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Strongly Extreme Points and Approximation Properties

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Abstract. We show that if x is a strongly extreme point of a bounded closed convex subset of a Banach space and the identity has a geometrically and topologically good enough local approximation at x, then x is already a denting point. It turns out that such an approximation of the identity exists at any strongly extreme point of the unit ball of a Banach space with the unconditional compact approximation property. We also prove that every Banach space with a Schauder basis can be equivalently renormed to satisfy the sufficient conditions mentioned.

1 Introduction

Let X be a (real) Banach space and denote its unit ball by B_X, its unit sphere by S_X, and its topological dual by X^*. Let A be a non-empty set in X. By a slice of A we mean a subset of A of the form

$$S(A, x^*, \varepsilon) := \{x \in A : x^*(x) > M - \varepsilon\},$$

where $\varepsilon > 0$, $x^* \in X^*$ with $x^* \neq 0$, and $M = \sup_{x \in A} x^*(x)$. We will simply write $S(x^*, \varepsilon)$ for a slice of a set A when A is clear from the setting.

Definition 1.1 Let B be a non-empty bounded closed convex set in a Banach space X and let $x \in B$.

(i) x is an extreme point of B if for any y, z in B we have

$$x = \frac{y+z}{2} \quad \Longrightarrow \quad y = z = x.$$

(ii) x is a strongly extreme point of B if for any sequences $(y_n)_{n=1}^{\infty}$, $(z_n)_{n=1}^{\infty}$ in B we have

$$\lim_n \Big\| x - \frac{y_n + z_n}{2} \Big\| = 0 \quad \Longrightarrow \quad \lim_n \| y_n - z_n \| = 0.$$

When B is the unit ball, it is easily seen that this condition can be replaced by

$$\lim_n \| x \pm x_n \| = 1 \quad \Longrightarrow \quad \lim_n \| x_n \| = 0.$$

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In this case we say that the norm is midpoint locally uniformly rotund (MLUR) at x.

(iii) x is a point of continuity for the map $\Phi: B \to X$ if Φ is weak-to-norm continuous at x. When Φ is the identity mapping we just say that x is a point of continuity (PC).

(iv) x is a denting point of B if for every $\varepsilon > 0$ there is $\delta > 0$ and a slice $S(x^*, \delta)$ of B containing x with diameter less than ε.

(v) x is a locally uniformly rotund (LUR) point of B_X if for any sequence $(x_n)_{n=1}^{\infty}$ we have

$$\lim_n \|x + x_n\| = 2\lim_n \|x_n\| = 2\|x\| = 2 \quad \Longrightarrow \quad \lim_n \|x - x_n\| = 0.$$

It is well known that LUR points are denting points and that denting points are strongly extreme points [8]. Trivially strongly extreme points are extreme points.

The importance of denting points became clear in the sixties when the Radon–Nikodým Property (RNP) got its geometric description. In particular, it became clear that extreme points in many cases are already denting, as every bounded closed convex set in a space with the RNP has at least one denting point. The "extra" an extreme point needs to become a denting point is precisely described in the following theorem.

Theorem 1.2 ([9]) *Let x be an extreme point of continuity of a bounded closed convex set C in X. Then x is a denting point of C.*

It is well known that all points of the unit sphere of ℓ_1 are points of continuity for the unit ball B_{ℓ_1}. So from Theorem 1.2 we get that every extreme point of the unit ball of any subspace of ℓ_1 automatically gets the "extra" to become denting.

However, despite the theoretical elegance of Theorem 1.2, it is not always easy to to check whether the identity mapping is weak-to-norm continuous at a certain point of a bounded closed convex set. For this reason it is natural to look for geometrical conditions that ensure weak-to-norm continuity of the identity operator at x when we approximate it strongly by maps that are weak-to-norm continuous at x.

One such idea could be to assume that x is strongly extreme (not just extreme as in Theorem 1.2) and that the identity map is approximated strongly by finite rank operators. But this is not enough to give the extreme point the "extra" needed to be denting. Consider $x = x(t) \equiv 1 \in B_{C(K)}$, where K is compact Hausdorff. Then x is strongly extreme in $B_{C(K)}$, but the identity map $I: B_{C(K)} \to B_{C(K)}$ is not weak-to-norm continuous if the cardinality of K is infinite (see the next paragraph). However, $\lim_n \|P_n x - x\| = 0$, where (P_n) are the projections corresponding to the Schauder basis in $C[0,1]$. Clearly, P_n is weak-to-norm continuous at any point of $B_{C[0,1]}$ (as any compact operator is).

Actually, whenever K is infinite compact Hausdorff, $C(K)$ belongs to the class D2 of Banach spaces where all non-empty relatively weakly open subsets of the unit ball have diameter 2. Naturally, in such spaces no point of the unit sphere can be a PC point. See *e.g.*, the references in [1] for more information about the class D2.

Assuming x is strongly extreme, we need to make stronger assumptions of the approximating sequence of the identity. One such condition that we impose is related to the behaviour of the approximating mappings close to the point x (see Theorem 2.1). In particular, we obtain as a corollary that in Banach spaces with the unconditional compact approximation property (UKAP) (see Definition 2.10), every strongly extreme point in the unit ball is PC and therefore denting. In particular, we have that this conclusion holds for Banach spaces with an unconditional basis with unconditional basis constant 1. Further, we show that every Banach space with a Schauder basis can be renormed to satisfy the conditions of Theorem 2.1.

The notation and conventions we use are standard and follow [7]. When considered necessary, notation and concepts are explained as the text proceeds.

2 Weak-to-norm Continuity of the Identity Map

Our most general result on how to force a strongly extreme point x to be denting in terms of approximating the identity map $I: X \to X$ at x is the following theorem.

Theorem 2.1 *Let C be a bounded closed convex set of a Banach space X and let x be a strongly extreme point of C. Assume that there is $\lambda \in (0,1]$ and a sequence of maps $\Phi_n: C \to X$, $n = 1, 2, \dots$ (not necessarily linear) that are weak-to-norm continuous at x and that satisfy*

$$\lim_n \|\Phi_n x - x\| = 0$$

and

$$\lim_n \lim_{\varepsilon \to 0^+} f_{n,\lambda}(\varepsilon) = 0, \tag{2.1}$$

where

$$f_{n,\lambda}(\varepsilon) = \sup\Big\{ \operatorname{dist}\Big((1+\lambda)\Phi_n y - \lambda y, C \Big) : y \in C, \|\Phi_n x - \Phi_n y\| \le \varepsilon \Big\},$$

Then x is a denting point of C.

Remark 2.2 Note that if (2.1) holds for some $\lambda \in (0,1]$, then it is also holds for any positive μ less than λ.

The proof follows from Theorem 1.2 and the next proposition, which is an interplay between weak and norm topology. With $B(x,\rho)$ we denote the ball with center at x and radius ρ.

Proposition 2.3 *Let x be a strongly extreme point of a convex set C of a normed space X and let $0 < \lambda \le 1$. Assume that for every $\eta > 0$ there exist a weak neighbourhood W of x and a map $\Phi: W \cap C \to X$ satisfying*

$$\Phi(W \cap C) \subset B(x, \eta), \tag{2.2}$$

$$\sup_{w \in W \cap C} \operatorname{dist}\Big((1+\lambda)\Phi w - \lambda w, C \Big) < \eta. \tag{2.3}$$

Then x is PC.

Proof Since x is a strongly extreme point, for every $\varepsilon > 0$, we can find $\delta > 0$ such that

$$\left\| x - \frac{u+v}{2} \right\| < \delta, \quad u, v \in C \quad \Longrightarrow \quad \|u - v\| < \lambda\varepsilon. \tag{2.4}$$

Set $\eta = \min\{\delta, \lambda\varepsilon\}/2$. There is a weak neighbourhood W of x and a map Φ satisfying (2.2) and (2.3). Set $\Psi = I - \Phi$ and pick an arbitrary $w \in W \cap C$. Put $y^+ = (1-\lambda)x + \lambda w$. Since $x, w \in C$, we get $y^+ \in C$ by convexity. Since

$$\Phi w + \lambda \Psi w - y^+ = (1-\lambda)(\Phi w - x),$$

we have from (2.2),

$$\|\Phi w + \lambda \Psi w - y^+\| \le (1-\lambda)\eta < \eta. \tag{2.5}$$

Having in mind (2.3), we can find $y^- \in C$ such that

$$\|(\Phi w - \lambda \Psi w) - y^-\| < \eta. \tag{2.6}$$

This and (2.5) imply

$$\begin{aligned} \left\|x - \frac{y^+ + y^-}{2}\right\| &\le \|x - \Phi w\| + \tfrac{1}{2}\left\|(\Phi w + \lambda\Psi w - y^+) + (\Phi w - \lambda \Psi w - y^-)\right\| \\ &< \|x - \Phi w\| + \eta \le 2\eta. \end{aligned}$$

From (2.4) we get $\|y^+ - y^-\| < \lambda\varepsilon$. On the other hand, using (2.5) and (2.6), we get

$$\begin{aligned} \|y^+ - y^-\| &= \left\|y^+ - (\Phi w + \lambda \Psi w) - y^- + (\Phi w - \lambda \Psi w) + 2\lambda \Psi w\right\| \\ &> 2\lambda \|\Psi w\| - 2\eta. \end{aligned}$$

Hence,

$$2\lambda\|\Psi w\| < \|y^+ - y^-\| + 2\eta < \lambda\varepsilon + 2\lambda\varepsilon = 3\lambda\varepsilon.$$

This and (2.2) imply

$$\|w - x\| \le \|\Phi w - x\| + \|\Psi w\| < 2\varepsilon.$$

Since w is an arbitrary element of $W \cap C$, we get that $W \cap C \subset B(x, 2\varepsilon)$. ∎

Remark 2.4 If x is PC for C, we get that x satisfies the hypotheses of Proposition 2.3 just taking $\Phi = I$, $\lambda \in (0,1]$.

Proof of Theorem 2.1 Let $\{\varepsilon_n\}$ be a sequence of positive numbers tending to 0. Since $\Phi_n \colon C \to X$, $n = 1, 2, \ldots$ is weak-to-norm continuous at x, there is a weak neighbourhood V_n of x such that

$$\Phi_n(V_n \cap C) \subset B(x, \varepsilon_n), n = 1, 2, \ldots$$

Thus, the conditions of Theorem 2.1 imply that for every $\eta > 0$, we can find $n = n(\eta)$ such that (2.2) and (2.3) hold for $W = V_n$ and $\Phi = \Phi_n$. Theorem 1.2 concludes the proof. ∎

Recall that every linear compact operator is weak-to-norm continuous on bounded sets. This together with Theorem 2.1 gives the following corollary.

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Corollary 2.5 *Let $(X, \|\cdot\|)$ be a Banach space and let $|||\cdot|||$ be an equivalent (not necessarily symmetric) norm on X with corresponding unit ball C. Let x be a strongly extreme point of C. Let $\lambda \in (0,1]$ and let $T_n \cdot X \to X, n = 1, 2, \dots$ be linear compact operators such that*

$$\lim_n \|T_n x - x\| = 0, \tag{2.7}$$

$$\lim_n \lim_{\varepsilon\to 0^+} \sup\big\{ |||(1+\lambda)T_n y - \lambda y||| : |||y||| \le 1, \|T_n(x-y)\| \le \varepsilon \big\} = 1. \tag{2.8}$$

In particular, the above is satisfied if

$$\lim_n |||(1+\lambda)T_n - \lambda I||| = 1. \tag{2.9}$$

Then x is a denting point of C.

Remark 2.6 If $|||\cdot|||$ is a non-symmetric norm in X and $T \cdot X \to X$ is a bounded linear operator, we get a non-symmetric norm $|||T||| = \sup\{|||Tu||| : |||u||| \le 1\}$.

Proof It is enough to prove that (2.8) implies (2.1) with $\Phi_n = T_n$. Indeed, since there exists $k > 0$ such that $\|\cdot\| \le k|||\cdot|||$, then for every $u \in X \setminus C$, we have

$$\begin{aligned} \operatorname{dist}(u, C) &= \inf\big\{ \|u - v\| : v \in C \big\} \le k \inf\big\{ |||u - v||| : v \in C \big\} \\ &\le k \Big|\Big|\Big| u - u/|||u||| \Big|\Big|\Big| = k(|||u||| - 1). \end{aligned}$$

■

Remark 2.7 The functions f_n defined in Theorem 2.1 can be discontinuous at 0. Indeed, let X be a Banach space, and let $e \in B_X$ and $e^* \in S_{X^*}$ be such that $e^*(e) = \|e\| = \|e^*\| = 1$. Define a (norm one) projection P on X by $Px = e^*(x)e$ and put

$$f(\varepsilon) = \sup\big\{ \|Py - Ry\| : \|y\| \le 1, \|P(e-y)\| \le \varepsilon \big\},$$

where $R = I - P$. Now, if the norm $\|\cdot\|$ on X is either strictly convex or Gâteaux differentiable at e^*, then f is discontinuous at 0. Indeed, let $\varepsilon = 0, \|y\| \le 1$, and $P(e-y) = 0$. We get $e^*(e-y)e = 0$. Hence, $e^*(y) = e^*(e) = 1$. By the strict convexity of the norm or the Gâteaux differentiability of the norm at e^*, we have $y = e$. This implies $Ry = Re = 0$, so $f(0) = 1$. In order to prove that f is discontinuous at 0, we simply apply Corollary 2.5 with $T_n = P$. Since e is strongly extreme, but not denting, we get $\lim_{\varepsilon\to 0^+} f(\varepsilon) > 1$.

Remark 2.7 shows that one cannot replace the limit condition (2.8) of Corollary 2.5, with

$$\lim_n \sup\big\{ |||(1+\lambda)T_n y - \lambda y||| : |||y||| \le 1, T_n x = T_n y \big\} = 1.$$

The conditions in Corollary 2.5 (and thus Theorem 2.1) are essential. Let us illustrate this by example.

Example 2.8 Recall from the introduction that $x = x(t) \equiv 1 \in B_{C(K)}$ is a strongly extreme point in $B_{C(K)}$ for any K compact Hausdorff.

Consider the space c of convergent sequences endowed with its natural norm. Let $e = (1, 1, \dots) \in S_c$ and let P_n be the projection on c that projects vectors onto their n

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first coordinates. Now e is a strongly extreme point of B_c that is not denting. Moreover, it is evident that the condition $\lim_n \|P_n e - e\| = 0$ fails and that condition (2.8) (even (2.9)) holds for $\lambda = 1$ (and thus for all $\lambda \in (0,1]$). It follows that the approximation condition in Corollary 2.5 is essential.

Example 2.9 Consider again c endowed with its natural norm. Let $e \in c$ be is as in Example 2.8. Define a projection P on c by $Px = \lim_n x(n)e$ and put $P_n = P$ for all n. By construction, $P_n e = e$. For $z = (0,1,1,\dots)$, we have $P_n z = Pz = e$. Now, for any $\lambda \in (0,1]$, we have

$$\|(1+\lambda)P_n z - \lambda z\| = \|(1+\lambda)e - \lambda z\| = 1 + \lambda.$$

Thus,

$$\lim_n \lim_{\varepsilon \to 0^+} \sup\big\{ \|(1+\lambda)P_n y - \lambda y\| : y \in B_X, \|P_n(e-y)\| < \varepsilon\| \big\} \geq 1 + \lambda.$$

It follows that condition (2.8) in Corollary 2.5 is essential.

We now present our results in terms of an approximation property introduced and studied by Godefroy, Kalton, and Saphar.

Definition 2.10 A Banach space X is said to have the *unconditional compact approximation property (UKAP)* if there exists a sequence (T_n) of linear compact operators on X such that $\lim_n \|T_n x - x\| = 0$ for every $x \in X$ and $\lim_n \|I - 2T_n\| = 1$ (see [6]). If the operators (T_n) are of finite rank, then X is said to have the *unconditional metric approximation property (UMAP)* (see [2]).

For examples of Banach spaces with the UKAP (in fact, UMAP) see [4,5]. In [4] a complete description of spaces with the UMAP is given. In that paper it is also proved that Banach spaces with UMAP actually have the commuting UMAP.

Clearly Banach spaces X with the UKAP satisfy condition (2.9) for $\lambda = 1$. Clearly also Banach spaces with an unconditional basis with basis constant 1 have the UKAP (simply put $T_n = P_n$ the projection onto the n first vectors of the basis). Thus, we immediately have the following corollary.

Corollary 2.11 *If X has the UKAP, in particular, if X has an unconditional basis with unconditional basis constant* 1*, then all strongly extreme points in B_X are denting points.*

Let us mention that the global condition (2.9) is much stronger than the local condition (2.8), even in the case when it holds for all x in S_X. This will be clear from the discussion below and, in particular, from Example 2.14, that shows that the condition (2.9) is strictly stronger than (2.8). For that example we will use the following result.

Proposition 2.12 *Let X be a Banach space and x a locally uniformly rotund (LUR) point in S_X. Let (T_n) be a sequence of linear bounded operators on X, with* $\lim_n \|T_n\| = 1$ *and satisfying condition* (2.7) *in Corollary* 2.5*. Then condition* (2.8) *holds for* $\lambda = 1$ *(and thus for all* $\lambda \in (0,1]$*) and* $C = B_X$.

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Proof First we show that for every sequence (ε_n) with $\varepsilon_n > 0$ and $\lim_n \varepsilon_n = 0$

$$\lim_n \operatorname{diam} D_n = 0, \tag{2.10}$$

where $D_n = \{y \in B_X : \|T_n(x-y)\| < \varepsilon_n\}$. To this end, note that it suffices to show that

$$y_n \in B_X,\ \lim_n \|T_n(x-y_n)\| = 0 \quad \Longrightarrow \quad \lim_n \|x-y_n\| = 0. \tag{2.11}$$

Indeed,

$$\begin{aligned}\|T_n\|\|x+y_n\| \ge \|T_n(x+y_n)\| &= \|2T_n x + T_n(y_n - x)\|\\ &\ge 2\|T_n x\| - \|T_n(y_n - x)\|.\end{aligned}$$

Hence, $\liminf_n \|x+y_n\| \ge 2$. Since $\|y_n\| \le 1$ we get $\lim_n \|x+y_n\| = 2$. Since x is a LUR point, we get that (2.11) holds, and thus (2.10) holds. In order to prove (2.8) for $\lambda = 1$, it is enough to show that $\lim_n d_n = 1$, where $d_n = \sup\{\|T_n x - R_n y\| : y \in D_n\}$, $R_n = I - T_n$. Since $x \in D_n$, we have $d_n \ge \|T_n x - R_n x\|$. So we get from (2.7) that $\liminf d_n \ge 1$. Now, pick an arbitrary $y \in D_n$. Then we have

$$\begin{aligned}\|T_n x - R_n y\| \le \|T_n x\| + \|R_n y\| &\le \|T_n x\| + \|R_n x\| + \|R_n(y-x)\|\\ &\le \|T_n x\| + \|R_n x\| + \|R_n\|\|y-x\|\\ &\le \|T_n x\| + \|R_n x\| + (\|T_n\|+1)\operatorname{diam} D_n.\end{aligned}$$

Hence, $\limsup d_n \le 1$. ∎

Proposition 2.13 *Let $(T_n)_0^\infty$ be a bounded sequence of linear compact operators on X, $T_0 = 0$, $R_n = I - T_n$, and let $(f_n)_0^\infty \subset S_{X^*}$ be a total family for X. Then the equivalent norm*

$$|||u||| = \Big(\sum_{n=0}^{\infty} 2^{-n}\big(\|R_n u\|^2 + f_n^2(u)\big)\Big)^{1/2}$$

is LUR at $x \in X$ provided $\lim_n \|R_n x\| = 0$. Moreover, if the operators $(T_n)_0^\infty$ commute and $\lim_n \|T_n\| = 1$, then $\lim_n |||T_n||| = 1$.

Proof Pick a sequence $(x_k) \subset X$ with $\lim_k |||(x_k + x)/2||| = |||x||| = |||x_k|||$. By convexity arguments ([3, Fact 2.3 p. 45]), we have

$$\lim_k \|R_n x_k\| = \|R_n x\|, \quad n = 1, 2, \dots, \tag{2.12}$$

$$\lim_k f_n(x_k) = f_n(x), \quad n = 1, 2, \dots. \tag{2.13}$$

First we show that (x_k) is norm relatively compact. Given $\varepsilon > 0$, we can find n with $\|R_n x\| < \varepsilon$. Using (2.12), we can find k_ε such that $\|R_n x_k\| < \varepsilon$ for $k > k_\varepsilon$. The set $K = \{x_1, x_2, \dots, x_{k_\varepsilon}\} \cup |||x||| T_n(C)$, where C is the unit ball corresponding to $|||\cdot|||$, is norm relatively compact. We show that K is an ε-net for (x_k). Indeed, pick x_k, $k > k_\varepsilon$. Then $\|x_k - T_n x_k\| = \|R_n x_k\| < \varepsilon$ and $T_n x_k \in |||x||| T_n(C)$. So (x_k) is norm relatively compact. Since (f_n) is total, we get from (2.13) that $\lim_k \|x_k - x\| = 0$. Thus, the norm $|||\cdot|||$ is LUR at the point $x \in X$.

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Now let us prove the moreover part. As (T_n) commute, we have

$$\begin{aligned}|||T_m u|||^2 &= \sum_{n=0}^{\infty} 2^{-n}\big(\|R_n T_m u\|^2 + f_n^2(T_m u)\big)\\ &= \sum_{n=0}^{\infty} 2^{-n}\big(\|T_m R_n u\|^2 + (T_m^* f_n(u))^2\big)\\ &\le \sum_{n=0}^{\infty} 2^{-n}\big(\|T_m\|^2\|R_n u\|^2 + \|T_m^*\|^2 f_n^2(u)\big) = \|T_m\|^2|||u|||^2.\end{aligned}$$

Hence, $|||T_m|||^2 \le \|T_m\|^2$ for all $m = 1, 2, \dots$, so $\limsup_m |||T_m||| \le \limsup_m \|T_m\|$. Since $\lim_m |||T_m x - x||| = 0$, we get $\liminf_m |||T_m||| \ge 1$, and so $\lim_m |||T_m||| = 1$, provided $\lim_m \|T_m\| = 1$. ■

It is now easy to give the announced example showing that condition (2.9) can fail, as condition (2.8) holds for every x in S_X.

Example 2.14 Consider c_0 endowed with the norm $\|\cdot\|$ defined by

$$\|x\| = \sup_{i,j\ge 1}\big(x(i) - x(j)\big),$$

where $x = (x(k)) \in c_0$. Clearly, $\|\cdot\|$ is equivalent to the canonical norm on c_0. Let P_n be the projection onto the n first vectors in the canonical basis (e_k) of c_0 and let $|||\cdot|||$ be the norm on c_0 given in Proposition 2.13 where $f_n = 0$ for every n. Then $(c_0, |||\cdot|||)$ fulfills the conditions of Proposition 2.12 and thus satisfies condition (2.8) for every x in S_{c_0}. Nevertheless, we have $|||P_k - \lambda R_k||| > 1$ for any $\lambda \in (0,1]$, so condition (2.9) fails. For the latter, just consider $(P_k - \lambda R_k)(\sum_{i=1}^{k+1} e_i)$.

From the two preceding propositions we also get the following corollary.

Corollary 2.15 *Let X be a Banach space with a Schauder basis. Then there exists an equivalent norm $\|\cdot\|$ on X for which the sequence of projections P_n onto the first n vectors of the basis satisfy (2.8) for $\lambda = 1$.*

On the other hand, we have the following proposition.

Proposition 2.16 *There exists an equivalent norm $\|\cdot\|$ on $C[0,1]$ such that (2.8) does not hold for any $\lambda > 0$ and any sequence (T_n) of compact linear operators on X when $x \in C[0,1]$ with $\|x\| = 1$ and $\lim_n \|T_n x - x\|_\infty = 0$.*

Proof The norm on $C[0,1]$ constructed in [1, Theorem 2.4] is midpoint locally uniformly rotund and has the diameter two property, *i.e.,* all non-empty relatively weakly open subsets of the unit ball have diameter 2. In particular, in this norm all points on the unit sphere are strongly extreme, but none are denting. Thus, the conclusion follows from Theorem 2.1. ■

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