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# MAXIMAL QUOTIENT RINGS AND $S$-RINGS

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Throughout, we assume all rings are associative with identity and all modules are unitary. See [**7**] for undefined terms and [**3**] for all homological concepts.

Let $R$ be a ring, $E(R)$ the injective envelope of $_RR$, and $H = \operatorname{Hom}_R(E(R), E(R))$. Then we obtain a bimodule $_RE(R)_H$. Let $Q = \operatorname{Hom}_H(E(R), E(R))$. $Q$ is called the *maximal left quotient ring of* $R$. $Q$ has the property that if $p, q \in Q$, $p \neq 0$, then there exists $r \in R$ such that $rp \neq 0$, $rq \in R$, i.e., $Q$ is a ring of left quotients of $R$.

A left ideal $I$ of $R$ is *dense* if for every $x, y \in R$, $x \neq 0$, there exists $r \in R$ such that $rx \neq 0$, $ry \in I$. An alternate description of $Q$ is $Q = \{x \in E(_RR) : (R : x)$ is a dense left ideal of $R\}$, where $(R : x) = \{r \in R : rx \in R\}$.

The *left singular ideal of* $R$ is $Z_l(R) = \{r \in R : l_R(r)$ is an essential left ideal of $R\}$, where $l_R(r) = \{x \in R : xr = 0\}$. If $Z_l(R) = (0)$, then $Q$ is a left self-injective von Neumann regular ring [**7**, § 4.5]. Most of the previous work on maximal left quotient rings has been done in this case. For example, $Q$ is semisimple Artinian if and only if $Z_l(R) = (0)$ and $R$ is finite-dimensional [**14**].

Our principal object is to study the maximal left quotient ring of a ring $R$ whose left singular ideal is not necessarily zero. We begin in § 1 by considering right $S$-rings and, more generally, $S$-modules. The class of right $S$-rings contains all quasi-Frobenius and commutative perfect rings. We restrict our attention to maximal left quotient rings which are right $S$-rings.

In § 2, we show that the maximal left quotient ring $Q$ of $R$ is a semiprimary right $S$-ring if and only if (1) $QA = Q$ for every dense left ideal $A$ of $R$, (2) dense left ideals can be lifted modulo $P(Q) \cap R$, and (3) $P(Q) \cap R$ is nilpotent, where $P(Q)$ denotes the prime radical of $Q$. Rings $R$ for which $Q$ is a left Artinian right $S$-ring are also characterized, and these results are applied to local rings. Various descriptions of the ideal $P(Q) \cap R$ are obtained.

In § 3, we consider maximal left quotient rings which are left self-injective semiprimary and quasi-Frobenius rings. We also show that if $R$ has a left Artinian classical left quotient ring $Q$, then $Q$ is a left and right $S$-ring if and only if $l_Rr_R(P(R)) = P(R)$ and $r_Rl_R(P(R)) = P(R)$. This result is applied to quasi-Frobenius rings.

**1. $S$-rings.** Let $_R\mathscr{M}$ denote the category of left $R$-modules. Then $\mathscr{T} = \{M \in {_R\mathscr{M}} : \operatorname{Hom}_R(M, E(_RR)) = (0)\}$ is a hereditary torsion class, i.e.,

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$\mathscr{T}$ is closed under homomorphic images, arbitrary direct sums, extensions, and submodules. The filter associated with $\mathscr{T}$ is $\mathscr{F} = \{I : I$ is a left ideal of $R$ such that $R/I \in \mathscr{T}\} = \{I : I$ is a dense left ideal of $R\}$. $\mathscr{F}$ has the following properties:

(1) If $L \in \mathscr{F}$ and $L \subseteq L' \subseteq R$, then $L' \in \mathscr{F}$.

(2) If $L, L' \in \mathscr{F}$, then $L \cap L' \in \mathscr{F}$.

(3) If L $\in \mathscr{F}$ and $r \in R$, then $(L : r) \in \mathscr{F}$.

(4) If $L, L' \in \mathscr{F}$, then $LL' \in \mathscr{F}$.

For a proof, see [**1**, Proposition 1.2.8].

If $M \in {}_R\mathscr{M}$, let $T(M)$ denote the $\mathscr{T}$-torsion submodule of $M$, i.e., $T(M)$ is the (unique) submodule of $M$ maximal in the set of submodules $A$ of $M$ for which $\mathrm{Hom}_R(A, E({}_RR)) = (0)$. Then the maximal left quotient ring $Q$ of $R$ is given by $Q = T(E({}_RR)/R) = \{x \in E({}_RR) : (R : x)$ is a dense left ideal of $R\}$ (see [**18**]). We will use this characterization of $Q$ extensively.

The dense left ideals of $R$ and $Q$ are related in the following way.

LEMMA 1.1. *Let $Q$ be the maximal left quotient ring of $R$, and let $T$ be a subring of $Q$ such that $R \subseteq T \subseteq Q$. Then:*

(1) *If $A$ is a dense left ideal of $R$, then $TA$ is a dense left ideal of $T$.*

(2) *If $B$ is a dense left ideal of $T$, then $B \cap R$ is a dense left ideal of $R$.*

Before proceeding with our study of quotient rings, we will introduce $S$-modules. In particular, we will consider $S$-rings and their relationship to dense left ideals and maximal quotient rings.

*Definition.* ${}_RM$ is an $S$-module if $M$ contains a copy of each of its simple images.

PROPOSITION 1.2. *Let $M$ be a left $R$-module and $E = \mathrm{End}_R(M)$. Consider the following conditions:*

(1) *$M$ is an $S$-module;*

(2) *for every submodule $N \neq M$, $r_E(N) \neq (0)$;*

(3) *for every essential submodule $N \neq M$, $r_E(N) \neq (0)$;*

(4) *for every maximal submodule $N$ of $M$, $r_E(N) \neq (0)$.*

*Then* (1) *if and only if* (4), (2) *if and only if* (3), *and* (2) *implies* (4). *Furthermore, if $M$ is finitely generated, then* (4) *implies* (2), *so all four conditions are equivalent.*

*Proof.* Trivially, (2) implies (3) and (4). That (1) and (4) are equivalent follows easily from the fact that a submodule $A$ of $M$ is maximal if and only if $M/A$ is simple. Assume (3) holds, and let $N$ be a proper submodule of $M$. Choose $K \subseteq M$ maximal with respect to $N \cap K = (0)$. Then $N + K$ is an essential submodule of $M$. If $N + K \neq M$, then $r_E(N) \supseteq r_E(N + K) \neq (0)$ by (3). If $N + K = M$, then $M = N \oplus K$, and there exists $0 \neq f \in E$ defined by $f(n, k) = k$, where $n \in N$, $k \in K$. Thus (3) implies (2). If $M$ is finitely generated, then every proper submodule of $M$ is contained in a maximal submodule of $M$, so (4) implies (2).

*Remark.* An arbitrary direct sum of $S$-modules is an $S$-module, but a direct summand of an $S$-module need not be an $S$-module.

PROPOSITION 1.3. *If $_RM$ is a finitely generated S-module such that $Z(M) = (0)$, then $M$ is completely reducible.*

*Proof.* Let $N$ be a maximal submodule of $M$. Then $M$ contains a copy of $M/N$, so $Z(M/N) = (0)$. Assume $N$ is an essential submodule of $M$. If $0 \neq m' \in M/N$, then $(N : m)$ is essential, so $0 \neq m' \in Z(M/N)$. But this is a contradiction, so $N$ is not essential. If $A$ is a proper essential submodule of $M$, then $A \subseteq K$ for some maximal submodule $K$ of $M$ since $M$ is finitely generated. Then $K$ is essential since $A$ is essential, but we proved above that no maximal submodule of $M$ is essential. Thus $M$ has no proper essential submodules, so $M$ is completely reducible.

PROPOSITION 1.4. *Let $R$ be a semiprime ring. If $_RM$ is a finitely generated projective S-module, then $M$ is completely reducible.*

*Proof.* Let $N$ be a maximal submodule of $M$. Since $M$ is an $S$-module, $M$ contains a copy of $M/N$. Also $M$ can be embedded in a finite direct sum of copies of $R$ since $M$ is finitely generated projective. Thus there exists a homomorphism $0 \neq f \in \mathrm{Hom}_R(M/N, R)$. Since $M/N$ is simple, $I = f(M/N)$ is a minimal left ideal of $R$. Then $I = Re$, $e$ an idempotent in $R$, since $R$ is semiprime. Thus $M/N \cong f(M/N) = Re$ is projective, so $N$ is a direct summand of $M$. Hence every maximal submodule of $M$ is a direct summand. Thus $M$ is completely reducible as in the previous proposition.

*Definition.* $_RM$ is *semiprime* if for every $0 \neq m \in M$, there exists $f \in \mathrm{Hom}_R(M, R)$ such that $f(m)m \neq 0$.

PROPOSITION 1.5. *If $_RM$ is a finitely generated semiprime S-module, then $M$ is completely reducible.*

*Proof.* Let $A$ be a maximal submodule of $M$. There exists a submodule $K$ of $M$ such that $K \cong M/A$. Let $0 \neq x \in K$. Then $K = Rx$ since $K$ is simple. Since $M$ is semiprime, there exists $f \in \mathrm{Hom}_R(M, R)$ such that $f(x)x \neq 0$. Then $f(K) \neq (0)$, so $f(K)$ is a minimal left ideal of $R$. $K = Rf(x)x$, so $(0) \neq f(K) = f(Rf(x)x) = Rf(x)^2$. Hence $f(x)^2 \neq 0$, so $f(K)^2 \neq (0)$. Therefore, $M/A \cong K \cong f(K) = Re$, $e$ an idempotent of $R$. Now proceed as in the proof of the previous proposition.

*Definition.* $R$ is a *right* (*left*) *S-ring* if each proper left (right) ideal of $R$ has nonzero right (left) annihilator.

PROPOSITION 1.6. *The following are equivalent.*

(1) *$R$ is a right S-ring.*

(2) *$R$ contains a copy of each simple left R-module.*

(3) *$R$ has no proper dense left ideals.*

*Proof.* See [**6**, Theorem 3.2].

COROLLARY. *$R$ is a right $S$-ring if and only if ${}_RR$ is an $S$-module.*

PROPOSITION 1.7. *Let $R$ be a commutative ring. Then $R$ is a right $S$-ring if and only if $R$ has a faithful finitely generated projective $S$-module.*

*Proof.* If $R$ is a right $S$-ring, then ${}_RR$ is a faithful finitely generated projective $S$-module. Assume $R$ has a faithful finitely generated projective $S$-module ${}_RM$. Then $M$ is a generator in the category of $R$-modules [**20**, Proposition 1.3], so every simple $R$-module is a homomorphic image of $M$. Let $K$ be a simple $R$-module. Since $M$ is an $S$-module and $K$ is a homomorphic image of $M$, there exists a submodule $L$ of $M$ such that $L \cong K$. Also there exists $f \in \mathrm{Hom}_R(M, R)$ such that $f(L) \neq (0)$ since $M$ is finitely generated projective. Thus $R$ contains a copy of every simple $R$-module, so $R$ is a right $S$-ring.

PROPOSITION 1.8. *If $R$ is a right $S$-ring which satisfies the descending chain condition on left annihilators, then $R/J(R)$ is semisimple Artinian.*

*Proof.* Since $R$ is a right $S$-ring, every simple left $R$-module is isomorphic to a minimal left ideal of $R$. Hence $J(R) = l_R(T)$, where $T$ is the left socle of $R$. But $R$ satisfies the descending chain condition on left annihilators, so there exist $x_1, \ldots, x_n \in T$ such that $J(R) = l_R(x_1) \cap \ldots \cap l_R(x_n)$. Then $Rx_1 \oplus \ldots \oplus Rx_n$ is a finite direct sum of simple left $R$-modules, and there exists a monomorphism from $R/J(R)$ into $Rx_1 \oplus \ldots \oplus Rx_n$. Thus $R/J(R)$ is semisimple Artinian.

*Definition.* Let $I$ be a two-sided ideal of $R$. We say that dense left ideals can be *lifted modulo $I$* if whenever $A$ is a left ideal of $R$ such that $(A + I)/I$ is a dense left ideal of $R/I$, then $A$ is a dense left ideal of $R$.

PROPOSITION 1.9. *Let $R$ be a left Noetherian right $S$-ring. The following are equivalent:*

(1) *$R$ is left Artinian.*

(2) *$R$ satisfies the regularity condition.*

(3) *Dense left ideals can be lifted modulo $P(R)$.*

(4) *$R$ satisfies the descending chain condition on left annihilators and $J(R)$ is nilpotent.*

*Proof.* (1) implies (2) by Small's theorem [**15**; **16**]. (4) implies (1) by Proposition 1.8, and clearly (1) implies (4). $R$ is its own maximal left quotient ring $Q$ since it is a right $S$-ring. Hence $P(Q) \cap R = P(R)$. Thus (3) implies (1) by Theorem 2.9 (see § 2). It remains only to show that (2) implies (3). Suppose $I$ is a left ideal of $R$ such that $(I + P(R))/P(R)$ is a dense left ideal of $R/P(R)$. Then $(I + P(R))/P(R)$ contains a regular element $a + P(R)$ since $R/P(R)$ is semiprime left Goldie [**12**, Lemma 5]. By Small's theorem, $R$ has a classical left quotient ring $Q$. Then $1 = a^{-1}a \in Q(I + P(R))$. Suppose $x \in (R : a^{-1})$.

Then $x = x \cdot 1 = xa^{-1}a \in Ra \subseteq I + P(R)$, so $(R : a^{-1}) \subseteq I + P(R)$. Hence $I + P(R)$ is a dense left ideal of $R$, so $I + P(R) = R$ since $R$ is a right $S$-ring. This implies that $I = R$ since $P(R)$ is small in $R$, so $I$ is a dense left ideal of $R$. Thus (2) implies (3).

The following theorem is known [**19**, Theorem 3.2].

THEOREM 1.10. *Let $Q$ be the maximal left quotient ring of $R$. The following are equivalent:*

(1) *$Q$ is a right $S$-ring.*

(2) *$QA = Q$ for every dense left ideal $A$ of $R$.*

(3) $\mathrm{Ker}(R \otimes {}_RM \rightarrow Q \otimes {}_RM) = T({}_RM)$ *for every left $R$-module $M$.*

COROLLARY. *Suppose the maximal left quotient ring $Q$ of $R$ is a right $S$-ring. If $A$ is a flat left $R$-module, then $T({}_RA) = (0)$.*

COROLLARY. *If the maximal left quotient ring $Q$ of $R$ is a right $S$-ring, then* $\mathrm{l.gl.dim.}R \geqq \mathrm{l.gl.dim.}Q$.

*Proof.* Let $M$ be a left $Q$-module, and let $\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow (0)$ be a projective resolution of $M$ as a left $R$-module. Then because $T(P_i) = (0)$ and $Q$ is a right $S$-ring, $\cdots \rightarrow Q \otimes P_2 \rightarrow Q \otimes P_1 \rightarrow Q \otimes P_0 \rightarrow Q \otimes M \rightarrow (0)$ is a projective resolution of $M \cong Q \otimes M$ as a left $Q$-module. Hence $\mathrm{pd}({}_QM) \leqq \mathrm{pd}({}_RM)$. The result follows.

*Example.* Let $0 \leqq n \leqq \infty$. Then there exists a ring $R$ with maximal left quotient ring $Q$ such that $\mathrm{l.gl.dim.}R = n$ and $\mathrm{l.gl.dim.}Q = 0$.

Case 1: $n = 0$. Let $R$ be any semisimple Artinian ring. Then $R = Q$, and $\mathrm{l.gl.dim.}R = 0$.

Case 2: $0 < n < \infty$. Let $K$ be a field and $R = K[x_1, \ldots, x_n]$. Then $\mathrm{gl.dim.}R = n$ [**3**, IX, 7.11]. $R$ is a commutative domain, so $R$ has a classical quotient ring $Q$ which is a field. Then $Q$ is the maximal quotient ring of $R$, and $\mathrm{gl.dim.}Q = 0$.

Case 3: $n = \infty$. Let $K$ be a field, and let $R$ be the subring of $K[[x]]$ consisting of all power series without terms of degree 1. $R$ is a local ring [**3**, Exercise 10, p. 160]. Also $R$ is a commutative domain, so it has a classical quotient ring $Q$ which is a field. Then $Q$ is the maximal quotient ring of $R$, and $\mathrm{gl.dim.}Q = 0$. Let $A$ be the $R$-module consisting of all power series without a constant term. Then $\mathrm{pd}_RA = \infty$ [**3**, Exercise 10, p. 160]. Hence $\mathrm{gl.dim.}R = \infty$.

If $R$ has a minimal dense left ideal $D$, then the maximal left quotient ring $Q$ of $R$ is isomorphic to $\mathrm{Hom}_R(D, D)$ [**7**, Corollary 3, p. 97]. For example, if $R$ is right perfect, then $R$ has a minimal dense left ideal, and this case has been studied by Storrer [**17**]. In particular, Storrer obtained the following result.

PROPOSITION 1.11. *Let $R$ be a ring which has a minimal dense left ideal $D$. Then the maximal left quotient ring $Q$ of $R$ is a right $S$-ring if and only if ${}_RD$ is finitely generated projective.*

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*Proof.* See [**17**, Theorem 5.6].

Proposition 1.11 has immediate consequences when combined with some results of Mares and Ware. The following definition is due to Mares.

*Definition.* Let $P$ be a projective $R$-module. $P$ is a *semiperfect* module if every homomorphic image of $P$ has a projective cover. $P$ is *perfect* if for every index set $I$, every homomorphic image of $\bigoplus \sum_{i \in I} P$ has a projective cover.

PROPOSITION 1.12. *Let $R$ be a ring which has a minimal dense left ideal $D$. Then the maximal left quotient ring $Q$ of $R$ is a semiperfect (left perfect) right $S$-ring if and only if ${}_RD$ is a finitely generated projective semiperfect (perfect) module.*

*Proof.* The proof follows immediately from Proposition 1.11, Proposition 5.1 of [**20**] and its converse due to Mares [**9**], and the fact that $Q = \text{Hom}_R(D, D)$.

*Definition* [**20**]. Let $P$ be a nonzero projective $R$-module. $P$ is *local* if $P$ has a unique maximal submodule which contains every proper submodule of $P$.

PROPOSITION 1.13. *Let $R$ be a ring which has a minimal dense left ideal $D$. The maximal left quotient ring $Q$ of $R$ is a local right $S$-ring if and only if $D$ is a finitely generated projective local $R$-module.*

*Proof.* The proof follows from Proposition 1.11 and [**20**, Theorem 4.2].

*Definition* [**20**]. Let $P$ be a projective $R$-module. $P$ is *regular* if every cyclic submodule of $P$ is a direct summand.

PROPOSITION 1.14. *Let $R$ be a commutative ring which has a minimal dense ideal $D$. The maximal quotient ring $Q$ of $R$ is a finite direct sum of fields if and only if ${}_RD$ is a finitely generated projective regular module.*

*Proof.* $Q = \text{Hom}_R(D, D)$ is a regular $S$-ring if and only if ${}_RD$ is a finitely generated projective regular module by Proposition 1.11 and [**20**, Corollary 3.10]. If $Q$ is a finite direct sum of fields, clearly $Q$ is a regular $S$-ring. If $Q$ is a regular $S$-ring, then $Z_l(Q) = (0)$, so $Q$ is semisimple Artinian by Proposition 1.3. Since $Q$ is also commutative, $Q$ is a finite direct sum of fields.

Let $R$ be a ring which has a classical left quotient ring $Q$. If $I$ is a left ideal of $Q$, then $I = Q(I \cap R)$. This property does not hold in general for maximal left quotient rings, even if they are left Artinian (see [**17**, Example 7.2]). However, this problem is eliminated when the maximal left quotient ring is a right $S$-ring.

LEMMA 1.15. *Suppose the maximal left quotient ring $Q$ of $R$ is a right $S$-ring, and let $J$ be a left ideal of $Q$. Then $J = Q(J \cap R)$.*

*Proof.* Clearly, $Q(J \cap R) \subseteq QJ \subseteq J$. Now let $x \in J$. $(R : x)$ is a dense left ideal of $R$, so $Q(R : x) = Q$ since $Q$ is a right $S$-ring. Hence $x \in Qx =$

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$Q(R:x)x = Q(Rx \cap R) \subseteq Q(J \cap R)$. Therefore $J \subseteq Q(J \cap R)$, so we have equality.

COROLLARY. *Suppose the maximal left quotient ring $Q$ of $R$ is a right $S$-ring, and let $J$ be a two-sided ideal of $Q$. Then $J^k = Q(J \cap R)^k$ for $k = 1, 2, 3, \ldots$ .*

*Proof.* We observe that

$$J^k = J^{k-1}Q(J \cap R) = J^{k-1}(J \cap R) = J^{k-2}Q(J \cap R)^2 = \ldots$$
$$= J(J \cap R)^{k-1} = Q(J \cap R)^k.$$

LEMMA 1.16. *Let $R$ be a ring whose maximal left quotient ring $Q$ is a right $S$-ring, and let $J$ be a two-sided ideal of $Q$. Then:*

(1) *$(R + J)/J$ is an essential left $((R + J)/J)$-submodule of $Q/J$.*

(2) *If $A/J$ is an essential left ideal of $Q/J$, then $(A/J) \cap (R + J)/J$ is an essential left ideal of $(R + J)/J$.*

(3) *If $B$ is a left ideal of $R$ such that $(B + J)/J$ is an essential left ideal of $(R + J)/J$, then $(Q/J) \cdot (B + J)/J$ is an essential left ideal of $Q/J$.*

(4) *$Q/J$ is a ring of left quotients of $(R + J)/J$.*

(5) *Let $\{A_i : i \in I\}$ be a collection of left ideals of $R$. Then $Q(\sum_{i \in I} A_i) = \sum_{i \in I} QA_i$.*

*Proof.* We will prove only statements (1) and (5) since the proofs of (2)–(4) use similar techniques. Let $x \in Q/J$. Then $Q(R:x) = Q$ since $Q$ is a right $S$-ring. Suppose $Rx \cap R \subseteq J$. Then $x \in Qx = Q(R:x)x = Q(Rx \cap R) \subseteq QJ = J$, which is a contradiction. Hence $Rx \cap R \nsubseteq J$. Thus (1) holds. Now let $\{A_i : i \in I\}$ be a collection of left ideals of $R$. Clearly $Q(\sum A_i) \subseteq \sum QA_i$. Let $x \in \sum QA_i$. Then $x = \sum q_j a_j$, $a_j \in A_{i_j}$ for some $i_j \in I$. Let $B = \bigcap (R : q_j)$. $B$ is a dense left ideal of $R$, so $QB = Q$. Hence there exist $p_k \in Q$, $b_k \in B$ such that $1 = \sum p_k b_k$. Then $x = 1 \cdot x = \sum p_k b_k q_j a_j \in Q(\sum A_i)$. Thus $\sum QA_i \subseteq Q(\sum A_i)$, so we have equality, and (5) holds.

## 2. Artinian maximal quotient rings which are $S$-rings.

PROPOSITION 2.1. *Let $Q$ be the maximal left quotient ring of $R$, and let $N = P(Q) \cap R$. $Q$ is a right $S$-ring such that $Q/P(Q)$ is semisimple Artinian if and only if*

(1) *$QA = Q$ for every dense left ideal $A$ of $R$, and*

(2) *dense left ideals can be lifted modulo $N$.*

*Proof.* Let us first assume that $R$ satisfies conditions (1) and (2). $Q$ is a right $S$-ring by (1). We claim that $Q/P(Q)$ is a right $S$-ring. Suppose $I/P$ is a dense left ideal of $Q/P$, where $P = P(Q)$. Then $(I \cap R)/N$ is a dense left ideal of $R/N$, so $I \cap R$ is a dense left ideal of $R$ by (2). Hence $I = Q(I \cap R) = Q$ by (1). Thus $Q/P$ has no proper dense left ideals, so $Q/P$ is a right $S$-ring. Then $Q/P$ is semisimple Artinian by Proposition 1.4. Conversely, assume $Q$ is a right $S$-ring such that $Q/P$ is semisimple Artinian. $R$ satisfies (1) since $Q$

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is a right $S$-ring. Now let $A$ be a left ideal of $R$ such that $(A + N)/N$ is a dense left ideal of $R/N$. Then $(Q/P)((A + P)/P) = (QA + P)/P$ is an essential left ideal of $Q/P$, so $Q/P = \text{socle}_l(Q/P) \subseteq (QA + P)/P$. Hence $QA + P = Q$, so $QA = Q$. Let $\sum q_i a_i + A \in (QA \cap R)/A$. Then $(A : \sum q_i a_i)$ is a dense left ideal of $R$, so $R(\sum q_i a_i + A) \cong R/(A : \sum q_i a_i) \in \mathscr{T}$. Hence $\text{Hom}_R(C, E(R)) = (0)$ for every cyclic submodule $C$ of $(QA \cap R)/A$, so $\text{Hom}_R((QA \cap R)/A, E(R)) = (0)$. Therefore, $R/A = (QA \cap R)/A \in \mathscr{T}$, so $A$ is a dense left ideal of $R$. Consequently, $R$ satisfies (2). This completes the proof.

THEOREM 2.2. *Let $Q$ be the maximal left quotient ring of $R$, and let $N = P(Q) \cap R$. $Q$ is a semiprimary right $S$-ring if and only if*
(1) *$QA = Q$ for every dense left ideal $A$ of $R$,*
(2) *dense left ideals can be lifted modulo $N$, and*
(3) *$N$ is nilpotent.*

*Proof.* Assume $R$ satisfies conditions (1)–(3). By Proposition 2.1, it suffices to show that $P(Q)$ is nilpotent. But this follows from (3) and the corollary to Lemma 1.15. The other half of the theorem follows immediately from Proposition 2.1.

THEOREM 2.3. *Let $Q$ be the maximal left quotient ring of $R$, and let $N_k = QN^k \cap R$, where $N = P(Q) \cap R$. $Q$ is a left Artinian right $S$-ring if and only if*
(1) *$QA = Q$ for every dense left ideal $A$ of $R$,*
(2) *dense left ideals can be lifted modulo $N$,*
(3) *$N$ is nilpotent, and*
(4) *$R/N_k$ is finite-dimensional* ($k = 1, 2, 3, \ldots$).

*Proof.* Suppose $R$ satisfies conditions (1)–(4). Then $Q$ is a semiprimary right $S$-ring by Theorem 2.2. $R/N_k = R/(QN^k \cap R) = R/(P(Q)^k \cap R) \cong (R + P(Q)^k)/P(Q)^k$. Denote $P(Q) = J(Q)$ by $J$. $(R + J^k)/J^k$ is essential in $Q/J^k$ by Lemma 1.16, so $Q/J^k$ is finite-dimensional by (4). $J^{k-1}/J^k \subseteq \text{socle}_l(Q/J^k)$, which is a finite direct sum of simples. Hence $J^{k-1}/J^k$ is either a finite direct sum of simples or zero. Since $Q$ is semiprimary, we get a composition series for $Q$, so $Q$ is left Artinian. Conversely, if $Q$ is a left Artinian right $S$-ring, then $R$ satisfies (1)–(3) by Theorem 2.2. $Q/J^k$ is a ring of left quotients of $(R + J^k)/J^k$ by Lemma 1.16, so $Q/J^k$ can be embedded in the maximal left quotient ring of $(R + J^k)/J^k$. Therefore, $R/N_k \cong (R + J^k)/J^k$ is finite-dimensional since $Q/J^k$ is finite-dimensional. Hence $R$ satisfies condition (4), and the proof is complete.

THEOREM 2.4. *The maximal left quotient ring of $R$ is a semiprimary* (*left Artinian*) *right $S$-ring if and only if the maximal left quotient ring of $R_n$ is a semiprimary* (*left Artinian*) *right $S$-ring for every $n \geqq 1$.*

*Proof.* Let $Q$ be the maximal left quotient ring of $R$. Half of the theorem is trivial, so let us suppose that $Q$ is a semiprimary (left Artinian) right $S$-ring.

Then $Q_n$ is the maximal left quotient ring of $R_n$, and $Q_n$ is semiprimary (left Artinian). Then $Q_n$ has a minimal dense left ideal $D$, and $D$ is a two-sided ideal [**17**, Proposition 1.1]. There exists an ideal $J$ of $Q$ such that $D = J_n$ [**10**, Theorem 2.24]. $r_Q(J) = (0)$ since $r_{Q_n}(J_n) = (0)$. Hence $J$ is a dense left ideal of $Q$, so $J = Q$ since $Q$ is a right $S$-ring. Thus $D = J_n = Q_n$, so $Q_n$ is a right $S$-ring.

*Example.* Rosenberg and Zelinsky [**13**, p. 375] have given a ring $R$ which has only the left ideals (0), $N$, and $R$, where $N \neq (0)$ and $N^2 = (0)$. $R$ is a left Artinian right $S$-ring which is not right Artinian.

*Example.* Let $F$ be a field, and let

$$R = \left\{ \begin{vmatrix} a & 0 & 0 \\ b & a & 0 \\ d & 0 & e \end{vmatrix} : a, b, d, e \in F \right\}.$$

$R$ is a left and right Artinian ring which is a right, but not a left, $S$-ring [**17**, Example 7.2].

*Example.* Let $F$ be a field, and let

$$R = \begin{vmatrix} F & F \\ 0 & F \end{vmatrix}.$$

Then $R$ is left and right Artinian, and $Q = F_2$ is the maximal left and maximal right quotient ring of $R$ [**4**]. Hence $R$ is neither a left nor a right $S$-ring (otherwise it would be left or right rationally complete). $R$ is also a left and right $QF$-3 ring by [**5**, Theorem 5]. However, $R$ is not quasi-Frobenius for then it would be both a left and a right $S$-ring.

We will now specialize some of the above results to the case where $Q$ is a local ring. This case is of particular interest for then $Q$ is the classical left quotient ring of $R$.

PROPOSITION 2.5. *Let $Q$ be the maximal left quotient ring of $R$, and let $N = P(Q) \cap R$. $Q$ is local with nilpotent radical if and only if*

(1) *$QA = Q$ for every dense left ideal $A$ of $R$,*

(2) *dense left ideals can be lifted modulo $N$,*

(3) *$N$ is nilpotent, and*

(4) *$R/N$ is a finite-dimensional domain.*

*In this case, $Q$ is the classical left quotient ring of $R$.*

*Proof.* Suppose $R$ satisfies (1)–(4). Then $Q$ is a semiprimary right $S$-ring by Theorem 2.2. Let $P = P(Q)$. Since $R/N \cong (R + P)/P$ is a finite-dimensional domain, it has a classical left quotient ring $Q'$ which is a division ring. Then $Q'$ is the maximal left quotient ring of $(R + P)/P$. $Q/P$ is a ring of left quotients of $(R + P)/P$, so we can assume that $Q/P \subseteq Q'$. Every nonzero element of $(R + P)/P$ is invertible in $Q/P$, so $Q/P = Q'$. Hence $Q/P$ is a division ring, so $Q$ is a local ring. Also $J(Q) = P(Q)$ is nilpotent since $Q$ is semiprimary. Conversely, suppose $Q$ is local with nilpotent radical. Then $Q$ is a semiprimary right $S$-ring, so $R$ satisfies (1)–(3) by Theorem 2.2. $R/N \cong (R + P)/P \subseteq Q/P$ and $Q/P$ is a division ring, so $R/N$ is a domain. Also $Q/P$ is a ring of left quotients of $(R + P)/P$ and $Q/P$ is finite-dimensional, so $R/N \cong (R + P)/P$ is finite-dimensional. Thus $R$ satisfies (4).

Let us now assume that $R$ satisfies the conditions and show that $Q$ is the classical left quotient ring of $R$. By the above proof, $Q/P$ is the classical left quotient ring of $(R + P)/P$. Let $x \in R$ be regular. Then $x \notin P$ since $P$ is nilpotent. Therefore, $x + P$ is invertible in $Q/P$, so there exists $q \in Q$ such that $xq - 1 \in P$ and $qx - 1 \in P$. But this implies that $xq$ and $qx$ are invertible in $Q$, so $x$ is invertible in $Q$. Thus every regular element of $R$ is invertible in $Q$. Let $q \in Q$. Then $(R : q)$ is a dense left ideal of $R$, so $Q(R : q) = Q$. Hence $(R : q) \nsubseteq P$, so there exists a unit $u \in Q$ such that $u \in (R : q)$ since $Q$ is a local ring. Then $u$ is a regular element of $R$ and $q = u^{-1}r$ for some $r \in R$. Thus $Q$ is the classical left quotient ring of $R$. This completes the proof.

COROLLARY. *Let $Q$ be the maximal left quotient ring of $R$ and let $N_k = QN^k \cap R$, where $N = P(Q) \cap R$. $Q$ is a local left Artinian ring if and only if*

(1) *$QA = Q$ for every dense left ideal $A$ of $R$,*
(2) *dense left ideals can be lifted modulo $N$,*
(3) *$N$ is nilpotent,*
(4) *$R/N_k$ is finite-dimensional* $(k = 1, 2, 3, \ldots)$, *and*
(5) *$R/N_1$ is a domain.*

*In this case, $Q$ is the classical left quotient ring of $R$.*

*Proof.* The proof follows immediately from Theorem 2.3 and Proposition 2.5.

PROPOSITION 2.6. *Let $Q$ be the maximal left quotient ring of $R$, and let $N = P(Q) \cap R$. $Q$ is a local, left Artinian, principal left ideal ring if the following conditions are satisfied:*

(1) *$QA = Q$ for every dense left ideal $A$ of $R$.*
(2) *Dense left ideals can be lifted modulo $N$.*
(3) *$R/N$ is a finite-dimensional domain.*
(4) *$N = Rx$ for some $x \in R$.*

*Proof.* $N$ is nil, so $x$ is nilpotent. Also $N^k = Rx^k$ for every positive integer $k$, so $N$ is nilpotent. Therefore, $Q$ is local with nilpotent radical by Proposition 2.5. $J(Q) = P(Q) = QN = QRx = Qx$, so every left ideal of $Q$ is of the form $J(Q)^k = Qx^k$ by [**11**, Proposition 2.1]. Hence $Q$ is a principal left ideal ring. Thus $Q$ is left Noetherian and semiprimary, so $Q$ is left Artinian.

In general, the ideal $N = P(Q) \cap R$ need not be the prime radical of $R$. We will now give various descriptions of this ideal. The following notation will be used:

$S_1 = \{A : A$ is a two-sided ideal of $R$, $A \subseteq P(R)$, and $AQ \subseteq QA\}$.
$S_2 = \{A : A$ is a left ideal of $R$ and $QA$ is nilpotent$\}$.
$S_3 = \{A : A$ is a two-sided ideal of $R$, $A \subseteq P(R)$, and $A$ is the right annihilator of a subset of $Q\}$.

PROPOSITION 2.7. *Assume $R$ is left Goldie and the maximal left quotient ring $Q$ of $R$ is a right $S$-ring. (Note that these conditions are satisfied if $Q$ is a left Artinian right $S$-ring.) Then $N = P(Q) \cap R$ can be described in the following ways:*

(1) $N = \sum \{A : A \in S_1\}$.
(2) $N = \sum \{A : A \in S_2\}$.
(3) $N = \sum \{l_R(Q/QA) : A \in S_1\}$.

*Also,*

(4) $N \supseteq \sum \{A : A \in S_3\}$ *with equality if $P(Q)$ is a right annihilator in $Q$, and*
(5) $P(Q)^n \cap R = \sum \{QA^n \cap R : A \in S_1\} = QN^n \cap R$.

*Proof.* (1) Let $A \in S_1$. Then $A$ is nilpotent since $A$ is nil and $R$ is left Goldie. Hence $QA$ is nilpotent since $AQ \subseteq QA$. Therefore,

$$A \subseteq QA \cap R \subseteq P(Q) \cap R = N.$$

Thus $\sum \{A : A \in S_1\} \subseteq N$. But $N \in S_1$, so we have equality.

(2) If $A \in S_2$, then $QA \subseteq P(Q)$, so $A \subseteq N$. Hence $\sum \{A : A \in S_2\} \subseteq N$. Also $N \in S_2$.

(3) Let $A \in S_1$, and let $B = l_R(Q/QA)$. Then $BQ \subseteq QA$, so

$$B \subseteq BQ \cap R \subseteq QA \cap R \subseteq QN \cap R = P(Q) \cap R = N.$$

Hence $N \supseteq \sum \{l_R(Q/QA) : A \in S_1\}$. Now let $x \in N$. By (1) there exist $A_1, \ldots, A_n \in S_1$ such that $x \in A_1 + \ldots + A_n$. But $(A_1 + \ldots + A_n)Q \subseteq A_1Q + \ldots + A_nQ \subseteq QA_1 + \ldots + QA_n = Q(A_1 + \ldots + A_n)$ by Lemma 1.16. Hence $A_1 + \ldots + A_n \in S_1$, so $x \in A$ for some $A \in S_1$. Then

$$x(Q/QA) \subseteq A(Q/QA) = (0)$$

since $AQ \subseteq QA$. Therefore, $x \in l_R(Q/QA)$, so $N \subseteq \sum \{l_R(Q/QA) : A \in S_1\}$.

(4) Let $A \in S_3$, say $A = r_R(X)$, where $X \subseteq Q$. Then

$$A \subseteq AQ \cap R \subseteq r_R(X) = A,$$

so $A = AQ \cap R$. Suppose $\sum x_i q_i \in AQ$, $x_i \in A$, $q_i \in Q$. $B = \bigcap (R : x_i q_i)$ is a dense left ideal of $R$, so $QB = Q$ since $Q$ is a right $S$-ring. Hence $1 = \sum p_j b_j$, $p_j \in Q$, $b_j \in B$. Now $\sum_i x_i q_i = \sum_{i,j} p_j b_j x_i q_i \in Q(AQ \cap R) = QA$. Hence $AQ \subseteq QA$, so $A \in S_1$. Hence $N \supseteq \sum \{A : A \in S_3\}$ by (1). If $P(Q)$ is a right annihilator in $Q$, then $N \in S_3$, so we have equality.

(5) $P(Q)^n \cap R = (QN)^n \cap R = QN^n \cap R$ by the corollary to Lemma 1.15. Now use (1).

In the remainder of this section, we will reconsider the results obtained in the first part of the section, this time restricting ourselves to certain classes of rings $R$. We begin by considering commutative rings.

THEOREM 2.8. *Let $R$ be a commutative ring, $Q$ the maximal quotient ring of $R$. $Q$ is semiprimary if and only if*
(1) *$QA = Q$ for every dense ideal $A$ of $R$,*
(2) *dense ideals can be lifted modulo $P(R)$, and*
(3) *$P(R)$ is nilpotent.*

*Proof.* First note that $P(R) = P(Q) \cap R$ since in a commutative ring the prime radical is just the set of nilpotent elements. Also a commutative semiprimary ring is a right $S$-ring. Hence the theorem follows immediately from Theorem 2.2.

THEOREM 2.9. *Let $R$ be a left Noetherian ring. Then the maximal left quotient ring $Q$ of $R$ is a left Artinian right $S$-ring if and only if*
(1) *$QA = Q$ for every dense left ideal $A$ of $R$, and*
(2) *dense left ideals can be lifted modulo $N = P(Q) \cap R$.*

*Proof.* Since $R$ is left Noetherian, $R/(QN^k \cap R)$ is finite-dimensional for all $k$. Also $N$ is nilpotent since $N$ is nil and $R$ is left Noetherian. The theorem now follows from Theorem 2.3.

PROPOSITION 2.10. *Let $R$ be a right Noetherian ring. Then the maximal left quotient ring $Q$ of $R$ is a semiprimary right $S$-ring if and only if*
(1) *$QA = Q$ for every dense left ideal $A$ of $R$, and*
(2) *$J(Q) \cap R$ is nilpotent.*

*Proof.* If $Q$ is a semiprimary right $S$-ring, then $R$ clearly satisfies (1) and (2). Now suppose conditions (1) and (2) are satisfied. $Q$ is a right $S$-ring by (1). Also $J(Q)$ is nilpotent by (2) and the corollary to Lemma 1.15. Hence it suffices to show that $Q/J(Q)$ is semisimple Artinian. By Proposition 1.8, we need only show that $Q$ satisfies the descending chain condition on left annihilators. Suppose $l_Q(X_1) \supseteq l_Q(X_2) \supseteq \ldots$ . Intersecting each term of the chain with $R$ we obtain the chain $l_R(X_1) \supseteq l_R(X_2) \supseteq \ldots$ . But then $r_R l_R(X_1) \subseteq r_R l_R(X_2) \subseteq \ldots$, so there exists an $n$ such that $r_R l_R(X_n) = r_R l_R(X_{n+1}) = \ldots$ since $R$ is right Noetherian. This implies that $l_R(X_n) = l_R(X_{n+1}) = \ldots$, so

$$l_Q(X_n) = Q(l_Q(X_n) \cap R) = Ql_R(X_n) = Ql_R(X_{n+1}) = l_Q(X_{n+1}) = \ldots .$$

Thus $Q$ satisfies the descending chain condition on left annihilators, so $Q/J(Q)$ is semisimple Artinian. This completes the proof.

PROPOSITION 2.11. *Let $R$ be a ring such that $Z_l(R) = P(R)$, and let $Q$ be the maximal left quotient ring of $R$. $Q$ is a semiprimary right $S$-ring if and only if*

(1) $QA = Q$ *for every dense left ideal* $A$ *of* $R$,
(2) *dense left ideals can be lifted modulo* $P(R)$,
(3) $P(R)$ *is nilpotent, and*
(4) $R/P(R)$ *is left Goldie.*

*Proof.* Suppose $Q$ is a semiprimary right $S$-ring. Then $P(Q)$ is nilpotent, so $P(Q) \cap R$ is nilpotent. Therefore, $P(Q) \cap R \subseteq P(R)$. Also $P(R) = Z_l(R) = Z_l(Q) \cap R$, so $P(R) \subseteq QP(R) = Q(Z_l(Q) \cap R) = Z_l(Q)$. Since $Q$ is semiprimary, $Z_l(Q)$ is a nil ideal, so $Z_l(Q) \subseteq J(Q) = P(Q)$. Hence

$$P(R) \subseteq Z_l(Q) \cap R \subseteq P(Q) \cap R,$$

so $P(R) = P(Q) \cap R$. Thus $R$ satisfies conditions (1)–(3) by Theorem 2.2. Also

$$R/P(R) \cong (R + P(Q))/P(Q),$$

and $Q/P(Q)$ is a ring of left quotients of $(R + P(Q))/P(Q)$. Therefore, $R/P(R)$ is finite-dimensional since $Q/P(Q)$ is finite-dimensional. In addition, $Q/P(Q)$ satisfies the ascending chain condition on left annihilators, so $(R + P(Q))/P(Q)$ has the property since it is a subring of $Q/P(Q)$. Thus $R/P(R)$ is left Goldie, so $R$ satisfies (4).

Assume conditions (1)–(4) are satisfied. By Theorem 2.2, it suffices to show that $P(R) = P(Q) \cap R$. Since $R/P(R)$ is a semiprime left Goldie ring and $((P(Q) \cap R) + P(R))/P(R)$ is a nil ideal in $R/P(R)$, we have

$$P(Q) \cap R \subseteq P(R).$$

Also $Z_l(Q) \cap R = Z_l(R) = P(R)$ is nilpotent, so $Z_l(Q)$ is nilpotent by the corollary to Lemma 1.15. Hence $Z_l(Q) \subseteq P(Q)$, so

$$P(R) = Z_l(Q) \cap R \subseteq P(Q) \cap R.$$

Consequently, $P(R) = P(Q) \cap R$. This completes the proof.

**3. Quasi-Frobenius quotient rings.** Since the maximal left quotient ring $Q$ of $R$ is given by $Q = \{x \in E(_R R) : (R : x) \text{ is a dense left ideal of } R\}$, $Q = E(_R R)$ if and only if $(R : x)$ is a dense left ideal of $R$ for every $x \in E(_R R)$. Also $Q = E(_R R)$ if and only if $Q$ is left self-injective [**7**, Proposition 3, p. 95]. Using this fact, one can obtain the following two theorems. The proofs are contained in [**2**].

THEOREM 3.1. *Let $Q$ be the maximal left quotient ring of $R$. $Q$ is a left self-injective semiprimary ring if and only if*
(1) $(R : x)$ *is a dense left ideal of* $R$ *for every* $x \in E(_R R)$,
(2) $QA = Q$ *for every dense left ideal* $A$ *of* $R$, *and*
(3) $Z_l(R)$ *is nilpotent.*

THEOREM 3.2. *Let $Q$ be the maximal left quotient ring of $R$. $Q$ is quasi-Frobenius if and only if*

(1) $(R : x)$ *is a dense left ideal of* $R$ *for every* $x \in E({}_R R)$,
(2) $QA = Q$ *for every dense left ideal* $A$ *of* $R$,
(3) $Z_l(R)$ *is nilpotent, and*
(4) $R/[QZ_l(R)^k \cap R]$ *is finite-dimensional* $(k = 1, 2, 3, \ldots)$.

We will now turn to classical left quotient rings. An element $a \in R$ is regular if $l_R(a) = r_R(a) = (0)$. $Q$ is the classical left quotient ring of $R$ if $Q \supseteq R$, every regular element of $R$ is invertible in $Q$, and given $q \in Q$, there exist $a, r \in R$, $a$ regular, such that $q = a^{-1}r$. Lance Small [**15**; **16**] has given necessary and sufficient conditions for $R$ to have a left Artinian classical left quotient ring $Q$.

THEOREM 3.3. *Suppose* $R$ *has a left Artinian classical left quotient ring* $Q$. *Then:*
(1) $Q$ *is a right S-ring if and only if* $l_R r_R(P(R)) = P(R)$.
(2) $Q$ *is a left S-ring if and only if* $r_R l_R(P(R)) = P(R)$.

*Proof.* Let $P = P(Q)$ and $N = P(R)$. By the proof of Small's theorem, we know that $P \cap R = N$. Hence $P = QN$.

(1) Assume $l_R r_R(N) = N$. Then $P = QN = Ql_R r_R(N) = l_Q r_R(N) = l_Q r_R(QN) = l_Q r_R(P)$. But $r_R(P) \subseteq r_Q(P)$, so $l_Q r_R(P) \supseteq l_Q r_Q(P)$. Hence $l_Q r_Q(P) \subseteq l_Q r_R(P) = P \subseteq l_Q r_Q(P)$, so $P = l_Q r_Q(P)$. Also $P = J(Q)$ since $Q$ is left Artinian. Therefore, $Q$ is a right $S$-ring [**17**, Proposition 5.1].

Conversely, assume $Q$ is a right $S$-ring. By [**17**, Proposition 5.1], P $= J(Q)$ is a left annihilator, say $P = l_Q(X)$. Since $Q$ satisfies the descending chain condition on left annihilators, there exist $q_1, \ldots, q_n \in X$ such that $P = l_Q(q_1) \cap \ldots \cap l_Q(q_n)$. There exist $a$, $y_i \in R$, $a$ regular, such that $q_i = a^{-1}y_i$. Let $r \in l_R(y_1, \ldots, y_n)$. Then $raq_i = raa^{-1}y_i = ry_i = 0$ for each $i$, so $ra \in P$. Hence $r \in Pa^{-1} \cap R \subseteq P \cap R = N$. Thus $l_R(y_1, \ldots, y_n) \subseteq N$. Suppose $s \in N$. $sy_i = saa^{-1}y_i = saq_i$ for each $i$. Also $s \in N$ implies $sa \in N \subseteq P$, so $saq_i = 0$ for each $i$. Therefore, $sy_i = 0$ for each $i$, so $s \in l_R(y_1, \ldots, y_n)$. Thus $N \subseteq l_R(y_1, \ldots, y_n)$, so $N = l_R(y_1, \ldots, y_n)$. Therefore, $N = l_R r_R(N)$.

(2) Suppose $r_R l_R(N) = N$. We will first show that $l_Q(N) \subseteq l_Q(P)$. Since $Q$ satisfies the descending chain condition on left annihilators, there exist $q_1, \ldots, q_n \in P$ such that $l_Q(P) = l_Q(q_1, \ldots, q_n)$. Each $q_i$ can be written as $q_i = a^{-1}x_i$, $a \in R$ regular, $x_i \in N$. Suppose $c^{-1}d \in l_Q(N)$. Then $(c^{-1}da)(a^{-1}x_i) = c^{-1}dx_i$ for each $i$. But $c^{-1}dN = (0)$ implies $dN = (0)$, so $dx_i = 0$ for each $i$. Hence $(c^{-1}da)q_i = 0$ for each $i$, so $c^{-1}da \in l_Q(P)$. Then $c^{-1}daP = (0)$ implies $daP = (0)$, so $daQP = (0)$ because $QP = P$. Also $aQ = Q$ since $a$ is regular, so $(0) = dQP = dP$. Hence $d \in l_Q(P)$, so $c^{-1}d \in l_Q(P)$. Thus $l_Q(N) \subseteq l_Q(P)$. But $N \subseteq P$ implies $l_Q(N) \supseteq l_Q(P)$. Hence $l_Q(N) = l_Q(P)$. Now $N = r_R l_R(N) = r_R(Ql_R(N)) = r_R l_Q(N) = r_R l_Q(P) = r_Q l_Q(P) \cap R$. Hence $P = QN = Q(r_Q l_Q(P) \cap R) = r_Q l_Q(P)$ since $r_Q l_Q(P)$ is a two-sided (therefore, left) ideal of $Q$. Then $Q$ is a left $S$-ring by [**17**, Proposition 5.1].

Now suppose $Q$ is a left $S$-ring. Then $r_Q l_Q(P) = P$ by [**17**, Proposition 5.1], so $N = r_Q l_Q(P) \cap R = r_R l_Q(P) = r_R[Q l_R(P)] = r_R l_R(P)$. Also $N \subseteq P$, so $l_R(N) \supseteq l_R(P)$. Hence $N = r_R l_R(P) \supseteq r_R l_R(N) \supseteq N$, so $N = r_R l_R(N)$. This completes the proof.

Suppose $R$ has a classical left quotient ring, and let $I$ be a left ideal of $R$. Then $I^T = \{r \in R : ar \in I$ for some regular element $a$ of $R\}$. See [**8**] for details regarding the operator $T$. $I$ is called a *closed left ideal* of $R$ if $I^T = I$.

As an application of the preceeding theorem, we obtain the following result.

THEOREM 3.4. *Suppose $R$ has a two-sided classical quotient ring $Q$ which is left Artinian. Then $Q$ is quasi-Frobenius if and only if $R$ satisfies*

(*) *if $I \subseteq P(R)$ is a closed left (right) ideal of $R$, then $I$ is a left (right) annihilator in $R$.*

*Proof.* First assume $Q$ is quasi-Frobenius. Then every left (right) ideal of $Q$ is a left (right) annihilator. Let $I \subseteq P(R)$ be a closed left ideal. Then $QI = l_Q(X)$ for some subset $X \subseteq Q$. Since $Q$ satisfies the descending chain condition on left annihilators, there exist $q_1, \ldots, q_n \in X$ such that $QI = l_Q(q_1, \ldots, q_n)$. Since $Q$ is a classical right quotient ring of $R$, there exist $x_i$, $b \in R$, $b$ regular, such that $q_i = x_i b^{-1}$. Then $I = I^T = QI \cap R = l_R(x_1 b^{-1}, \ldots, x_n b^{-1}) = l_R(x_1, \ldots, x_n)$, so $I$ is a left annihilator in $R$. Similarly, if $I \subseteq P(R)$ is a closed right ideal of $R$, then $I$ is a right annihilator in $R$.

Now assume $R$ satisfies (*). $P(R) = P(Q) \cap R$ since $Q$ is left Artinian. Therefore, $P(R)$ is a closed left ideal and a closed right ideal, so $P(R)$ is a left and right annihilator by (*). Hence $l_R r_R(P(R)) = r_R l_R(P(R)) = P(R)$, so $Q$ is a left and right $S$-ring by Theorem 3.3. $P(Q)$ is therefore a left annihilator and a right annihilator, so left and right annihilators can be lifted modulo $P(Q)$. Also $Q/P(Q)$ is semisimple Artinian. Therefore, if $I$ is a left (right) ideal of $Q$ such that $I \not\subseteq P(Q)$, then $I$ is a left (right) annihilator in $Q$. Now let $I \subseteq P(Q)$ be a left ideal of $Q$. Then $I \cap R$ is a closed left ideal of $R$, and $I \cap R \subseteq P(R)$. By (*), $I \cap R = l_R(X)$ for some subset $X$ of $R$. Then $I = l_Q(X)$, so $I$ is a left annihilator in $Q$. Similarly, if $J \subseteq P(Q)$ is a right ideal in $Q$, then $J$ is a right annihilator. Thus $Q$ is a left Artinian ring such that every left ideal of $Q$ is a left annihilator and every right ideal of $Q$ is a right annihilator, so $Q$ is quasi-Frobenius. This completes the proof.

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