ON GENERALIZED POWERS OF THE DIFFERENCE OPERATOR

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1. Introduction. Sumner **(3)** discussed $\nabla_h^\lambda f(z)$ for arbitrary real λ and h, where the averaging operator ∇_h is defined by

$$\nabla_h f(z) = \tfrac{1}{2}[f(z+h) + f(z)] \tag{1.1}$$

when $f(z)$ is an entire function of exponential type $<2\pi/|h|$. Boas **(2)** gave an alternative definition of ∇_h which gave Sumner's results quickly and showed that his definition is equivalent to that of Sumner.

Let the difference operator with span h be defined by

$$h\Delta_h f(z) = f(z+h) - f(z) = (\exp hD - 1)f(z), \tag{1.2}$$

where $D = d/dz$. Sumner **(4)** gave a definition of Δ_h^λ for all real λ and h when $f(z)$ is an entire function of exponential type $<2\pi/|h|$ and obtained some important properties of Bernoulli numbers and Bernoulli polynomials. It is the purpose of this paper to show that the method given by Boas **(2)** may be used to obtain some general properties of Bernoulli numbers and the polynomials associated with the Bernoulli numbers from which Sumner's results follow as a special case and show that we do get a definition equivalent to that of Sumner.

2. Method of Boas. Let $f(z)$ be an entire function of exponential type τ. If

$$f(z) = \sum_{n=0}^{\infty} a_n z_n,$$

its Borel-Laplace transform $F(w)$ is defined by

$$F(w) = \sum_{n=0}^{\infty} \frac{n!\, a_n}{w^{n+1}}. \tag{2.1}$$

If S is the conjugate indicator diagram of $f(z)$ and $F(w)$ its Borel-Laplace transform, then

$$f(z) = (2\pi i)^{-1} \int_C F(w) e^{zw} dw, \tag{2.2}$$

where C is a contour surrounding S; in particular we may choose C to be the circle $|w| = \tau + \epsilon$ if necessary; see Boas **(1**, pp. 73–74**)**.

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Let D denote d/dz. If $\phi(w)$ is regular on S and on the contours C which are close to the boundary of S, the operator $\phi(D)$ is defined by

$$\phi(D)f(z) = (2\pi i)^{-1}\int_C F(w)\phi(w)e^{zw}dw, \tag{2.3}$$

where the definition is independent of C if we consider only those contours which lie in the domain of regularity of $\phi(w)$. If $E(S)$ denotes the set of all entire functions whose conjugate indicator diagrams are subsets of S, we notice that $\phi(D)$ is a transformation of $E(S)$ into itself. If ψ is regular over the range of $\phi(w)$ for w in S, we have

$$\psi[\phi(D)]f(z) = (2\pi i)^{-1}\int_C F(w)\psi[\phi(w)]e^{zw}dw. \tag{2.4}$$

Thus, if λ is a positive integer, we have

$$[\phi(D)]^\lambda f(z) = (2\pi i)^{-1}\int_C F(w)[\phi(w)]^\lambda e^{zw}dw. \tag{2.5}$$

If $\phi(w)$ has no zeros on S, (2.5) holds for all λ. If we define

$$[\phi(w)]^\lambda = e^{\lambda \log\phi(w)} \tag{2.6}$$

with the same branch of $\log\phi(w)$ for all λ, then

$$[\phi(D)]^\lambda\{[\phi(D)]^\mu f(z)\} = [\phi(D)]^{\lambda+\mu}f(z), \tag{2.7}$$

which also implies that $[\phi(D)]^{-\lambda}$ inverts $[\phi(D)]^\lambda$.

3. General Bernoulli numbers and polynomials of general order. Let $g(z)$ be an entire function of exponential type such that $g(0) \neq 0$ and $g'(0) \neq 0$. We shall define the general Bernoulli numbers $B_n^{-\lambda}(g)$ and the general Bernoulli polynomials $B_n^{-\lambda}(g, z)$ with respect to g of general order λ by

$$\left[\frac{g(t) - g(0)}{t}\right]^\lambda = \sum_{n=0}^{\infty} t^n B_n^{-\lambda}(g)/n!, \tag{3.1}$$

$$\left[\frac{g(t) - g(0)}{t}\right]^\lambda \exp zt = \sum_{n=0}^{\infty} t^n B_n^{-\lambda}(g, z)/n!. \tag{3.2}$$

If ρ is the absolute value of the zero of $g(z)$ which is nearest to the origin so that $g(z)$ is zero-free inside the circle $|z| = \rho$, then the series on the right converge for all z and all t when $\lambda > 0$ and for all z and $|t| < \rho$ when $\lambda < 0$.

Comparing coefficients of t^n in (3.2), we obtain

$$B_n^{-\lambda}(g, 0) = B_n^{-\lambda}(g) \tag{3.3}$$

and

$$B_n^{-\lambda}(g, z) = \sum_{p=0}^{n}\binom{n}{p}B_p^{-\lambda}(g)z^{n-p}. \tag{3.4}$$

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Considering the function

$$\left[\frac{g(t)-g(0)}{t}\right]^{\lambda+\mu} \exp(z+w)t,$$

by virtue of Cauchy's theorem on double power series, we obtain the identity

$$B_n^{-\lambda-\mu}(g, z+w) = \sum_{p=0}^{n} \binom{n}{p} B_{n-p}^{-\lambda}(g,z) B_p^{-\mu}(g,w). \tag{3.5}$$

4. THEOREM. *If we set*

$$\delta_g^{\lambda}(t) = \left[\frac{g(t)-g(0)}{t}\right]^{\lambda},$$

then the operator $\delta_g^{\lambda}(hD)$ *satisfies the following:*

(A) *if*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n/n!$$

is an entire function of exponential type $\tau < \rho/|h|$, *then*

$$\delta_g^{\lambda}(hD)f(z) = \sum_{n=0}^{\infty} c_n h^n B_n^{-\lambda}(g, z/h)/n!,$$

(B) $$\delta_g^{\lambda}(hD)z^n = h^n B_n^{-\lambda}(g, z/h),$$

(C) $$\delta_g^{\lambda}(hD)e^{\tau z} = e^{\tau z}\delta_g^{\lambda}(h\tau), \qquad 0 < \tau < \rho/|h|,$$

(D) $$\delta_g^{\lambda}(hD)B_n^{-\mu}(g, z/h) = B_n^{-\lambda-\mu}(g, z/h),$$

where λ, μ *are real numbers.*

Proof. First we notice that $\delta_g(hw)$ is an entire function. The absolute value of the zero of $\delta_g(hw)$ nearest to the origin is $\rho/|h|$. Since $\tau < \rho/|h|$, $\delta_g(hw)$ has no zeros on S; hence (2.5) holds for all λ.

By (2.5),

$$\begin{aligned}
\delta_g^{\lambda}(hD)f(z) &= (2\pi i)^{-1} \int_C F(w)\, \delta_g^{\lambda}(hw) e^{zw} dw \\
&= (2\pi i)^{-1} \int_C F(w) \left[\frac{g(hw)-g(0)}{hw}\right]^{\lambda} e^{zw} dw \\
&= (2\pi i)^{-1} \int_C F(w) \left[\sum_{n=0}^{\infty} \frac{h^n w^n}{n!} B_n^{-\lambda}(g, z/h)\right] dw \quad \text{by (3.2)} \\
&= \sum_{n=0}^{\infty} \frac{h^n B_n^{-\lambda}(g, z/h)}{n!} (2\pi i)^{-1} \int_C F(w) w^n dw
\end{aligned}$$

$$= \sum_{n=0}^{\infty} c_n h^n B_n^{-\lambda}(g, z/h)/n!. \tag{4.2}$$

This is (A).

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Taking $f(z) = z^n$ so that $c_n = n!$ and $c_m = 0$ for $m \neq n$, (4.2) gives

$$\delta_g^{\lambda}(hD)z^n = h^n B_n^{-\lambda}(g, z/h). \tag{4.3}$$

This is (B).

If $f(z) = e^{\tau z}$ so that $c_n = \tau^n$, then (4.2) gives

$$\begin{aligned} \delta_g^{\lambda}(hD)e^{\tau z} &= \sum_{n=0}^{\infty} \tau^n h^n B_n^{-\lambda}(g, z/h)/n! \\ &= \left[\frac{g(\tau h) - g(0)}{\tau h}\right] \exp(\tau h . z/h) \quad \text{by (3.2)} \\ &= \delta_g^{\lambda}(h\tau) \exp \tau z. \end{aligned} \tag{4.4}$$

This is (C).

$$\text{If } f(z) = B_n^{-\mu}(g, z/h) = \sum_{p=0}^{n} \binom{n}{p} (z/h)^{n-p} B_p^{-\mu}(g) \quad \text{by (3.4)},$$

then by (4.2), we have

$$\begin{aligned} \delta_g^{\lambda}(hD)B_n^{-\mu}(g, z/h) &= \sum_{p=0}^{n} \binom{n}{p} B_p^{-\mu}(g) B_{n-p}^{-\lambda}(g, z/h) \\ &= B_n^{-\lambda-\mu}(g, z/h) \quad \text{by (3.5)} \end{aligned} \tag{4.5}$$

and this is (D).

We notice that the exponential property of this operator follows from (2.7).

To obtain Sumner's results (**4**, pp. 535, 539) we have only to take $g(z) = e^z$. Then

$$\delta_g^{\lambda}(t) = \delta^{\lambda}(t) \text{ defined by Sumner.}$$

5. Sumner's definition of the operator $\delta^{\lambda}(hD)$. When $\lambda \geqslant 0$, let

$$\alpha_{\lambda}(t) = \sum_{n=0}^{\infty} \binom{\lambda}{n} \beta_n(t), \tag{5.1}$$

where

$$\beta_n(t) = \sum_{p=0}^{n} \binom{n}{p} (-1)^{n-p} \alpha_p(t) \tag{5.2}$$

and $\alpha_p(t)$ is defined inductively by

$$\alpha_p(t) = \int_{t-1}^{t} \alpha_{p-1}(v)dv, \qquad \alpha_0(t) = \begin{cases} 0 & \text{if } t < C \\ 1 & \phantom{\text{if }} t > C \end{cases} \tag{5.3}$$

for $p = 1, 2, 3, \ldots$. Then $\delta^{\lambda}(t)$ has the representation

$$\delta^{\lambda}(t) = \int_{-\infty}^{\infty} e^{tu}\, d\alpha_{\lambda}(u) \tag{5.4}$$

and the operator $\delta^{\lambda}(hD)$ is then defined by

$$\delta^{\lambda}(hD)f(z) = \int_{-\infty}^{\infty} f(z + hu)d\alpha_{\lambda}(u); \tag{5.5}$$

see (**4**, p. 534).

When n is a positive integer, Sumner defines the functions $\phi_n(y)$ inductively by

$$\phi_{n+1}(y) = \int_{-\infty}^{\infty} \phi_n(y - v)d\phi_1(v), \qquad n = 1, 2, 3, \ldots, \tag{5.6}$$

where

$$\phi_1(y) = \frac{e^{\pi y}}{e^{\pi y} + e^{-\pi y}}. \tag{5.7}$$

Then, using the representation

$$\epsilon(t) = \frac{t}{2\sinh(t/2)} = \int_{-\infty}^{\infty} e^{ity}d\phi_1(y),$$

which gives

$$\epsilon^n(t) = \int_{-\infty}^{\infty} e^{ity}d\phi_n(y)$$

and

$$\delta^{-n}(t) = \epsilon^n(t)e^{-nt/2} = \int_{-\infty}^{\infty} e^{t(iy-n/2)}d\phi_n(y), \tag{5.8}$$

he defines

$$\delta^{-n}(hD)f(z) = \int_{-\infty}^{\infty} f[z + h(iy - n/2)]d\phi_n(y). \tag{5.9}$$

6. We shall now show that the definitions (5.5) and (5.9) can be deduced from (2.5).

If $\lambda \geqslant 0$, (2.5) becomes, by virtue of (5.4),

$$\begin{aligned}\delta^{\lambda}(hD)f(z) &= (2\pi i)^{-1}\int_C F(w)\delta^{\lambda}(hw)e^{zw}dw \\ &= (2\pi i)^{-1}\int_C F(w)\left[\int_{-\infty}^{\infty} e^{hwu}d\alpha_{\lambda}(u)\right]e^{zw}dw \\ &= \int_{-\infty}^{\infty}\left[(2\pi i)^{-1}\int_C F(w)e^{w(hu+z)}dw\right]d\alpha_{\lambda}(u) \\ &= \int_{-\infty}^{\infty} f(z + hu)d\alpha_{\lambda}(u) \quad \text{by (2.2)}\end{aligned}$$

and this is (5.5).

If n is a positive integer, (2.5) still holds when $\lambda = -n$, so that

$$\begin{aligned}
\delta^{-n}(hD)f(z) &= (2\pi i)^{-1} \int_C F(w)\delta^{-n}(hw)e^{zw}dw \\
&= (2\pi i)^{-1} \int_C F(w)\left[\int_{-\infty}^{\infty} e^{(iy-n/2)hw} d\phi_n(y) \right] e^{zw}dw \\
&= \int_{-\infty}^{\infty} \left\{ (2\pi i)^{-1} \int_C F(w) e^{[z+h(iy-n/2)]w} dw \right\} d\phi_n(y) \\
&= \int_{-\infty}^{\infty} f[z + h(iy - n/2)] d\phi_n(y) \quad \text{by (2.2)}
\end{aligned}$$

and this is (5.9).

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