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# COMPLICATED BIFURCATIONS OF PERIODIC SOLUTIONS IN SOME SYSTEM OF ODE

A. SOLEEV

ABSTRACT. In a vicinity of a stationary solution we consider a real analytic system of ODE of order four, depending on a small parameter. We look for families of periodic solutions which contract to the stationary solution, when the parameter tends to zero. We apply the general methods developed in [2] for the study of complex bifurcations and in [4] for local resolutions of singularities.

Near a stationary point we consider a real system of ODE of order four, depending on a small parameter. We look for families of periodic solutions which contract to the stationary point, when the parameter tends to zero. The computations and investigations in this paper are based on two methods: a method introduced in [2] to analyze complicated bifurcations and a method presented in [4] to compute local resolutions of singularities. We briefly describe these methods as follows.

First of all, we bring the system to a normal form in a vicinity of a fixed point, then we compute the set $\mathcal{A}$ containing all the families of periodic solutions that contract to this fixed point. These families can be written as asymptotic power series in a small parameter. To obtain the first few terms of these series from the normal form, we single out the first approximation of the system (*truncated system*) and study it in detail.

In the nondegenerate case it is the truncated system that determines character of the bifurcations and their asymptotics. The higher terms in the normal form allow one to make the asymptotic expansion of the family more precise. Thus, the computation of these families of periodic solutions is performed over the coefficients of the terms of the normal form. For concrete systems, the computation of the coefficients of terms in the normal form can be made only up to terms of some finite degree. In this case it is important to compute all coefficients of the terms of the lowest degree (that appear in the truncated system).

We consider a real analytic system whose expression in complex conjugate coordinates is:

$$\begin{aligned} dy_1/dt &= a(\varepsilon)y_1 + f_1(\varepsilon, y_1, y_2, \overline{y_1}, \overline{y_2}), \\ dy_2/dt &= a(\varepsilon)y_2 + f_2(\varepsilon, y_1, y_2, \overline{y_1}, \overline{y_2}) \end{aligned} \tag{1}$$

and the corresponding complex conjugate equations. We assume that $a(0) = i = \sqrt{-1}$ and the functions $f_1$ and $f_2$ are expanded into power series without any free and linear

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terms in $y_j, \overline{y_j}, j = 1, 2$. We look for families of periodic solutions for (1), which contract to the stationary point $y_1 = y_2 = \overline{y_1} = \overline{y_2} = 0$ when the small parameter $\varepsilon$ tends to zero (see [3]).

Then the normal form of the system (1) is as follows

$$(2) \qquad \begin{aligned} du_1/dt &= a(\varepsilon)u_1 + \Phi_1(\varepsilon, u_1, u_2, \overline{u_1}, \overline{u_2}), \\ du_2/dt &= a(\varepsilon)u_2 + \Phi_2(\varepsilon, u_1, u_2, \overline{u_1}, \overline{u_2}) \end{aligned}$$

and the corresponding conjugate equations, where

$$(3) \qquad \begin{aligned} a(\varepsilon) &= i + d_1\varepsilon + \cdots, \\ \Phi_j(\varepsilon, u_1, u_2, \overline{u_1}, \overline{u_2}) &= \sum_Q a_{jQ} u_1^{q_1} u_2^{q_2} \overline{u_1}^{q_3} \overline{u_2}^{q_4} \end{aligned}$$

with $Q = (q_1, q_2, q_3, q_4)$ and $q_1 + q_2 - q_3 - q_4 = 1$.

For small $|y_1|, |y_2|$ and $\varepsilon$, all desired families of periodic solutions of the system (1) are in the set $\mathcal{A}$ [2] which is determined from the normal form (2) by the system of four equations

$$(4) \qquad \begin{aligned} a(\varepsilon)u_j + \Phi_j(\varepsilon, u_1, u_2, \overline{u_1}, \overline{u_2}) &= a(0)\alpha u_j, \\ \bar{a}(\varepsilon)\overline{u_j} + \bar{\Phi}_j(\varepsilon, u_1, u_2, \overline{u_1}, \overline{u_2}) &= \bar{a}(0)\alpha\overline{u_j}, \quad j = 1, 2 \end{aligned}$$

where $\alpha$ is a parameter. Eliminating $\alpha$, we obtain a system of three analytical equations in four independent variables:

$$(5) \qquad \begin{aligned} g_1 &\overset{\text{def}}{=} u_2\Phi_1 - u_1\Phi_2 = 0, \\ g_2 &\overset{\text{def}}{=} \overline{g_1} = \overline{u_2}\overline{\Phi_1} - \overline{u_1}\overline{\Phi_2} = 0, \\ g_3 &\overset{\text{def}}{=} \big(a(\varepsilon) + \bar{a}(\varepsilon)\big)u_2\overline{u_2} + \overline{u_2}\Phi_2 + u_2\overline{\Phi_2} = 0. \end{aligned}$$

In a small vicinity near the stationary point $u_1 = u_2 = \overline{u_1} = \overline{u_2} = 0$, the set of solutions of system (5) have branches. We shall find all these branches by means of the method developed in [4] (see also [1]). Taking into account the first terms of the power series (3), we find the supports of the polynomials $g_i$ for the system (5):

$D(g_1) = \{Q_1^1 = (2, 1, 1, 0), Q_2^1 = (1, 2, 0, 1), Q_3^1 = (1, 2, 1, 0), Q_4^1 = (2, 1, 0, 1),$
$Q_5^1 = (0, 3, 0, 1), Q_6^1 = (0, 3, 1, 0), Q_7^1 = (3, 0, 1, 0), Q_8^1 = (3, 0, 0, 1), \cdots\};$

$D(g_2) = \{Q_1^2 = (0, 1, 0, 2), Q_2^2 = (0, 1, 1, 2), Q_3^2 = (1, 0, 1, 2), Q_4^2 = (0, 1, 2, 1),$
$Q_5^2 = (0, 1, 0, 3), Q_6^2 = (1, 0, 0, 3), Q_7^2 = (1, 0, 3, 0), Q_8^2 = (0, 1, 3, 0), \cdots\};$

$D(g_3) = \{Q_1^3 = (1, 0, 0, 1), Q_2^3 = (2, 0, 1, 1), Q_3^3 = (1, 1, 0, 2), Q_4^3 = (1, 1, 1, 1),$
$Q_5^3 = (0, 2, 0, 2), Q_6^3 = (0, 2, 1, 1), Q_7^3 = (2, 0, 2, 0), Q_8^3 = (1, 1, 2, 0), \cdots\}.$

For the supports $D(g_i)$ obtained above, we can compute the corresponding *Newton polyhedra* and *normal cones* (see [4]). The computation shows that the system (5) has only one truncation whose normal cone is $\mathbb{R}_+\Omega$, where $\Omega = (-1,-1,-1,-1)$. ($\mathbb{R}_+ = \{t \in \mathbb{R}, t \geq 0\}$). The truncated subsystem associated with the cone $\mathbb{R}_+\Omega$ consists of

$$\begin{aligned} \hat{g_1} \stackrel{\text{def}}{=} & b_1u_1^2u_2\overline{u_1} + b_2u_1u_2^2\overline{u_2} + b_3u_1u_2^2\overline{u_1} + b_4u_1^2u_2\overline{u_2} \\ & + b_5u_2^3\overline{u_2} + b_6u_2^3\overline{u_1} - b_7u_1^3\overline{u_1} - b_8u_1^3\overline{u_2} = 0 \end{aligned} \tag{6}$$

and its conjugate equation $\hat{g_2} = 0$. Considering the vectors $T_1 = Q_7^1 - Q_1^1 = (1,-1,0,0)$, $T_2 = Q_7^2 - Q_1^2 = (0,0,1,-1)$ and $T_3 = Q_3^3 - Q_1^3 = (0,1,0,1)$, we construct a unimodular matrix (by adding on an extra vector $T_4 = (1,0,0,0)$)

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{with inverse} \quad \alpha^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{pmatrix}.$$

The *power transformations* corresponding to these matrices are

$$u_1 = u_1, \quad \begin{cases} z = u_1u_2^{-1}, \\ \bar{z} = \bar{u}_1\bar{u}_2^{-1}, \\ r = u_2\bar{u}_2, \end{cases} \quad \text{and} \quad \begin{cases} u_2 = u_1z^{-1}, \\ \bar{u}_1 = u_1^{-1}z\bar{z}r, \\ \overline{u_2} = u_1^{-1}zr. \end{cases} \tag{7}$$

Under the power transformation (7) and the reduction by $u_1u_2$ in the first equation, by $\bar{u}_1\bar{u}_2$ in the second one and by $u_2\bar{u}_2$ in the third one, the system (5) can be converted into

$$\begin{aligned} G_1 &\stackrel{\text{def}}{=} \psi_1(\varepsilon, z, \bar{z}, r) - \psi_2(\varepsilon, z, \bar{z}, r) = 0, \\ G_2 &\stackrel{\text{def}}{=} \overline{\psi_1} - \overline{\psi_2} = 0, \\ G_3 &\stackrel{\text{def}}{=} \big(a(\varepsilon) + \bar{a}(\varepsilon)\big) + \big(\psi_2(\varepsilon, z, \bar{z}, r) + \bar{\psi}_2(\varepsilon, z, \bar{z}, r)\big) = 0, \end{aligned} \tag{8}$$

where

$$\Phi_j(\varepsilon, u_1, u_2, \bar{u}_1, \bar{u}_2) = u_j\psi_j(\varepsilon, z, \bar{z}, r).$$

After the reduction by $u_1u_2z^{-1}r$ in the first equation and by $\bar{u}_1\bar{u}_2\bar{z}^{-1}r$ in the second one, the truncated system (6) is translated into

$$\hat{G} \stackrel{\text{def}}{=} b_1z^2\bar{z} + b_2z + b_3z\bar{z} + b_4z^2 + b_5 + b_6\bar{z} - b_7z^3\bar{z} - b_8z^3 = 0 \tag{9}$$

and its conjugate equation. From the first equation of system (9) we find

$$\bar{z} = \frac{b_5 + b_2z + b_4z^2 - b_8z^3}{b_7z^3 - b_1z^2 - b_3z - b_6}. \tag{10}$$

If we substitute $\bar{z}$ into the second equation of system (9) we obtain an algebraic equation of degree 10 in $z$. Consequently, system (9) has ten complex roots $(z_0, \bar{z}_1)$, but not for all of them $\bar{z}_0 = \bar{z}_1$.

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THEOREM 1. *There exists such system (1), that the system (9) has 10 simple roots* $(z_0, \bar{z}_0)$, *i.e. they are real in real coordinates.*

PROOF. Let for system (9) $b_3 = b_4 = b_5 = b_7 = 0$. Then (10) is

(11) $$\bar{z} = -z\frac{b_2 - b_8 z^2}{b_6 + b_1 z^2}.$$

Denote

(12) $$x = \frac{b_2 - b_8 z^2}{b_6 + b_1 z^2}.$$

Then

(13) $$z^2 = \frac{b_2 - b_6 x}{b_8 + b_1 x}$$

and the equation (11) becomes

(14) $$\bar{z}/z = -x.$$

From this we see that

(15) $$|x| = 1$$

for solutions which are interesting for us *i.e.* $x\bar{x} = 1$. By squaring both sides of (14) we obtain the equation

$$\bar{z}^2 = x^2 z^2.$$

According to (13) and $(\overline{13})$ after the change $\bar{x} = 1/x$, it turns into

$$\frac{\bar{b}_2 x - \bar{b}_6}{\bar{b}_8 x + \bar{b}_1} = x^2 \frac{b_2 - b_6 x}{b_8 + b_1 x},$$

which is equivalent to an equation of degree 4. We need solutions that satisfy the relations (14) and (15). To make them more explicit, we multiply the equation (11) $\bar{z} = -xz$ by $z$. Then according to (13) for $x\bar{x} = 1$ we have

$$z\bar{z} = -z^2 x = -x\frac{b_2 - b_6 x}{b_1 x + b_8} = -\frac{(b_2 - b_6 x)(\bar{b}_1 + \bar{b}_8 x)}{(b_1 x + b_8)(\bar{b}_1 \bar{x} + \bar{b}_8)}.$$

Since $\operatorname{Im} z\bar{z} = 0$ and $\operatorname{Re} z\bar{z} > 0$, we obtain

(16) $$\operatorname{Im}(b_2 - b_6 x)(\bar{b}_8 x + \bar{b}_1) = 0, \quad |x| = 1,$$

(17) $$\operatorname{Re}(b_2 - b_6 x)(\bar{b}_8 x + \bar{b}_1) < 0.$$

The equations (16) are two quadratic equations with respect to $\operatorname{Re} x$ and $\operatorname{Im} x$. After the elimination one of them, we obtain equation of degree 4 such that from its roots we can choose only those that satisfy the inequality (17).

Now we prove that there exists such system (16), (17) with 4 solutions. For that in the complex $x$-plain we consider the points of intersection of the circle $|x| = 1$ and the hyperbola $\operatorname{Im}(b_2 - b_6 x)(\bar{b}_8 x + \bar{b}_1) = 0$. Here the points $x = b_2/b_6$ and $x = -\bar{b}_1/\bar{b}_8$ lie in this hyperbola and are used for boundary of those its points that satisfy the inequality (17), whereas the point

$$x = \big((b_2/b_6) - (\bar b_1/\bar b_8)\big)/2$$

is the center of the hyperbola.

For simplicity, we restrict ourself with the case $b_6 = b_8 = 1$. Then

$$(x - b_2)(x + \bar b_1) = \Big(x - \frac{b_2 - \bar b_1}{2}\Big)^2 - \Big(\frac{b_2 + \bar b_1}{2}\Big)^2$$

and

$$\begin{aligned}\operatorname{Re}(x - b_2)(x + \bar b_1) = {} & \Big[\operatorname{Re}\Big(x - \frac{b_2 - \bar b_1}{2}\Big)\Big]^2 - \Big[\operatorname{Im}\Big(x - \frac{b_2 - \bar b_1}{2}\Big)\Big]^2 \\ & + \frac{1}{4}[\operatorname{Re}(b_2 + \bar b_1)]^2 - \frac{1}{4}[\operatorname{Im}(b_2 + \bar b_1)]^2,\end{aligned}$$

$$\operatorname{Im}(x - b_2)(x + \bar b_1) = \operatorname{Re}\Big(x - \frac{b_2 - \bar b_1}{2}\Big)\operatorname{Im}\Big(x - \frac{b_2 - \bar b_1}{2}\Big) + \frac{1}{4}\operatorname{Re}(b_2 + \bar b_1)\operatorname{Im}(b_2 + \bar b_1).$$

On the hyperbola $\operatorname{Im}(x - b_2)(x + \bar b_1) = 0$ the inequality $\operatorname{Re}(x - b_2)(x + \bar b_1) < 0$ means that the $\operatorname{Re} x$ lies on the interval

$$J = (\min[\operatorname{Re} b_2, -\operatorname{Re} b_1], \max[\operatorname{Re} b_2, -\operatorname{Re} b_1]),$$

if $\operatorname{Re} b_2 \neq \operatorname{Re} b_1$. Further, we restrict ourselves to the case $\operatorname{Re} b_2 \neq \operatorname{Re} b_1$ and $\operatorname{Im} b_2 = \operatorname{Im} b_1$. Then the first equation in (16) defines two perpendicular lines

$$(18) \qquad \operatorname{Re} x = \frac{1}{2}\operatorname{Re}(b_2 - \bar b_1), \quad \operatorname{Im} x = \frac{1}{2}\operatorname{Im}(b_2 - \bar b_1)$$

The condition (17) is satisfied on the whole first line and in the interval $J$ on the second line.

Now we consider the case, when both lines (18) intersect the unit circle and both points $b_2$ and $-\bar b_1$ lie outside it, *i.e.*,

$$|\operatorname{Re} b_2 - \operatorname{Re} b_1| < 2, \quad |\operatorname{Im} b_2| < 1, \quad |b_2| > 1, \quad |b_1| > 1.$$

Then the first line intersects the unit circle in two points and the interval $J$ intersects it also in two points, *i.e.* we have 4 solutions of the system (16), (17).

According to (13), to each suitable value $x^0$ there corresponds two values $\pm z_0$. Hence we have 8 different solutions $(z_0, \bar z_1)$ with $\bar z_1 = \bar z_0 \neq 0$ and $\infty$.

In addition the equation (9) has the root $z_0 = 0$ since $b_5 = 0$ and the root $z_0 = \infty$ since $b_7 = 0$. Evidently $\bar z_1 = \bar z_0$ for them. So the equation (9) has 10 roots with that property. This finishes the proof of the theorem.

Now we shall go back, and solve the system (8) with respect to four variables $z, \bar z, r, \varepsilon$. For small $\varepsilon$ and $r$ solutions of system (8) belong to the vicinity of the point $(z_0, \bar z_0)$. We assume that the point $(z_0, \bar z_0)$ is the simple root of the system (9), *i.e.* in it the Jacobian

$D(\hat{G},\hat{\hat{G}})/D(z,\bar{z}) \neq 0$. Then taking $z = z_0$, $\bar{z} = \bar{z}_0$, $r = 0$, $\varepsilon = 0$ and applying the Implicit Function Theorem we obtain the roots of the system (8) in the form of expansions

$$(19) \qquad \begin{cases} z = z_0 + O(r), \\ \bar{z} = \bar{z}_0 + O(r), \\ \varepsilon = -\frac{r}{2\operatorname{Re} d_1}\left[2\operatorname{Re}\left(\hat{\psi}_1(0,z_0,\bar{z}_0,1)\right) + O(r)\right]. \end{cases}$$

Substituting these expansions into (7) we obtain

$$(20) \qquad \begin{cases} u_2 = u_1\left(z_0 + O(r)\right)^{-1}, \\ \bar{u}_1 = u_1^{-1} r\left(z_0\bar{z}_0 + O(r)\right), \\ \bar{u}_2 = u_1^{-1} r\left(z_0 + O(r)\right), \\ \varepsilon = -r\left(M + O(r)\right), \end{cases}$$

where $M = \operatorname{Re}[\hat{\psi}_1(0,z_0,\bar{z}_0,1)]/\operatorname{Re} d_1$.

After substituting (20) into the sum (3), the first equation of system (2) implies

$$\frac{d\ln u_1}{dt} = a\Big(-r\big(M + O(r)\big)\Big) + \psi^0(z_0,\bar{z}_0,r),$$

which yields $u_1 = Ce^{\Theta t}$ where

$$\Theta = a\Big(-r\big(M + o(r)\big)\Big) + \psi^0(z_0,\bar{z}_0,r), \quad C = \text{constant}.$$

Consequently, from (20) we can obtain a family of periodic solutions of the system (2), corresponding to the roots (19) of the system (5):

$$(21) \qquad \begin{cases} u_2 = e^{\Theta t}\left(z_0 + O(r)\right)^{-1}, \\ \bar{u}_1 = e^{-\Theta t} r\left(z_0\bar{z}_0 + O(r)\right), \\ \bar{u}_2 = e^{-\Theta t} r\left(z_0 + O(r)\right), \\ \varepsilon = -r\left(M + O(r)\right), \end{cases}$$

As all solutions $(z_0,\bar{z}_0)$ obtained by the Theorem 1 are simple, so for each of them we can apply the Implicit Function Theorem and find corresponding series (20) and families of periodic solutions in form (21). So we have proved the following theorem.

THEOREM 2. *There exist systems (1), in which 10 families of real periodic solutions bifurcate from the stationary point $y = 0$, when $\varepsilon$ passes through zero.*

If $(z_0,\bar{z}_0)$ is not a simple root of the system (9), we substitute $z = z_0 + \eta$, $\bar{z} = \bar{z}_0 + \bar{\eta}$ into the system (8), which produces

$$(22) \qquad \begin{aligned} H_1(\varepsilon,\eta,\bar{\eta},r) &\overset{\text{def}}{=} G_1(\varepsilon,z_0+\eta,\bar{z}_0+\bar{\eta},r) = 0, \\ H_2(\varepsilon,\eta,\bar{\eta},r) &\overset{\text{def}}{=} G_2(\varepsilon,z_0+\eta,\bar{z}_0+\bar{\eta},r) = 0, \\ H_3(\varepsilon,\eta,\bar{\eta},r) &\overset{\text{def}}{=} G_3(\varepsilon,z_0+\eta,\bar{z}_0+\bar{\eta},r) = 0. \end{aligned}$$

To this system we apply the toroidal blowing up process used to pass from system (5) to system (8). In our case the singularity is concentrated at the point $\varepsilon = \eta = r = 0$. After the application of the procedure, the point will be blown up into a plane, and we must find several roots of a new truncated system. Sum of their multiplicities is exactly the multiplicity of the root $(z_0, \bar{z}_0)$. So each of new roots is simpler than the initial root. We can iterate this process until we obtain a non singular system. This way we can determine all the components of the families of periodic solutions of system (2) which contract to the singular point (see [1], [4], [5], [8]).

In the same manner, one can study periodic solutions of the Hamiltonian system with two degrees of freedom near a resonant periodic solution (see [9]). Generally, bifurcations of periodic modes in resonant cases from Poiseuille flow, Couette flow and other flows were investigated by this way (see [6, 7]).

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*Faculty of Mathematics and Mechanics*
*Samarkand State University*
*University Boulevard 15,*
*Samarkand 703004*
*Uzbekistan*
*e-mail: soleev@samarkand.silk.glas.apc.org*