $\mathfrak{s}_\nu$ is the section given in § 2.1.1. This is not a section of $G_n(\mathbb{A})$, but can be used with the following subgroups.

Claim 3.1. *The section $\mathfrak{s}$ is a splitting of $G_n(F)$ and $N_n(\mathbb{A})$. It is well defined on $T_{n+1}(\mathbb{A})$.*

Proof. First, observe that $\mathfrak{s}$ is indeed well defined on these subgroups. Indeed, let $g \in G_n(F)$ and write $g = utwu'$ as in (2.12). Then, for any place ν, $\mathfrak{s}_\nu(g) = \mathfrak{s}_\nu(u)\mathfrak{s}_\nu(t)\mathfrak{s}_\nu(w)\mathfrak{s}_\nu(u')$. For almost all ν, the elements u, t, w and u' lie in K_ν. Then Claim 2.3 implies that $\mathfrak{s}_\nu(g) \in K_\nu^*$. The same claim implies that, if $u \in N_n(\mathbb{A})$ and $t \in T_{n+1}(\mathbb{A})$, $\mathfrak{s}_\nu(u_\nu) = \kappa_\nu(u_\nu) \in K_\nu^*$ and $\mathfrak{s}_\nu(t_\nu) = \kappa_\nu(t_\nu) \in K_\nu^*$ for almost all ν.

Since, for all ν, $\mathfrak{s}_\nu$ is a splitting of $N_n(F_\nu)$, $\mathfrak{s}$ is a splitting of $N_n(\mathbb{A})$. To show that $\mathfrak{s}$ is a splitting of $G_n(F)$, one uses (2.12), the relations in [10, § 1] defining the multiplication in $G_n(F_\nu)$ and $\widetilde{G}_n(F_\nu)$, and the fact that c is trivial on $F^* \times F^*$. □

Remark 3.1. Let H be a subgroup of $G_n(\mathbb{A})$ such that, for any $h \in H$, for almost all ν, $\mathfrak{s}(h_\nu) = \kappa(h_\nu)$. Then, if $h, h' \in H$, we have $\mathfrak{s}_\nu(h_\nu)\mathfrak{s}_\nu(h'_\nu) = \mathfrak{s}_\nu(h_\nu h'_\nu)$ almost everywhere. Hence we can define the 2-cocycle $\sigma(h, h') = \mathfrak{s}(h)\mathfrak{s}(h')\mathfrak{s}(hh')^{-1}$ on H, and then $\sigma = \prod_\nu \sigma_\nu$, where σ_ν is the block-compatible cocycle of § 2.1.1. According to Claim 3.1, σ is well defined on $T_n(\mathbb{A})$, and trivial on $G_n(F)$ and $N_n(\mathbb{A})$.

Henceforth $r = 2$.

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3.2. Representations, intertwining operators and Eisenstein series

We study irreducible genuine representations of $\widetilde{T}_{n+1}(\mathbb{A})$ formed by extending a genuine character of $C_{\widetilde{T}_{n+1}(\mathbb{A})} = \mu_2^{\times}\backslash \prod'_\nu C_{\widetilde{T}_{n+1}(F_\nu)}$ to a maximal abelian subgroup, and then inducing to $\widetilde{T}_{n+1}(\mathbb{A})$. The resulting representation is independent of the choices of abelian subgroup and extension [47, § 0.3].

Let χ be a genuine character of $C_{\widetilde{T}_{n+1}(\mathbb{A})}$, which is trivial on $C_{\widetilde{T}_{n+1}(\mathbb{A})} \cap \mathfrak{s}(T_{n+1}(F))$. Then $\chi = \otimes'_\nu \chi_\nu$, where, for almost all ν, χ_ν is unramified, because it is trivial on $C_{\widetilde{T}_{n+1}(F_\nu)} \cap K^*_\nu$ (see § 2.2.2).

Following Kazhdan and Patterson [47, § II.1], we will use two maximal subgroups; the resulting representation will be the same. One choice is $X = \mu_2^{\times}\backslash \prod'_\nu X_\nu$, where $C_{\widetilde{T}_{n+1}(F_\nu)} < X_\nu < \widetilde{T}_{n+1}(F_\nu)$ is a local maximal abelian subgroup such that, for almost all ν, $X_\nu = C(\widetilde{T}_{n+1}(F_\nu), \widetilde{T}_{n+1}(F_\nu) \cap K^*_\nu)$ (see Claim 2.9). The subgroup X is suitable for relating a global representation to its local pieces. Globally it is simpler to work with a subgroup containing $\mathfrak{s}(T_{n+1}(F))$. As in [47, Lemma II.1.1], we introduce the following alternative.

Claim 3.2. *The subgroup $C(\widetilde{T}_{n+1}(\mathbb{A}), \mathfrak{s}(T_{n+1}(F)))$ is a maximal abelian subgroup of $\widetilde{T}_{n+1}(\mathbb{A})$, which contains $\mathfrak{s}(T_{n+1}(F))$. We have $C(\widetilde{T}_{n+1}(\mathbb{A}), \mathfrak{s}(T_{n+1}(F))) = \mathfrak{s}(T_{n+1}(F))$ $C_{\widetilde{T}_{n+1}(\mathbb{A})}$.*

Proof. The containment from right to left is clear, because by Claim 3.1 the section $\mathfrak{s}$ splits $G_n(F)$; hence the elements of $\mathfrak{s}(T_{n+1}(F))$ commute. Now we proceed as in the proof of Claim 2.9, and note that, by Weil [86, § XIII.5, Propositions 7 and 8], $c(x, y) = 1$ for all $y \in \mathbb{A}^*$ if and only if $x \in (\mathbb{A}^*)^2$, and $c(x, y) = 1$ for all $y \in F^*$ if and only if $x \in F^*(\mathbb{A}^*)^2$ (compare to § 1.3.3). Observe that the proof of Claim 2.9 uses the local cocycle σ_ν defined using $\mathfrak{s}_\nu$. As explained in Remark 3.1, the argument extends to the global setting. □

For $\underline{s} = (s_1, \ldots, s_{n+1}) \in \mathbb{C}^{n+1}$, define a non-genuine character $\mathrm{e}_{\underline{s}}$ of $\widetilde{T}_{n+1}(\mathbb{A})$ by requiring that $\mathrm{e}_{\underline{s}}(\alpha^\vee(x)) = |x|^{(\alpha,\underline{s})/\mathfrak{l}(\alpha)}$ for all $\alpha \in \Delta_{G'_{n+1}}$ and $x \in \mathbb{A}^*$. For example, $\mathrm{e}_{\underline{s}}(\alpha_1^\vee(x)) = |x|^{(s_1-s_2)/2}$, $\mathrm{e}_{\underline{s}}(\alpha_{n+1}^\vee(x)) = |x|^{s_{n+1}}$. The character $\mathrm{e}_{\underline{s}}$ is trivial on $\mathfrak{s}(T_{n+1}(F))$.

If $d \in \mathbb{C}$, denote $\underline{d} = (d, \ldots, d)$ (the length of $\underline{d}$ will be clear from the context).

Let χ be a genuine character of $C_{\widetilde{T}_{n+1}(\mathbb{A})}$, which is trivial on $C_{\widetilde{T}_{n+1}(\mathbb{A})} \cap \mathfrak{s}(T_{n+1}(F))$. Denote $\chi_{\underline{s}} = \mathrm{e}_{\underline{s}} \cdot \chi$. Assume that we are given an abelian subgroup $C_{\widetilde{T}_{n+1}(\mathbb{A})} < H < \widetilde{T}_{n+1}(\mathbb{A})$ and an extension χ' of χ to H. We will always assume that $(\chi')_{\underline{s}} = (\chi_{\underline{s}})'$. Therefore we continue to denote the extension by χ (instead of χ'), and the notation $\chi_{\underline{s}}$ is not ambiguous.

Let $V_H(\chi_{\underline{s}})$ be the space of the representation $Ind_{HN_n(\mathbb{A})}^{\widetilde{G}_n(\mathbb{A})}(\chi_{\underline{s}})$. Specifically, this is the space of functions $f : \widetilde{G}_n(\mathbb{A}) \to \mathbb{C}$ which are $\widetilde{K}$-smooth on the right, and $f(h\mathfrak{s}(u)g) = \delta_{B_n(\mathbb{A})}^{1/2}(h)\chi_{\underline{s}}(h) f(g)$ for all $h \in H$, $u \in N_n(\mathbb{A})$ and $g \in \widetilde{G}_n(\mathbb{A})$. Set $V_H(\chi) = V_H(\chi_{\underline{0}})$. A standard section $f_{\underline{s}}$ is a function such that $f_{\underline{s}}|_{\widetilde{K}}$ is independent of $\underline{s}$, and, for all $\underline{s}$, $f_{\underline{s}} \in V_H(\chi_{\underline{s}})$. For any $f \in V_H(\chi)$ there is a standard section $f_{\underline{s}}$ with $f_{\underline{0}} = f$.

We extend χ to $C(\widetilde{T}_{n+1}(\mathbb{A}), \mathfrak{s}(T_{n+1}(F)))$ and to X. The extension to $C(\widetilde{T}_{n+1}(\mathbb{A}), \mathfrak{s}(T_{n+1}(F)))$ is obtained by acting trivially on $\mathfrak{s}(T_{n+1}(F))$. The extension to X is arbitrary, as long as both extensions agree on $C(\widetilde{T}_{n+1}(\mathbb{A}), \mathfrak{s}(T_{n+1}(F))) \cap X$ (a suitable extension to X with this property always exists; see [47, p. 108]).

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For brevity, put $\Xi = C(\widetilde{T}_{n+1}(\mathbb{A}), \mathfrak{s}(T_{n+1}(F)))$.

The space $V_X(\chi_{\underline{s}})$ decomposes as the restricted tensor product $\otimes'_\nu V(\chi_{\underline{s},\nu})$ with respect to $\{V(\chi_{\underline{s},\nu})^{K_\nu^*}\}_{\nu \notin S_0}$, where $V(\chi_{\underline{s},\nu})$ is the space of $Ind_{\widetilde{B}_n(F_\nu)}^{\widetilde{G}_n(F_\nu)}(\rho_\nu(\chi_{\underline{s},\nu}))$, defined in § 2.2.1.

The spaces $V_X(\chi)$ and $V_\Xi(\chi)$ are isomorphic (see [47, p. 109]). Specifically, for $f \in V_X(\chi)$, define $f' \in V_\Xi(\chi)$ via

$$f'(g) = \sum_{\delta' \in (C_{\widetilde{T}_{n+1}(\mathbb{A})} \cap \mathfrak{s}(T_{n+1}(F))) \backslash T_{n+1}(F)} f(\mathfrak{s}(\delta')g), \quad g \in \widetilde{G}_n(\mathbb{A}). \tag{3.1}$$

For $\mathbf{w} \in W_n$, let $M'(w, \chi_{\underline{s}}) : V_\Xi(\chi_{\underline{s}}) \to V_\Xi({}^{\mathbf{w}}\chi_{\underline{s}})$ be the standard intertwining operator defined by (the meromorphic continuation of)

$$M'(w, \chi_{\underline{s}}) f'(g) = \int_{N_n^w(\mathbb{A}) \backslash N_n(\mathbb{A})} f'_{\underline{s}}(\mathfrak{s}(w)^{-1}\mathfrak{s}(u)g)\, du. \tag{3.2}$$

Note that $\mathfrak{W}_n \subset G_n(F)$. Using the isomorphism $V_\Xi \cong V_X$, we can define the corresponding intertwining operator $M(w, \chi_{\underline{s}}) : V_X(\chi_{\underline{s}}) \to V_X({}^{\mathbf{w}}\chi_{\underline{s}})$. The poles and zeros (with multiplicities) of these operators coincide.

Now we are ready to define the Eisenstein series. For $f \in V_X(\chi)$, let

$$E_{B_n}(g; f, \underline{s}) = \sum_{\delta \in B_n(F) \backslash G_n(F)} f'_{\underline{s}}(\mathfrak{s}(\delta)g).$$

The sum is absolutely convergent in a suitable cone and has a meromorphic continuation. Analogously to [47, Proposition II.1.2], or using the general formulation of Mœglin and Waldspurger [63, § II.1.7], if

$$E_{B_n}^{N_n}(g; f, \underline{s}) = \int_{N_n(F) \backslash N_n(\mathbb{A})} E_{B_n}(\mathfrak{s}(u)g; f, \underline{s})\, du$$

is the constant term of $E_{B_n}(g; f, \underline{s})$ along N_n,

$$E_{B_n}^{N_n}(g; f, \underline{s}) = \sum_{\mathbf{w} \in W_n} M'(w, \chi_{\underline{s}}) f'_{\underline{s}}(g).$$

3.3. Induction from M_n

Throughout this section, the groups GL_n and G_0 are regarded as subgroups of M_n. Since the local subgroups $\widetilde{GL}_n(F_\nu)$ and $\widetilde{G}_0(F_\nu)$ commute, $\widetilde{GL}_n(\mathbb{A})$ and $\widetilde{G}_0(\mathbb{A})$ commute. Therefore, genuine automorphic representations of $\widetilde{M}_n(\mathbb{A})$ can be described using the usual tensor product.

Let χ be a genuine character of $C_{\widetilde{T}_{n+1}(\mathbb{A})}$ which is trivial on $C_{\widetilde{T}_{n+1}(\mathbb{A})} \cap \mathfrak{s}(T_{n+1}(F))$. According to the results of § 2.1.6,

$$C_{\widetilde{T}_{n+1}(\mathbb{A})} = \{(\zeta, \zeta) : \zeta \in \mu_2\} \backslash (C_{\widetilde{T}_{GL_n}(\mathbb{A})} \times \widetilde{G}_0(\mathbb{A})).$$

Therefore we can write uniquely $\chi = \chi^{(1)} \otimes \chi^{(2)}$, where $\chi^{(1)}$ and $\chi^{(2)}$ are genuine characters of $C_{\widetilde{T}_{GL_n}(\mathbb{A})}$ and $\widetilde{G}_0(\mathbb{A})$. The character $\chi^{(1)}$ is trivial on $C_{\widetilde{T}_{GL_n}(\mathbb{A})} \cap \mathfrak{s}(T_{GL_n}(F))$, and $\chi^{(2)}$ is trivial on $\mathfrak{s}(G_0(F))$.

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The description of § 3.2 was adapted from the exposition in [47, § II.1], which we briefly recall. For $\underline{s} \in \mathbb{C}^n$, define a non-genuine character $\mathrm{e}_{\underline{s}}$ of $\widetilde{T}_{GL_n}(\mathbb{A})$ by requiring that $\mathrm{e}_{\underline{s}}(\alpha_i^\vee(x)) = |x|^{(\alpha_i, \underline{s})/2}$ for all $2 \leqslant i \leqslant n$ and $x \in \mathbb{A}^*$. Let σ be a genuine character of $C_{\widetilde{T}_{GL_n}(\mathbb{A})}$. For $C_{\widetilde{T}_{GL_n}(\mathbb{A})} < H < \widetilde{T}_{GL_n}(\mathbb{A})$, the space $V_H(\sigma_{\underline{s}})$ consists of functions on $\widetilde{GL}_n(\mathbb{A})$.

Denote $K_{GL_n(F_\nu)} = K_\nu \cap GL_n(F_\nu)$, and, for $\nu \notin S_0$, $K^*_{GL_n(F_\nu)} = \kappa_\nu(K_{GL_n(F_\nu)})$. We have the subgroup $X^{GL_n} = \mu_2^\times \backslash \prod'_\nu X_\nu^{GL_n}$, where, for almost all ν, $X_\nu^{GL_n} = C(\widetilde{T}_{GL_n}(F_\nu), \widetilde{T}_{GL_n}(F_\nu) \cap K^*_{GL_n(F_\nu)})$. Also, let $\Xi^{GL_n} = C(\widetilde{T}_{GL_n}(\mathbb{A}), \mathfrak{s}(T_{GL_n}(F)))$. We alternate between $V_{X^{GL_n}}(\sigma)$ and $V_{\Xi^{GL_n}}(\sigma)$, where $f \in V_{X^{GL_n}}(\sigma)$ is mapped to $f' \in V_{\Xi^{GL_n}}(\sigma)$ using a summation over $(C_{\widetilde{T}_{GL_n}(\mathbb{A})} \cap \mathfrak{s}(T_{GL_n}(F))) \backslash T_{GL_n}(F)$.

Claim 3.3. *Let χ be as above, and denote by $V_{Q_n}(V_{\Xi^{GL_n}}(\chi^{(1)}) \otimes \chi^{(2)})$ the subspace of right $\widetilde{K}$-smooth functions in $Ind_{\widetilde{Q}_n(\mathbb{A})}^{\widetilde{G}_n(\mathbb{A})}(Ind_{\Xi^{GL_n} N_{GL_n}(\mathbb{A})}^{\widetilde{GL}_n(\mathbb{A})}(\chi^{(1)}) \otimes \chi^{(2)})$. Then*

$$V_\Xi(\chi) \cong V_{Q_n}(V_{\Xi^{GL_n}}(\chi^{(1)}) \otimes \chi^{(2)}).$$

Proof. According to § 2.1.6 (and [47, p. 59]),

$$C_{\widetilde{T}_{n+1}(F_\nu)} = \{(\zeta, \zeta) : \zeta \in \mu_2\} \backslash (C_{\widetilde{T}_{GL_n}(F_\nu)} \times \widetilde{G}_0(F_\nu)), \tag{3.3}$$

$$T_{n+1}(F_\nu)^{\mathrm{m}} = \{(\zeta, \zeta) : \zeta \in \mu_2\} \backslash (T_{GL_n}(F_\nu)^{\mathrm{m}} \times \widetilde{G}_0(F_\nu)), \tag{3.4}$$

and, for $\nu \notin S_0$,

$$\begin{aligned} &C(\widetilde{T}_{n+1}(F_\nu), \widetilde{T}_{n+1}(F_\nu) \cap K_\nu^*) \\ &= \{(\zeta, \zeta) : \zeta \in \mu_2\} \backslash (C(\widetilde{T}_{GL_n}(F_\nu), \widetilde{T}_{GL_n}(F_\nu) \cap K^*_{GL_n(F_\nu)}) \times \widetilde{G}_0(F_\nu)). \end{aligned} \tag{3.5}$$

In general, let σ be a genuine character of $C_{\widetilde{T}_{n+1}(F_\nu)}$. Using (3.3), we can write $\sigma_\nu = \sigma_\nu^{(1)} \otimes \sigma_\nu^{(2)}$ for unique genuine characters $\sigma_\nu^{(1)}$ and $\sigma_\nu^{(2)}$ of $C_{\widetilde{T}_{GL_n}(F_\nu)}$ and $\widetilde{G}_0(F_\nu)$. Then

$$Ind_{\widetilde{B}_n(F_\nu)}^{\widetilde{G}_n(F_\nu)}(\rho_\nu(\sigma_\nu)) \cong Ind_{\widetilde{Q}_n(F_\nu)}^{\widetilde{G}_n(F_\nu)}(Ind_{\widetilde{B}_{GL_n}(F_\nu)}^{\widetilde{GL}_n(F_\nu)}(\rho_\nu(\sigma_\nu^{(1)})) \otimes \sigma_\nu^{(2)}). \tag{3.6}$$

Assume that

$$X_\nu = \begin{cases} C(\widetilde{T}_{n+1}(F_\nu), \widetilde{T}_{n+1}(F_\nu) \cap K_\nu^*) & \nu \notin S_0, \\ \widetilde{T}_{n+1}(F_\nu)^{\mathrm{m}} & \text{otherwise.} \end{cases}$$

Define $X^{GL_n} = \mu_2^\times \backslash \prod'_\nu X_\nu^{GL_n}$, where

$$X_\nu^{GL_n} = \begin{cases} C(\widetilde{T}_{GL_n}(F_\nu), \widetilde{T}_{GL_n}(F_\nu) \cap K^*_{GL_n(F_\nu)}) & \nu \notin S_0, \\ \widetilde{T}_{GL_n}(F_\nu)^{\mathrm{m}} & \text{otherwise.} \end{cases}$$

Now (3.4)–(3.6) yield

$$Ind_{XN_n(\mathbb{A})}^{\widetilde{G}_n(\mathbb{A})}(\chi) \cong Ind_{\widetilde{Q}_n(\mathbb{A})}^{\widetilde{G}_n(\mathbb{A})}(Ind_{X^{GL_n} N_{GL_n}(\mathbb{A})}^{\widetilde{GL}_n(\mathbb{A})}(\chi^{(1)}) \otimes \chi^{(2)}),$$

and the isomorphism between the spaces follows because

$$(C_{\widetilde{T}_{n+1}(\mathbb{A})} \cap \mathfrak{s}(T_{n+1}(F))) \backslash T_{n+1}(F) \cong (C_{\widetilde{T}_{GL_n}(\mathbb{A})} \cap \mathfrak{s}(T_{GL_n}(F))) \backslash T_{GL_n}(F). \qquad \square$$

3.4. Global exceptional (small) representations

3.4.1. Definition and basic properties. Let χ be a genuine character of $C_{\widetilde{T}_{n+1}(\mathbb{A})}$, which is trivial on $C_{\widetilde{T}_{n+1}(\mathbb{A})} \cap \mathfrak{s}(T_{n+1}(F))$. Analogously to the local case, call χ exceptional if, for all $\alpha \in \Delta_{G_n}$, $\chi(\alpha^{\vee *}(x^{\mathfrak{l}(\alpha)})) = |x|$ for all $x \in \mathbb{A}^*$. We have $\chi = \otimes' \chi_\nu$, where χ_ν is a local exceptional character of $C_{\widetilde{T}_{n+1}(F_\nu)}$.

Put $\mathrm{Res}_{\underline{s}=\underline{0}} = \lim_{\underline{s}\to\underline{0}} \prod_{i=2}^{n}(s_i - s_{i+1})s_{n+1}$. For $f \in V_X(\chi)$, define

$$\theta_f(g) = \mathop{\mathrm{Res}}_{\underline{s}=\underline{0}} E_{B_n}^{N_n}(g; f, \underline{s}), \quad g \in \widetilde{G}_n(\mathbb{A}).$$

Proposition 3.4. *Let χ be an exceptional character, and let $f \in V_X(\chi)$.*

(1) *The residue $\mathrm{Res}_{\underline{s}=\underline{0}} M'(w_0, \chi_{\underline{s}}) f'_{\underline{s}}(g)$ is finite and non-zero. If $w \neq w_0$, $\mathrm{Res}_{\underline{s}=\underline{0}} M'(w, \chi_{\underline{s}}) f'_{\underline{s}} = 0$.*

(2) *The constant term $\theta_f^{N_n}$ of θ_f along N_n is the image of $M'(w_0, \chi)$:*

$$\int_{N_n(F)\backslash N_n(\mathbb{A})} \theta_f(\mathfrak{s}(u)g)\, du = M'(w_0, \chi) f'(g).$$

Here, we normalize the measure by requiring the volume of $F\backslash\mathbb{A}$ to be 1.

(3) *The representation $\Theta_{G_n,\chi}$ spanned by θ_f as f varies in $V_X(\chi)$ is irreducible, automorphic, genuine, and belongs to $L^2(\mathfrak{s}(G_n(F))\backslash\widetilde{G}_n(\mathbb{A}))$. It is isomorphic to $\otimes'_\nu \Theta_{G_n,\chi_\nu}$.*

Proof. The proof follows exactly as in [17, Proposition 3.1 and Theorem 3.2] and [47, Proposition II.1.2 and Theorems II.1.4 and II.2.1]; we briefly reproduce the argument. The assertions follow from the calculation of the poles of $M'(w, \chi_{\underline{s}}) f'_{\underline{s}}(g)$. When we take a pure tensor f, $M(w, \chi_{\underline{s}}) f_{\underline{s}} = (\otimes_{\nu\in S}\varphi_\nu) \otimes (\otimes'_{\nu\notin S}\varphi_\nu)$, where $S \supset S_0$ is a finite set of places depending on χ and f. The unramified part is $\otimes'_{\nu\notin S}\varphi_\nu$. For any fixed place ν, the local intertwining operator is holomorphic, since $\chi_{\underline{s},\nu}$ belongs to the positive Weyl chamber when $\underline{s}$ tends to $\underline{0}$. Thus the poles are located in the unramified part; they are determined by the product $\prod_{\nu\notin S} c(\mathbf{w}, \chi_{\underline{s},\nu})$. If $\mathbf{w} = \mathbf{w_0}$, according to Claim 2.22, the poles belong to

$$\frac{\prod_{2\leqslant i<j\leqslant n+1} \zeta(j-i+s_j-s_i)\zeta(-j-i+2(n+2)+s_i+s_j) \prod_{i=2}^{n+1} \zeta(n+2-i+s_i)}{\prod_{2\leqslant i<j\leqslant n+1} \zeta(1+j-i+s_j-s_i)\zeta(1-j-i+2(n+2)+s_i+s_j) \prod_{i=2}^{n+1} \zeta(n+3-i+s_i)}.$$

Here, ζ denotes the partial Dedekind zeta function of F with respect to S. Recall that ζ is holomorphic except at $s = 1$, where it has a simple pole, and is non-zero on the right half-plane $\Re(s) \geqslant 1$. When $\underline{s} \to \underline{0}$, precisely n zeta functions contribute a pole. If $\mathbf{w} \neq \mathbf{w_0}$, one of these functions will be omitted. Finally, the image $M'(w_0, \chi) V_\Xi(\chi)$ is isomorphic to $\otimes'_\nu M_\nu(w_0, \chi_\nu) V(\chi_\nu)$ which is $\otimes'_\nu \Theta_{G_n,\chi_\nu}$. □

The representation $\Theta_{G_n,\chi}$ is the global exceptional representation.

3.4.2. Explicit construction of exceptional characters. Let η be a Hecke character, i.e., a character of $F^*\backslash\mathbb{A}^*$, and let ψ be a non-trivial character of $F\backslash\mathbb{A}$. Let γ_ψ be the corresponding global Weil factor. One can mimic the local construction of

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§ 2.3.3 and construct a global exceptional character. Define, for $t = \prod_{i=1}^{n} \eta_i^\vee(a_i^2)\beta_1^\vee(t_1) \in T_{n+1}(\mathbb{A})^2$, $d \in (\mathbb{A}^*)^{2/\gcd(2,n+1)}$, $z = \prod_{i=1}^{n} \eta_i^\vee(d)$, and $\zeta \in \mu_2$,

$$\chi(\zeta\mathfrak{s}(t)\mathfrak{s}(z)) = \zeta \prod_{i=1}^{n} |a_i|^{n-i+1} \eta(\Upsilon(t))|d|^{n(n+1)/4}\eta(d)^n \gamma_\psi^{\lceil n/2 \rceil}(d). \tag{3.7}$$

This is a global exceptional character, and it decomposes as $\otimes' \chi_\nu$, where χ_ν is given by (2.44). For almost all ν, χ_ν is unramified.

If n is odd, in contrast with the local case, this description does not exhaust all exceptional characters, as one may replace γ_ψ with a more general class of functions. However, $\chi|_{\mathfrak{s}(T_{n+1}(\mathbb{A})^2)}$ is uniquely described by η and (3.7).

If we write $\chi = \chi^{(1)} \otimes \chi^{(2)}$ in the notation of § 3.3,

$$\chi^{(1)}\left(\zeta\mathfrak{s}\left(\prod_{i=1}^{n}\eta_i^\vee(a_i^2)\right)\mathfrak{s}\left(\prod_{i=1}^{n}\eta_i^\vee(d)\right)\right) = \zeta\prod_{i=1}^{n}|a_i|^{n-i+1}\eta\left(d^n\prod_{i=1}^{n}a_i^2\right)|d|^{n(n+1)/4}\gamma_\psi^{\lceil n/2\rceil}(d),$$

$$\chi^{(2)}(\zeta\mathfrak{s}(\beta_1^\vee(t_1))) = \zeta\eta(t^{-2}). \tag{3.8}$$

3.4.3. The constant term. Let χ be an exceptional character, and let θ be an automorphic form in the space of $\Theta_{G_n,\chi}$. For any $1 \leqslant k \leqslant n$, define the constant term of θ along U_k by

$$\theta^{U_k}(g) = \int_{U_k(F)\backslash U_k(\mathbb{A})} \theta(\mathfrak{s}(u)g)\,du, \quad g \in \widetilde{G}_n(\mathbb{A}). \tag{3.9}$$

Let χ be an exceptional character. We prove Theorem 3. Namely, the function $m \mapsto \theta^{U_n}(m)$ ($m \in \widetilde{M}_n(\mathbb{A})$) belongs to the space of $\Theta_{GL_n,|\det|^{-1/2}\chi^{(1)}} \otimes \Theta_{G_0,\chi^{(2)}}$, where $\chi^{(1)}$ and $\chi^{(2)}$ are exceptional characters such that $\chi = \chi^{(1)} \otimes \chi^{(2)}$.

Proof of Theorem 3. Write $\theta = \theta_f$ for some $f \in V_X(\chi)$, $\theta_f(g) = \mathrm{Res}_{\underline{s}=\underline{0}}\, E_{B_n}^{N_n}(g; f, \underline{s})$ (see § 3.4.1). Let $\mathcal{W} \subset W_n$ be a set satisfying $G_n = \coprod_{\mathbf{w}\in\mathcal{W}} B_n w^{-1} Q_n$. For $X < Q_n$ and $\mathbf{w} \in \mathcal{W}$, set $X^w = {}^w B_n \cap X$. According to Mœglin and Waldspurger [63, § II.1.7],

$$\int_{U_n(F)\backslash U_n(\mathbb{A})} \sum_{\delta\in B_n(F)\backslash G_n(F)} f'_{\underline{s}}(\mathfrak{s}(\delta)\mathfrak{s}(u)g)\,du = \sum_{\mathbf{w}\in\mathcal{W}} \sum_{m\in M_n^w(F)\backslash M_n(F)} M'(w,\chi_{\underline{s}}) f'_{\underline{s}}(\mathfrak{s}(m)g).$$

Here, $M'(w, \chi_{\underline{s}})$ is defined by meromorphic continuation of an integral over $U_n^w(F)\backslash U_n(\mathbb{A})$. As we argue below, the domain of integration can be changed, and $M'(w, \chi_{\underline{s}})$ is actually given by (3.2).

We describe a choice of the set $\mathcal{W}$. There are 2^n representatives to consider. For $\underline{b} \in \{0, 1\}^n$, set $\mathbf{w}_{\underline{b}} = \prod_{i=1}^{n} \mathbf{w}_{\epsilon_{i+1}}{}^{b_i}$ (a product of commuting reflections). The elements $\{\mathbf{w}_{\underline{b}}\}_{\underline{b}}$ are a set of representatives, but $w_{\underline{b}} N_n w_{\underline{b}}^{-1}$ might not contain N_{GL_n}. It will be convenient to select elements $\mathbf{w}$ which satisfy

$$N_{GL_n} < {}^w N_n \tag{3.10}$$

(see below). Write $\underline{b} = (1^{m_1}0^{l_1}\cdots 1^{m_k}0^{l_k})$, where $m_i, l_i \geqslant 0$, $m = \sum_{i=1}^{k} m_i$, $l = \sum_{i=1}^{k} l_i$, $m + l = n$, and we assume that k is minimal such that $\underline{b}$ can be written in this form (e.g., if

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$m_i, m_{i+1} > 0$, $l_i > 0$). For any $a, b, c \geqslant 0$ such that $a+b+c \leqslant n$, put

$$\omega_{a,b,c} = diag\left(I_a, \begin{pmatrix} 0 & I_{n-a-b-c} \\ J_b & 0 \end{pmatrix}, I_c\right) \in GL_n(F),$$
$$\alpha_{\underline{b},j} = \omega_{\sum_{i=1}^{j-1} l_i, m_j, \sum_{i=1}^{j-1} m_i} \quad \forall 1 \leqslant j \leqslant k$$

(if $m_j = 0$, $\alpha_{\underline{b},j} = I_n$), $\mathbf{z}_{\underline{b}} = \alpha_{\underline{b},k} \cdots \alpha_{\underline{b},1}$. Our set of representatives is $\mathcal{W} = \{\mathbf{z}_{\underline{b}} \mathbf{w}_{\underline{b}} : \underline{b} \in \{0,1\}^n\}$. The elements $\mathbf{w} \in \mathcal{W}$ satisfy (3.10).

Since $U_n^w < N_n$, we may write the du-integration of $M'(w, \chi_{\underline{s}})$ over $U_n^w(\mathbb{A})\backslash U_n(\mathbb{A})$. The condition (3.10) implies that ${}^w N_n \cap N_n = N_{GL_n}({}^w N_n \cap U_n)$, whence $U_n^w \backslash U_n = ({}^w N_n \cap N_n)\backslash N_n$. Therefore $M'(w, \chi_{\underline{s}}) f'_{\underline{s}} \in V_{\Xi}({}^{\mathbf{w}}\chi)$.

We use the notation of § 3.3. Write ${}^{\mathbf{w}}\chi = \chi_{\mathbf{w}}^{(1)} \otimes \chi_{\mathbf{w}}^{(2)}$, where $\chi_{\mathbf{w}}^{(1)}$ and $\chi_{\mathbf{w}}^{(2)}$ are genuine characters of $C_{\widetilde{T}_{GL_n}(\mathbb{A})}$ and $\widetilde{G}_0(\mathbb{A})$. According to Claim 3.3, we can regard $M'(w, \chi_{\underline{s}}) f'_{\underline{s}}$ as an element of $V_{Q_n}(V_{\Xi^{GL_n}}(\chi_{\mathbf{w}}^{(1)})_{\underline{z}} \otimes (\chi_{\mathbf{w}}^{(2)})_{s_{n+1}})$, where $\underline{z} \in \mathbb{C}^n$ satisfies $\mathrm{e}_{\underline{z}} = {}^{\mathbf{w}}\mathrm{e}_{\underline{s}}|_{T_{GL_n}(\mathbb{A})}$. Put $\underline{s}^{(n)} = (s_1, \ldots, s_n)$.

The mapping

$$\{f, w, g, s_{n+1}\}(\underline{s}^{(n)}, b) = M'(w, \chi_{\underline{s}}) f'_{\underline{s}}(bg) \quad (b \in \widetilde{GL}_n(\mathbb{A}))$$

belongs to $V_{\Xi^{GL_n}}(|\det|^{n/2}(\chi_{\mathbf{w}}^{(1)})_{\underline{z}})$ (we used $\delta_{Q_n(\mathbb{A})}^{1/2} = |\det|^{n/2}$).

By (3.10), $GL_n(F)^w = B_{GL_n}(F)$; hence $M_n^w(F)\backslash M_n(F) \cong GL_n(F)^w \backslash GL_n(F)$, and the summation is an Eisenstein series with respect to B_{GL_n}, applied to $\{f, w, g, s_{n+1}\}$. Thus

$$\theta_f(g) = \sum_{\mathbf{w} \in \mathcal{W}} \operatorname*{Res}_{\underline{s}=\underline{0}} E_{B_{GL_n}}(I_n; \{f, w, g, s_{n+1}\}, \underline{s}^{(n)}).$$

Fix $\mathbf{w} = \mathbf{z}_{\underline{b}} \mathbf{w}_{\underline{b}}$, and let m_i, l_i, m, l be as above. Let $\mathcal{U}_{\alpha_{n+1}}$ be the unipotent subgroup of $N_n(\mathbb{A})$ generated by $n_{\alpha_{n+1}}$. Then $\mathcal{U}_{\alpha_{n+1}} < ({}^w N_n \cap N_n)(\mathbb{A})$ if and only if $m = 0$.

Claim 3.5. *The function* $M'(w, \chi_{\underline{s}}) f'_{\underline{s}}$ *is holomorphic at* $\underline{s}$, *except perhaps for a simple pole at* $s_{n+1} = 0$. *The pole exists for some data* f' *if and only if* $m > 0$.

Claim 3.6. *The series* $E_{B_{GL_n}}(I_n; \{f, w, g, s_{n+1}\}, \underline{s}^{(n)})$ *is holomorphic at* $\underline{s}^{(n)}$, *except perhaps for simple poles at* $s_i - s_{i+1}$ *for* $1 \leqslant i < n$. *If* $l > 0$ *and* $m > 0$, *it has at most* $n-2$ *poles. If* $l = 0$, *the function* $y \mapsto E_{B_{GL_n}}(I_n; \{f, w, yg, s_{n+1}\}, \underline{s}^{(n)})$ *on* $\widetilde{GL}_n(\mathbb{A})$ *belongs to the space of* $\Theta_{GL_n, |\det|^{-1/2}\chi^{(1)}}$.

Before proving these claims, we use them to complete the proof of the theorem. According to Claim 3.5, $s_{n+1} \cdot \{f, w, g, s_{n+1}\}$ is holomorphic and vanishes at $s_{n+1} = 0$ if $m = 0$. Hence we may assume that $m > 0$. Then, by Claim 3.5, the series $E_{B_{GL_n}}(I_n; \{f, w, g, s_{n+1}\}, \underline{s}^{(n)})$ multiplied by $\prod_{i=1}^{n-1}(s_i - s_{i+1})$ vanishes for $\underline{s} \to \underline{0}$, unless $l = 0$, and then $\mathbf{w} = \mathbf{z}_{\underline{1}} \mathbf{w}_{\underline{1}}$. Therefore

$$\theta_f(g) = \operatorname*{Res}_{\underline{s}=\underline{0}} E_{B_{GL_n}}(I_n; \{f, w, g, s_{n+1}\}, \underline{s}^{(n)}).$$

Now Claim 3.5 shows that $y \mapsto \theta_f(yg)$ belongs to the space of $\Theta_{GL_n, |\det|^{-1/2}\chi^{(1)}}$. Clearly $h \mapsto \theta_f(hg)$ $(h \in \widetilde{G}_0(\mathbb{A}))$ lies in the space of $\Theta_{G_0, \chi^{(2)}}$. The theorem is proved.

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Proof of Claim 3.5. As in the proof of Proposition 3.4, and with the same notation, we analyze the poles of $M'(w,\chi_{\underline{s}})f'_{\underline{s}}$ by looking at $M(w,\chi_{\underline{s}})f_{\underline{s}}$ for a pure tensor $f\in V_X(\chi)$. Denote $M(w,\chi_{\underline{s}})f_{\underline{s}}=(\otimes_{\nu\in S}\varphi_\nu)\otimes(\otimes'_{\nu\notin S}\varphi_\nu)$. At each $\nu\in S$, the local intertwining operator is holomorphic around $\underline{0}$, because, when $\underline{s}\to\underline{0}$, $\chi_{\underline{s},\nu}$ belongs to the positive Weyl chamber. The poles of the intertwining operator coincide with the poles of the infinite product. If $m=0$, the integration is trivial; hence clearly there is no pole. Otherwise, the poles are contained in the set of poles of

$$\frac{\prod_{2\leqslant i<j\leqslant n+1}\zeta(-j-i+2(n+2)+s_i+s_j)\prod_{i=2}^{n+1}\zeta(n+2-i+s_i)}{\prod_{2\leqslant i<j\leqslant n+1}\zeta(1-j-i+2(n+2)+s_i+s_j)\prod_{i=2}^{n+1}\zeta(n+3-i+s_i)}.$$

This product is holomorphic, except for a simple pole at $s_{n+1}=0$. Furthermore, since $\mathcal{U}_{\alpha_{n+1}}$ is not a subgroup of $({}^{w}N_n\cap N_n)(\mathbb{A})$, the factor $\zeta(n+2-i+s_i)$ with $i=n+1$ contributes to the poles of the intertwining operator. Hence there is a pole at $s_{n+1}=0$. □

Proof of Claim 3.6. To compute the poles of $E_{B_{GL_n}}(I_n;\{f,w,g,s_{n+1}\},\underline{s}^{(n)})$, we consider the constant term of $\{f,w,g,s_{n+1}\}$ along N_{GL_n}. According to Kazhdan and Patterson [47, Proposition II.1.2],

$$\begin{aligned}\int_{N_{GL_n}(F)\backslash N_{GL_n}(\mathbb{A})} & E_{B_{GL_n}}(\mathfrak{s}(u);\{f,w,g,s_{n+1}\},\underline{s}^{(n)})\,du\\ &=\sum_{\omega\in W_{GL_n}}(M^{GL_n})'(\omega,\underline{s}^{(n)})\{f,w,g,s_{n+1}\}'_{\underline{s}^{(n)}}(I_n).\end{aligned}$$

Here, W_{GL_n} is the Weyl subgroup of GL_n, and

$$(M^{GL_n})'(\omega,\underline{s}^{(n)}):V_{\Xi^{GL_n}}(|\det|^{n/2}(\chi^{(1)}_{\mathbf{w}})_{\underline{z}})\to V_{\Xi^{GL_n}}(|\det|^{n/2}\ {}^{\mathfrak{s}(\omega)}((\chi^{(1)}_{\mathbf{w}})_{\underline{z}}))$$

is the corresponding intertwining operator.

We may assume that $\chi|_{\mathfrak{s}(T_{n+1}(\mathbb{A})^2)}$ is defined with respect to a Hecke character η according to (3.7). Since $g\mapsto\eta(\Upsilon(g))$ is an automorphic character of $G_n(\mathbb{A})$ which extends to a non-genuine automorphic character of $\widetilde{G}_n(\mathbb{A})$, and $(\eta\Upsilon\cdot\chi)^{(1)}_{\mathbf{w}}=\eta\Upsilon\cdot(\chi^{(1)}_{\mathbf{w}})$, we may prove the claim for $(\eta^{-1}\Upsilon)\cdot\chi$. This means that we assume that $\eta=1$. Then, for $t=\prod_{i=1}^{n}\eta_i^\vee(a_i^2)\beta_1^\vee(t_1)\in T_{n+1}(\mathbb{A})^2$, $d\in(\mathbb{A}^*)^{2/\gcd(2,n+1)}$, $z=\prod_{i=1}^{n}\eta_i^\vee(d)$, and $\zeta\in\mu_2$,

$$\chi(\zeta\mathfrak{s}(t)\mathfrak{s}(z))=\zeta\prod_{i=1}^{n}|a_i|^{n-i+1}|d|^{n(n+1)/4}\gamma(d). \tag{3.11}$$

Here, γ is a complex-valued function on the quotient of $(\mathbb{A}^*)^{2/\gcd(2,n+1)}$ by $((\mathbb{A}^*)^{2/\gcd(2,n+1)}\cap F^*)\mathbb{A}^{*2}$, which satisfies $\gamma(dd')=c(d,d')^{\lceil n/2\rceil}\gamma(d)\gamma(d')$ and is non-zero. Additionally, by Claim 3.5, we may assume that $s_{n+1}=0$.

The element $\mathbf{z}_{\underline{b}}$ acts on the cocharacters $\eta_i^\vee$. Let τ be the permutation such that

$$\mathbf{z}_{\underline{b}}^{-1}\left(\prod_{i=1}^{n}\eta_i^\vee(a_i)\right)=\prod_{i=1}^{n}\eta_i^\vee(a_{\tau^{-1}(i)}).$$

Also define $\underline{c} \in \{-1, 1\}^n$ by $c_i = 1 - 2b_i$. Then

$$(\chi_{\mathbf{w}}^{(1)})_{\underline{z}}\left(\mathfrak{s}\left(\prod_{i=1}^{n}\eta_i^\vee(a_i^2)\right)\right) = \prod_{i=1}^{n}|a_i|^{c_{\tau(i)}(n-\tau(i)+1+s_{\tau(i)})}.$$

Note that here we used the fact that $\chi(\mathfrak{s}(\beta_1^\vee(t_1))) = 1$, because (see, e.g., (1.7))

$$\mathbf{w}_{\underline{b}}\left(\prod_{i=1}^{n}\eta_i^\vee(a_i)\right) = \prod_{i=1}^{n}\eta_i^\vee(a_i^{\mp 1})\beta_1^\vee(\cdots).$$

We show that $\chi_{\mathbf{w}}^{(1)}$ belongs to the positive Weyl chamber. Denote by $\sqcup$ the concatenation of tuples; e.g., $(2, 5, 1) \sqcup (2, 8, -1, 0) = (2, 5, 1, 2, 8, -1, 0)$. For r integers $i_1, \ldots, i_r$, set ${}^J(i_1, i_2, \ldots, i_r) = (i_r, \ldots, i_2, i_1)$. Let $(1, \ldots, n) = [l_1] \sqcup \cdots \sqcup [l_k] \sqcup [m_k] \sqcup \cdots \sqcup [m_1]$, where, for $r \geqslant 0$, $[r]$ is an ascending sequence of r integers. Then

$$(\tau^{-1}(1), \ldots, \tau^{-1}(n)) = {}^J[m_1] \sqcup [l_1] \sqcup \cdots \sqcup {}^J[m_k] \sqcup [l_k].$$

It follows that, if $1 \leqslant i < j < l$, $\tau(i) < \tau(j)$ and $c_{\tau(i)} = c_{\tau(j)} = 1$. Thus, for $x \in \mathbb{A}^*$,

$$\chi_{\mathbf{w}}^{(1)}(\mathfrak{s}(\eta_i^\vee(x^2)\eta_j^\vee(x^{-2}))) = |x|^{n-\tau(i)+1-(n-\tau(j)+1)} = |x|^{\tau(j)-\tau(i)}. \tag{3.12}$$

For $l + 1 \leqslant i < j < n$, we have $\tau(i) > \tau(j)$ and $c_{\tau(i)} = c_{\tau(j)} = -1$, whence

$$\chi_{\mathbf{w}}^{(1)}(\mathfrak{s}(\eta_i^\vee(x^2)\eta_j^\vee(x^{-2}))) = |x|^{\tau(i)-\tau(j)}. \tag{3.13}$$

If $l \leqslant i \leqslant l$ and $l + 1 \leqslant j \leqslant n$, then $c_{\tau(i)} = 1$ and $c_{\tau(j)} = -1$, and hence

$$\chi_{\mathbf{w}}^{(1)}(\mathfrak{s}(\eta_i^\vee(x^2)\eta_j^\vee(x^{-2}))) = |x|^{2n+2-\tau(i)-\tau(j)}. \tag{3.14}$$

We conclude that $\chi_{\mathbf{w}}^{(1)}$ belongs to the positive Weyl chamber. Therefore, for decomposable data, at any place ν the local intertwining operator is holomorphic. Then the poles of the intertwining operator $(M^{GL_n})'(\omega, \underline{s}^{(n)})$ appear only in the unramified part.

Consider such poles. For the purpose of a bound, we may assume that $\omega = J_n$. Let $S \supset S_0$ be a finite set. For a positive root (i, j), where $1 \leqslant i < j \leqslant n$, and $\nu \notin S$, let

$$d_{(i,j),\nu} = (\chi_{\mathbf{w}}^{(1)})_\nu(\mathfrak{s}_\nu(\eta_i^\vee(\varpi_\nu^2)\eta_j^\vee(\varpi_\nu^{-2}))) = q_\nu^{-C_{(i,j)}}.$$

Because $(\chi_{\mathbf{w}}^{(1)})_\nu$ belongs to the positive Weyl chamber, $C_{(i,j)} > 0$. Hence the poles appear (with multiplicities) in the product

$$\prod_{1\leqslant i<j\leqslant n}\zeta(C_{(i,j)} + s_{\tau(j)} - s_{\tau(i)}).$$

For a pair (i, j) appearing in this product, there is a pole at $\underline{s}^{(n)} = \underline{0}$ if and only if $C_{(i,j)} = 1$. For any i, there is at most one j such that $\tau(j) - \tau(i) = 1$, and, similarly, at most one j with $\tau(i) - \tau(j) = 1$. Looking at (3.12)–(3.14), we see that the only possible poles are simple poles at $s_i - s_{i+1} = 0$ for each $1 \leqslant i < n$ such that $i \neq l$. If $l = 0$ or $m = 0$ (in which case $l = n$), these are $n - 1$ poles. Otherwise we have at most $n - 2$ poles.

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Now assume that $l = 0$. Then

$$|\det|^{n/2}\chi_{\mathbf{w}}^{(1)}\left(\mathfrak{s}\left(\prod_{i=1}^{n}\eta_i^{\vee}(a_i^2)\right)\mathfrak{s}\left(\prod_{i=1}^{n}\eta_i^{\vee}(d)\right)\right) = \prod_{i=1}^{n}|a_i|^{n-i}|d|^{n(n-1)/4}\gamma(d^{-1}).$$

This is an exceptional character of $C_{\widetilde{T}_{GL_n}(\mathbb{A})}$, and therefore, according to the results of Kazhdan and Patterson [47, Proposition II.1.2 and Theorems II.1.4 and II.2.1], the mapping

$$y \mapsto \lim_{\underline{s}^{(n)}\to\underline{0}}\prod_{i=1}^{n-1}(s_i - s_{i+1})E_{B_{GL_n}}(y; \{f, w, g, 0\}, \underline{s}^{(n)})$$

(we assumed that $s_{n+1} = 0$) belongs to the space of $\Theta_{GL_n,|\det|^{n/2}\chi_{\mathbf{w}}^{(1)}}$. Since

$$E_{B_{GL_n}}(y; \{f, w, g, s_{n+1}\}, \underline{s}^{(n)}) = E_{B_{GL_n}}(I_n; \{f, w, yg, 0, \underline{s}^{(n)}),$$

it is left to show that $|\det|^{n/2}\chi_{\mathbf{w}}^{(1)} = |\det|^{-1/2}\chi^{(1)}$. Equality (3.11) implies that

$$\chi^{(1)}\left(\mathfrak{s}\left(\prod_{i=1}^{n}\eta_i^{\vee}(a_i^2)\right)\mathfrak{s}(z)\right) = \prod_{i=1}^{n}|a_i|^{n-i+1}|d|^{n(n+1)/4}\gamma(d).$$

Because $\gamma(d^{-1}) = \gamma(d)$, the result follows. □

Remark 3.2. The analogous result for SO_{2n+1} was proved in [17, 45] by induction, using a parabolic subgroup with a Levi part isomorphic to $GL_1 \times SO_{2(n-1)+1}$. This approach does not seem to work, because the representation of $M_1(\mathbb{A})$ cannot be described using the tensor product. Note that the twist of $\chi^{(1)}$ by $|\det|^{-1/2}$ is compatible with the result on SO_{2n+1}.

3.4.4. Vanishing results. Let ψ be a non-trivial character of $F\backslash\mathbb{A}$. We use the notation of § 2.3.2. The global counterpart of the twisted Jacquet modules is a class of Fourier coefficients, for generic characters. Any character of $V_{\mathcal{O}}(\mathbb{A})$ which is trivial on $V_{\mathcal{O}}(F)$ is the lift of a character of $V_{\mathcal{O}}^{\mathrm{a}}(\mathbb{A})$, trivial on $V_{\mathcal{O}}^{\mathrm{a}}(F)$. Such a character can be identified with a point $b \in V_{\mathcal{O}}^{\mathrm{a}}(F)$. The character ψ_b is generic if b belongs to the open orbit with respect to the action of $C(G_n(\overline{F}), h_{\mathcal{O}}(\overline{F}^*))$ on $V_{\mathcal{O}}^{\mathrm{a}}(\overline{F})$. The Fourier coefficient of an automorphic function φ on $\widetilde{G}_n(\mathbb{A})$ or $G_n(\mathbb{A})$, with respect to $\mathcal{O}$ and ψ_b, is given by

$$\int_{V_{\mathcal{O}}(F)\backslash V_{\mathcal{O}}(\mathbb{A})}\varphi(\mathfrak{s}(v))\psi_b(v)\,dv.$$

This is the Fourier coefficient of Theorem 2. This theorem follows immediately from Theorem 1 using a local–global principle (see, e.g., [39, Proposition 1]). In particular, as in [17, Proposition 4.3], we have the following result (directly implied by Lemma 2.25).

Proposition 3.7. *Let χ be an exceptional character, and let $b \in F^{2n-1}$ be with $\ell(b) \neq 0$. Then, for any θ in the space of $\Theta_{G_n,\chi}$, $\int_{U_1(F)\backslash U_1(\mathbb{A})}\theta(\mathfrak{s}(u))\psi(r(u)b)\,du = 0$.*

4. Application — the theta period

In this section, we prove Theorem 4. Let F be a number field with a ring of adèles $\mathbb{A}$. Denote by A^+ the subgroup of idèles of F whose finite components are trivial and Archimedean components are equal, real, and positive; it can be identified with $\mathbb{R}_{>0}$.

Regard GL_n and G_0 as subgroups of M_n. Let $GL_n(\mathbb{A})^1 < GL_n(\mathbb{A})$ be the subgroup of matrices b such that $|\mathrm{det} b| = 1$. If $g = bhuk \in G_n(\mathbb{A})$ with $b \in GL_n(\mathbb{A})$, $h \in G_0(\mathbb{A})$, $u \in U_n(\mathbb{A})$, and $k \in K$, define $H(g) = |\det b|$.

Let $\widetilde{G}_n(\mathbb{A})$ be the double cover of $G_n(\mathbb{A})$, defined in § 3.1. We identify $G_n(F)$ and $N_n(\mathbb{A})$ with their images in $\widetilde{G}_n(\mathbb{A})$ under $\mathfrak{s}$ (see Claim 3.1). Furthermore, regard $G_n(\mathbb{A})$ as a subgroup of $\widetilde{G}_n(\mathbb{A})$ by fixing a section $G_n(\mathbb{A}) \to \widetilde{G}_n(\mathbb{A})$. For example, one can take the section κ defined in § 3.1. We suppress the actual section from the notation.

Let τ be an irreducible unitary automorphic cuspidal representation of $GL_n(\mathbb{A})$ with a central character ω_τ, and let η be a unitary Hecke character. As in the statement of the theorem, assume that $\omega_\tau^2(t)\eta^n(t) = 1$ for all $t \in A^+$.

Let ρ be a smooth complex-valued function on $G_n(\mathbb{A})$, satisfying the following properties: it is K finite; for any $b^1 \in GL_n(\mathbb{A})^1$, $h \in G_0(\mathbb{A})$, $u \in U_n(\mathbb{A})$, $t \in A^+$, $a = t \cdot I_n$, and $g \in G_n(\mathbb{A})$, $\rho(b^1 ahug) = \delta_{Q_n(\mathbb{A})}^{\frac{1}{2}}(a)\omega_\tau(a)\eta(h)\rho(b^1 g)$; and, for any $k \in K$, the mapping $b^1 \mapsto \rho(b^1 k)$ is a $K \cap GL_n(\mathbb{A})^1$-finite vector in the space of τ. The standard section corresponding to ρ is defined by $\rho_s(g) = H(g)^s \rho(g)$, for any $s \in \mathbb{C}$.

We have the following Eisenstein series:

$$E_{Q_n}(g; \rho, s) = \sum_{\delta \in Q_n(F)\backslash G_n(F)} \rho_s(\delta g), \quad g \in G_n(\mathbb{A}).$$

According to Hundley and Sayag [37, Proposition 18.0.4], the series $E_{Q_n}(g; \rho, s)$ is holomorphic when $\Re(s) > 0$, except perhaps at $s = 1/2$, where it may have a simple pole. This pole exists (for some data ρ, g) if and only if $L(s, \tau, Sym^2 \otimes \eta)$ has a pole at $s = 1$. Equivalently, one can use the partial L-function $L^S(s, \tau, Sym^2 \otimes \eta)$, where S is any finite set of places; see [37, Remark 2.2.2 and § 19.3].

Remark 4.1. In [37], the L-function was actually $L^S(s, \tau, Sym^2 \otimes \eta^{-1})$. This is because their embedding of G_0 in G_n was different. In the notation of § 1.2, they sent $\theta_1^\vee$ to $-\epsilon_1^\vee$ (compare (1.7) to their formula for wmw^{-1} in [37, Lemma 13.2.4]).

Put $E_{1/2}(g; \rho) = \mathrm{Res}_{s=1/2} E_{Q_n}(g; \rho, s)$, where $\mathrm{Res}_{s=1/2} = \lim_{s\to 1/2}(s - 1/2)$.

In general, if $H < G_n(\mathbb{A})$, and we are given two complex-valued genuine functions φ and φ' on $\widetilde{H}$, the function $h \mapsto \varphi(h)\varphi'(h)$ is independent of the actual choice of section $H \to \widetilde{G}_n(\mathbb{A})$. Therefore it can be regarded as a function on H.

Let χ and χ' be global exceptional characters, constructed as explained in § 3.4.2, and form the exceptional representations $\Theta_{G_n,\chi}$ and $\Theta_{G_n,\chi'}$. We take χ and χ' such that $\chi \cdot \chi' \cdot \eta = 1$ on $C_{G_n(\mathbb{A})}$, which means that

$$(\chi \cdot \chi')(\beta^\vee(t))\eta(t) = 1, \quad \forall t \in \mathbb{A}^*. \tag{4.1}$$

This is possible (see § 3.4.2). Let θ (respectively, θ') be an automorphic form in the space of $\Theta_{G_n,\chi}$ (respectively, $\Theta_{G_n,\chi'}$).

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For any complex-valued function Ξ on $G_n(F)\backslash G_n(\mathbb{A})$, satisfying $\Xi(\beta^\vee(t)g) = \eta(t)\Xi(g)$ for all $g \in G_n(\mathbb{A})$ and $t \in \mathbb{A}^*$, the co-period of Ξ, θ, and θ' is the following integral:

$$\mathcal{I}(\Xi, \theta, \theta') = \int_{C_{G_n(\mathbb{A})}G_n(F)\backslash G_n(\mathbb{A})} \Xi(g)\theta(g)\theta'(g)\, dg,$$

provided it is absolutely convergent. Note that $H(\beta^\vee(t)g) = H(g)$, and hence ρ_s, $E_{Q_n}(\cdot;\rho,s)$, and $E_{1/2}(\cdot;\rho)$ are, at least formally, possible candidates for Ξ.

Let dt be the standard Lebesgue measure on $\mathbb{R}$. We fix measures such that the following formulas hold.

$$\begin{aligned} \int_{Q_n(F)\backslash G_n(\mathbb{A})} \varphi(g)\, dg &= \int_K \int_{M_n(F)\backslash M_n(\mathbb{A})} \int_{U_n(F)\backslash U_n(\mathbb{A})} \varphi(umk) du \delta^{-1}_{Q_n(\mathbb{A})}(m)\, dm\, dk, \\ \int_{GL_n(F)\backslash GL_n(\mathbb{A})} \varphi(b)\, db &= n \int_{GL_n(F)\backslash GL_n(\mathbb{A})^1} \int_{\mathbb{R}_{>0}} \varphi((t\cdot I_n)b)t^{-1}\, dt\, db. \end{aligned} \tag{4.2}$$

The co-period $\mathcal{I}(E_{1/2}(\cdot;\rho),\theta,\theta')$ is absolutely convergent (see [45, Claim 3.1]). In order to compute it, we apply the truncation operator of Arthur [1, 2] to $E_{Q_n}(g;\rho,s)$ as in, for example, [28, 33, 39, 40]. For the complete argument, see [45]; we provide a sketch. For a real number $d > 1$, let $ch_{>d} : \mathbb{R}_{>0} \to \{0,1\}$ be the characteristic function of $\mathbb{R}_{>d}$, and set $ch_{\leqslant d} = 1 - ch_{>d}$. Denote

$$\begin{aligned} \mathcal{E}_1^d(g;s) &= \sum_{\delta \in Q_n(F)\backslash G_n(F)} \rho_s(\delta g) ch_{\leqslant d}(H(\delta g)), \\ \mathcal{E}_2^d(g;s) &= \sum_{\delta \in Q_n(F)\backslash G_n(F)} M(w,s)\rho_s(\delta g) ch_{>d}(H(\delta g)). \end{aligned}$$

Here, $M(w,s)$ is the global intertwining operator corresponding to a representative w such that $wU_nw^{-1} = U_n^-$. The co-periods $\mathcal{I}(\mathcal{E}_1^d(\cdot;s),\theta,\theta')$ and $\mathcal{I}(\mathcal{E}_2^d(\cdot;s),\theta,\theta')$ are absolutely convergent for $\Re(s) \gg 0$ and

$$\mathcal{I}(\Lambda_d E_{Q_n}(\cdot;\rho,s),\theta,\theta') = \mathcal{I}(\mathcal{E}_1^d(\cdot;s) - \mathcal{E}_2^d(\cdot;s),\theta,\theta'), \tag{4.3}$$

where $\Lambda_d E_{Q_n}(\cdot;\rho,s)$ is the application of the truncation operator to $E_{Q_n}(\cdot;\rho,s)$.

The following proposition describes $\mathcal{I}(\mathcal{E}_i^d(\cdot;s),\theta,\theta')$, and essentially completes the proof.

Proposition 4.1. *The co-periods $\mathcal{I}(\mathcal{E}_1^d(\cdot;s),\theta,\theta')$ and $\mathcal{I}(\mathcal{E}_2^d(\cdot;s),\theta,\theta')$ have meromorphic continuation to the whole plane. Moreover,*

$$\mathcal{I}(\mathcal{E}_1^d(\cdot;s),\theta,\theta') = \frac{d^{s-1/2}}{s-1/2} \int_K \int_{GL_n(F)\backslash GL_n(\mathbb{A})^1} \rho(bk)\theta^{U_n}(bk)\theta'^{U_n}(bk)\, db\, dk, \tag{4.4}$$

$$\mathcal{I}(\mathcal{E}_2^d(\cdot;s),\theta,\theta') = \frac{-d^{-s-1/2}}{s+1/2} \int_K \int_{GL_n(F)\backslash GL_n(\mathbb{A})^1} M(w,s)\rho_s(bk)\theta^{U_n}(bk)\theta'^{U_n}(bk)\, db\, dk. \tag{4.5}$$

Here, θ^{U_n} and θ'^{U_n} are given by (3.9). In particular, $\mathcal{I}(\mathcal{E}_1^d(\cdot;s),\theta,\theta')$ is holomorphic, except perhaps at $s = 1/2$, where it may have a simple pole.

Remark 4.2. We corrected two typos from the corresponding formulas of [45]: the normalization of the measures should be with n (instead of n^{-1} in [45, (3.1)]) and in the statement of the proposition [45, (3.4)] should be $\frac{-c^{-s-1/2}}{s+1/2}$.

Now taking the residue in (4.3) yields

$$\mathcal{I}(\Lambda_d E_{1/2}(\cdot;\rho),\theta,\theta') = \operatorname*{Res}_{s=1/2} \mathcal{I}(\mathcal{E}_1^d(\cdot;s),\theta,\theta') - \operatorname*{Res}_{s=1/2} \mathcal{I}(\mathcal{E}_2^d(\cdot;s),\theta,\theta').$$

Here, $\Lambda_d E_{1/2}(\cdot;\rho)$ is the application of the truncation operator to $E_{1/2}(\cdot;\rho)$. Taking $d \to \infty$ (and using [53, Proposition 4.3.3]), we obtain

$$\mathcal{I}(E_{1/2}(\cdot;\rho),\theta,\theta') = \int_K \int_{GL_n(F)\backslash GL_n(\mathbb{A})^1} \rho(bk)\theta^{U_n}(bk)\theta'^{U_n}(bk)\, db\, dk.$$

This proves part (1) of the theorem. Part (2) follows from [28, Theorem 3.2] (see also [39, Proposition 2] and [40]).

Proof of Proposition 4.1. The proof is a simple modification of the proof of [45, Proposition 3.2]. We briefly present the argument. Consider $\mathcal{I}(\mathcal{E}_1^d(\cdot;s),\theta,\theta')$. Collapsing the summation into the integral gives

$$\int_{C_{G_n(\mathbb{A})}Q_n(F)\backslash G_n(\mathbb{A})} \rho_s(g) ch_{\leqslant d}(H(g))\theta(g)\theta'(g)\, dg. \tag{4.6}$$

We write the Fourier expansion of θ' along the derived group C_{U_n} of U_n. Assume that $n > 1$; the case when $n = 1$ is trivial because the cover of $G_1(\mathbb{A})$ splits (by Claim 2.2). The group $M_n(F)$ acts on the set of characters of $C_{U_n}(F)\backslash C_{U_n}(\mathbb{A})$ with $\lfloor n/2 \rfloor + 1$ orbits. For $0 \leqslant j \leqslant \lfloor n/2 \rfloor$, define a character ψ_j of $C_{U_n}(F)\backslash C_{U_n}(\mathbb{A})$ by $\psi_j(c) = \psi(\sum_{i=1}^{j} c_{n-2i+1,n+2i})$, where ψ is a non-trivial character of $F\backslash\mathbb{A}$. The stabilizer $St_{\psi_j}(F)$ of ψ_j in $M_n(F)$ is equal to $St'_{\psi_j}(F) \times G_0(F)$, where $St'_{\psi_j}(F)$ is the stabilizer computed in [45],

$$St'_{\psi_j}(F) = \left\{ \begin{pmatrix} a & z \\ 0 & b \end{pmatrix} : a \in GL_{n-2j}(F), b \in Sp_j(F), z \text{ is any } (n-2j)\times 2j \text{ matrix} \right\} < GL_n(F),$$

and Sp_j is the symplectic group in $2j$ variables. Then

$$\theta'(g) = \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{\lambda \in St_{\psi_j}(F)\backslash GL_n(F)} \int_{C_{U_n}(F)\backslash C_{U_n}(\mathbb{A})} \theta'(c\lambda g)\psi_j(c)\, dc.$$

Plugging this into (4.6) gives a sum of integrals $\sum_{j=0}^{\lfloor n/2 \rfloor} \mathcal{I}_j$ with

$$\begin{aligned}\mathcal{I}_j = &\int_K \int_{C_{G_n(\mathbb{A})}St_{\psi_j}(F)\backslash M_n(\mathbb{A})} \rho_s(mk) ch_{\leqslant d}(H(m)) \\ &\times \int_{U_n(F)\backslash U_n(\mathbb{A})} \theta(umk) \left(\int_{C_{U_n}(F)\backslash C_{U_n}(\mathbb{A})} \theta'(cumk)\psi_j(c)\, dc \right) du\, \delta_{Q_n(\mathbb{A})}^{-1}(m)\, dm\, dk.\end{aligned}$$

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Set $V_j = (C_{U_n} \cap U_n \cap U_{n-2j})\backslash(U_n \cap U_{n-2j})$ and

$$X_j(F) = \left\{ \begin{pmatrix} I_{n-2j} & 0 & v_1 & c_1 & c_2 \\ & I_{2j} & 0 & c_3 & c_1' \\ & & 1 & 0 & v_1' \\ & & & I_{2j} & 0 \\ & & & & I_{n-2j} \end{pmatrix} \in U_n(F) \right\}.$$

The constant term of θ' along C_{U_n} is a function on $V_j(F)\backslash V_j(\mathbb{A}) \cong (F\backslash\mathbb{A})^{n-2j}$, and, by considering a Fourier expansion of this function and using Proposition 3.7, one sees that

$$\int_{C_{U_n}(F)\backslash C_{U_n}(\mathbb{A})} \theta'(c)\psi_j(c)\,dc = \int_{X_j(F)\backslash X_j(\mathbb{A})} \theta'(c)\psi_j(c)\,dc.$$

Here, ψ_j was extended to a character of $X_j(\mathbb{A})$, trivially on the coordinates of v_1 (and v_1'). Applying this to $\mathcal{I}_j$ and factoring the du-integration through X_j yields

$$\begin{aligned}\mathcal{I}_j = &\int_K \int_{C_{G_n(\mathbb{A})}St_{\psi_j}(F)\backslash M_n(\mathbb{A})} \rho_s(mk)ch_{\leqslant d}(H(m)) \\ &\times \int_{X_j(\mathbb{A})\backslash(U_n(F)\backslash U_n(\mathbb{A}))} \left(\int_{X_j(F)\backslash X_j(\mathbb{A})} \theta(xu_1mk)\psi_j^{-1}(x)\,dx\right) \\ &\times \left(\int_{X_j(F)\backslash X_j(\mathbb{A})} \theta'(xu_1mk)\psi_j(x)\,dx\right) du_1\delta_{Q_n(\mathbb{A})}^{-1}(m)\,dm\,dk.\end{aligned}$$

The following lemma is the heart of the unfolding argument.

Lemma 4.2. *For any $j > 0$, $\mathcal{I}_j = 0$.*

Proof. The proof of [45, Lemma 3.5] is separated into two cases, $j < n/2$ and $j = n/2$. In the case when $j < n/2$, one introduces an inner integration along a unipotent radical of GL_n, into $\mathcal{I}_j$, then uses the fact that $\rho_s|_{GL_n(\mathbb{A})}$ is a cusp form to show that $\mathcal{I}_j$ vanishes. The arguments include a Fourier expansion, an 'exchange of roots', and a few vanishing results. The vanishing results we use here are Proposition 3.7, Claim 2.26, and Proposition 2.29. Note that in [45] we also used a result on a global constant term (with respect to U_k and any $k \geqslant 3$); this can be replaced with a local result: Proposition 2.29 (as already observed in [45, Remark 3.8]).

Assume that $j = n/2$ (in particular, n is even). In this case, $St_{\psi_{n/2}} = Sp_{n/2}$. Utilizing a result of Ikeda [38, Proposition 1.3] on functions on Jacobi groups along with Proposition 3.7, one can introduce an inner integration $\int_{Sp_{n/2}(F)\backslash Sp_{n/2}(\mathbb{A})} \rho_s(ymk)\,dy$ into $\mathcal{I}_j$, which vanishes according to Jacquet and Rallis [39, Proposition 1]. The arguments in [45] are applicable. We need to observe that, as in the case of the cover of $SO_{2n+1}(\mathbb{A})$, the restriction of the cover of $G_n(\mathbb{A})$ to $Sp_{n/2}(\mathbb{A})$ is a non-trivial double cover, which is therefore isomorphic to the usual metaplectic cover of $Sp_{n/2}(\mathbb{A})$. □

Therefore, integral (4.6) is equal to $\mathcal{I}_0$, and, since $St_{\psi_0}(F) = M_n(F)$ and

$$(C_{G_n(\mathbb{A})}M_n(F))\backslash M_n(\mathbb{A}) \cong GL_n(F)\backslash GL_n(\mathbb{A}),$$

integral $\mathcal{I}_0$ becomes

$$\int_K \int_{GL_n(F)\backslash GL_n(\mathbb{A})} \rho_s(bk) ch_{\leqslant d}(H(b))\theta^{U_n}(bk)\theta'^{U_n}(bk)\delta^{-1}_{Q_n(\mathbb{A})}(b)\, db\, dk. \tag{4.7}$$

Regarding the case $n = 1$, Lemma 2.24 immediately implies that $\theta' = \theta'^{U_1}$, and we also get that (4.6) equals (4.7). Now assume that $n \geqslant 1$.

Finally, to extract the dependence on s, one uses (4.2). Assume that χ (respectively, χ') is given by (3.7) with respect to η_χ, ψ_χ (respectively, $\eta_{\chi'}$, $\psi_{\chi'}$). Then (3.7) and (4.1) imply that

$$\eta_\chi(t^{-2})\eta_{\chi'}(t^{-2})\eta(t) = 1, \quad \forall t \in \mathbb{A}^*. \tag{4.8}$$

Let $t \in A^+$. Then $\gamma_\psi(t) = 1$, and, according to (3.8),

$$\chi^{(1)}\left(\mathfrak{s}\left(\prod_{i=1}^{n} \eta_i^\vee(t)\right)\right) = t^{n(n+1)/4}\eta_\chi(t)^n. \tag{4.9}$$

By Theorem 3, (4.8), and (4.9),

$$\theta^{U_n}(t\cdot I_n)\theta'^{U_n}(t\cdot I_n) = t^{n(n-1)/2}\eta(t)^{n/2}\theta^{U_n}(I_n)\theta'^{U_n}(I_n).$$

Since also

$$\rho_s(t\cdot I_n)\delta^{-1}_{Q_n(\mathbb{A})}(t\cdot I_n) = t^{n(s-n/2)}\omega_\tau(t)\rho(I_n),$$

when we apply (4.2) to (4.7), we get

$$n\int_K \int_{GL_n(F)\backslash GL_n(\mathbb{A})^1} \rho(bk)\theta^{U_n}(bk)\theta'^{U_n}(bk)\int_{0<t^n\leqslant d} \omega_\tau(t)\eta(t)^{n/2}t^{n(s-1/2)}t^{-1}\, dt\, db\, dk.$$

Because $\omega_\tau(t)\eta(t)^{n/2} = 1$, we reach (4.4).

Similar arguments apply to $\mathcal{I}(\mathcal{E}_2^d(\cdot;s),\theta,\theta')$, and we obtain (4.5). Note that

$$M(w,s)\rho_s(t\cdot I_n)\delta^{-1}_{Q_n(\mathbb{A})}(t\cdot I_n) = t^{n(-s-n/2)}\omega_\tau^{-1}(t)\eta(t)^{-n}\rho(I_n). \qquad \square$$

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