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Spectrally Bounded Linear Maps on $\mathcal{B}(X)$

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Abstract. We characterize surjective linear maps on $\mathcal{B}(X)$ that are spectrally bounded and spectrally bounded below.

1 Introduction and Background

Let X be a complex Banach space and $\mathcal{B}(X)$ the algebra of all bounded linear operators on X. For $A \in \mathcal{B}(X)$ we denote by $r(A)$ the spectral radius of A. A linear map $\phi\colon \mathcal{B}(X) \to \mathcal{B}(X)$ preserves the spectral radius if $r(\phi(A)) = r(A)$, $A \in \mathcal{B}(X)$. We say that ϕ is spectrally bounded if there exists a positive constant M such that $r(\phi(A)) \leq Mr(A)$ for every $A \in \mathcal{B}(X)$. It is spectrally bounded below if there exists a positive constant m such that $r(\phi(A)) \geq mr(A)$, $A \in \mathcal{B}(X)$. In [4], Brešar and the second author proved that every surjective linear map on $\mathcal{B}(X)$ preserving the spectral radius is a Jordan automorphism multiplied by a complex constant of modulus one. This result was motivated by Kaplansky's problem [6] on possible generalizations of the Gleason-Kahane-Żelazko theorem. For a nice introduction to this topic we refer to [1] or [2].

It turns out that in the case of Hilbert spaces the result of Brešar and Šemrl can be obtained under the weaker assumption of spectral boundedness [12]. It is perhaps more surprising that in [12] we had to restrict ourselves to Hilbert spaces because the result obtained there does not hold for general Banach spaces (see [12, Remark 3]).

Recently, the above result has been extended to more general von Neumann algebras [9]. Some recent related results on spectrally bounded maps can be found in [8]. A breakthrough on the way to the final solution of Kaplansky's problem was made by Aupetit [3] who confirmed Kaplansky's conjecture for von Neumann algebras. Based on some of the ideas in these papers we are now able to improve the result of Brešar and Šemrl in the following way.

Theorem 1.1 *Let X be an infinite-dimensional complex Banach space and $\phi\colon \mathcal{B}(X) \to \mathcal{B}(X)$ a surjective linear map. Suppose there exist positive constants m and M such that $mr(A) \leq r(\phi(A)) \leq Mr(A)$ for every $A \in \mathcal{B}(X)$. Then there exist a spectrally bounded linear functional φ on $\mathcal{B}(X)$, a nonzero complex number c, and either a bounded invertible linear operator $T\colon X \to X$ such that*

$$\phi(A) = cTAT^{-1} + \varphi(A)I, \quad A \in \mathcal{B}(X), \tag{1}$$

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or a bounded invertible linear operator $T\colon X' \to X$ *such that*

$$\phi(A) = cTA'T^{-1} + \varphi(A)I, \quad A \in \mathcal{B}(X). \tag{2}$$

The aim of this paper is not only to generalize the result from [4], but mainly to give much shorter and simpler proof.

In the case that $X = H$ is an infinite-dimensional Hilbert space, the only spectrally bounded linear functional $\varphi\colon \mathcal{B}(H) \to \mathbb{C}$ is the zero functional. This is not so for general Banach spaces. Namely, there exists an infinite-dimensional Banach space X such that the algebra $\mathcal{B}(X)$ has a nonzero multiplicative linear functional [10, 13] (see also [5] for a more general result). Every nonzero linear multiplicative functional maps every element of the algebra into its spectrum, and must be therefore spectrally bounded. Some further information on spectrally bounded functionals can be found in [7].

The case that X is finite-dimensional has been treated in [12, Remark 4].

We will conclude this section by a brief remark concerning the converse of the statement of our theorem. Let ϕ be of the form (1) with φ spectrally bounded. Such a map is obviously spectrally bounded, but in general need not be surjective nor spectrally bounded below. However, if $c \neq -\varphi(I)$, then ϕ is bijective with the inverse $A \mapsto \frac{1}{c}T^{-1}AT - b\varphi(A)I$, where $b = 1/c(c + \varphi(I))$. To verify this we need [7, Lemma 2.1] which claims that $\varphi(AB) = \varphi(BA)$ for every pair $A, B \in \mathcal{B}(X)$. Now, ϕ^{-1} is also spectrally bounded which implies that ϕ is spectrally bounded below.

2 Proof

We will first show that ϕ is injective. Assume to the contrary that there exists a nonzero $A \in \mathcal{B}(X)$ such that $\phi(A) = 0$. Then obviously, $r(A) = 0$. If x and Ax were linearly dependent for every $x \in X$ then the operator A would be a scalar operator dI for some $d \in \mathbb{C}$ which would yield $A = 0$, a contradiction. So, we can find $x \in X$ such that x and $Ax = y$ are linearly independent. Let V be a topological complement of the linear span of x and y and define a square-zero operator $S \in \mathcal{B}(X)$ by $Sx = Sy = x - y$ and $Sv = 0$, $v \in V$. Because $S^2 = 0$ we have $0 = r(\phi(S)) = r(\phi(S + A))$, and consequently, $r(S + A) = 0$ which is in a contradicton with $(S + A)x = x$.

Hence, both ϕ and ϕ^{-1} are spectrally bounded operators. By [9, Lemma 3.1], both ϕ and ϕ^{-1} preserve nilpotents. Therefore, ϕ maps $\mathcal{B}_0(X)$, the linear span of the set of all nilpotent operators in $\mathcal{B}(X)$, onto itself. Here we can apply [11, Main Theorem] in order to conclude that there is a nonzero complex number c and either a bounded bijective linear operator $T\colon X \to X$ such that $\phi(A) = cTAT^{-1}$ for every $A \in \mathcal{B}_0(X)$, or a bounded bijective linear operator $T\colon X' \to X$ such that $\phi(A) = cTA'T^{-1}$ for every $A \in \mathcal{B}_0(X)$. In the second case X must be reflexive, and therefore, the natural embedding $K\colon X \to X''$ is bijective. Replacing ϕ by $A \mapsto \frac{1}{c}T^{-1}\phi(A)T$ in the first case or by $A \mapsto \frac{1}{c}K^{-1}T'\phi(A)'(T^{-1})'K$ in the second case, we may assume that $\phi\colon \mathcal{B}(X) \to \mathcal{B}(X)$ is a bijective linear map satisfying $\phi(A) = A$, $A \in \mathcal{B}_0(X)$, which is spectrally bounded and spectrally bounded below, say, $kr(A) \leq r(\phi(A)) \leq Kr(A)$, $A \in \mathcal{B}(X)$, for some positive constants k and K.

In our next step we will show that for arbitrary $A \in \mathcal{B}(X)$ and $x \in X$ there exists a complex number $\varphi(x, A)$ such that $\phi(A)x = Ax + \varphi(x, A)x$.

We will first consider the special case that $Ax = \mu x$ for some complex number μ. In this special case we have to show that x and $\phi(A)x$ are linearly dependent. Assume to the contrary that they are linearly independent. Take any complex $\lambda_0 > Kr(A)$. Then x and $(\lambda_0 - \phi(A))^{-1}x$ are linearly independent. Thus, there exists $f \in X'$ with $f(x) = 0$ and $f((\lambda_0 - \phi(A))^{-1}x) = 1$. Then $(\phi(A) + x \otimes f)(\lambda_0 - \phi(A))^{-1}x = ((\phi(A) - \lambda_0) + \lambda_0 + x \otimes f)(\lambda_0 - \phi(A))^{-1}x = \lambda_0(\lambda_0 - \phi(A))^{-1}x$, and consequently, $r(\phi(A) + x \otimes f) > Kr(A)$. On the other hand, if for some scalar λ the operator $\lambda - A$ is invertible, then from $Ax = \mu x$ and $f(x) = 0$ we get that $f((\lambda - A)^{-1}x) = 0$ which further yields that $\lambda - A - x \otimes f = (\lambda - A)(I - (\lambda - A)^{-1}x \otimes f)$ is invertible. Hence, if a scalar λ does not belong to the spectrum of A then it does not belong to the spectrum of $A + x \otimes f$ as well, and consequently, $r(\phi(A + x \otimes f)) = r(\phi(A) + x \otimes f) \leq Kr(A)$, a contradiction.

Consider now the general case. Let x be any vector from X and $A \in \mathcal{B}(X)$ any operator. Because of the previous step we may assume that x and $Ax = y$ are linearly independent. Then we can find $f \in X'$ with $f(x) = f(y) = 1$. Set $R = (x - y) \otimes f$. Then R is a square-zero operator and thus, $\phi(R) = R$. Moreover, $(A + R)x = x$. Applying the previous step we see that there is a scalar $\varphi(x, A)$ such that $\phi(A)x = Ax + \varphi(x, A)x$. Applying the linearity of $\phi(A)$ and A one can easily prove that $\varphi(x, A) = \varphi(A)$ is independent of x.

It is easy to check that $\varphi \colon \mathcal{B}(X) \to \mathbb{C}$ is a linear functional. To see that it is spectrally bounded note that $Kr(A) \geq r(\phi(A)) = r(A + \varphi(A)I) \geq |\varphi(A)| - r(A)$. This completes the proof.

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