A NOTE ON NON-DISTRIBUTIVE SUBLATTICES OF DEGREES AND HYPERDEGREES

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In (**1**, §§ 2.3 and 2.4) we proved that certain distributive lattices are simultaneously lattice-embeddable in the degrees of recursive unsolvability and in the hyperdegrees. Let $\mathscr{L}$ be the non-distributive lattice $\{0, 1, a_0, a_1, \ldots\}$, where $a_i \cup a_j = 1$ and $a_i \cap a_j = 0$ whenever $i \neq j$. We shall prove the following theorem.

THEOREM. *The lattice $\mathscr{L}$ is simultaneously lattice-embeddable in the degrees and hyperdegrees.*

For $A \subseteq N$, let $\mathbf{deg}(A)$ and $\mathbf{hyp}(A)$ be the degree and hyperdegree of A, respectively. To prove the theorem we must construct hyperarithmetically incomparable sets $A_0, A_1, \ldots$ such that for $\Delta = \mathbf{deg}, \mathbf{hyp}$ and for all distinct i, j:

$$\Delta(A_i) \cup \Delta(A_j) = \Delta(A_0) \cup \Delta(A_1), \tag{1}$$

$$\Delta(A_i) \cap \Delta(A_j) \text{ exists and equals } \Delta(N). \tag{2}$$

Now, if each $\langle A_i, A_j\rangle$ were a generic pair in the sense of (**1**), then (2) would hold. (For $\Delta = \mathbf{hyp}$, (2) is the same as (**1**, Theorem 13); for $\Delta = \mathbf{deg}$, the proof is similar (cf. **1**, Corollary 2 to Theorem 14).) In order that (1) hold, it would be sufficient that each A_i be the (lower) Dedekind cut of a real number x_i and that there be rational numbers a_i, b_i ($i \in N$) with

$$\begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} \neq 0 \quad \text{whenever } i \neq j,$$

and real numbers s, t such that $(i)(x_i = a_i s + b_i t)$.

Thus, we are led to modification of the forcing method of (**1**). Let $\mathscr{M}^*$ be like the language $\mathscr{L}^*$ of (**1**, § 1.3) except that only two pairs $\mathrm{A}_0, \mathrm{A}_0'$ and $\mathrm{A}_1, \mathrm{A}_1'$ of "generic set constants" are adjoined. Let ρ be an effective 1-1 correspondence between N and the rationals, and let $D(x) = \{\rho^{-1}(r) \mid r$ is rational and less than $x\}$. Change the definition of *consistent* set of conditions (**1**, § 1.3) to require that if $\rho(m) \leqq \rho(n)$, then not both $\mathrm{A}_i(\mathbf{n}, \mathbf{0})$ and $\mathrm{A}_i(\mathbf{m}, \mathbf{1})$ are in the set. Then a set of conditions P determines a closed rational rectangle $|P|$ in the plane (rather than a basic closed set in $2^N \times 2^N$). All the remainder

Received July 2, 1967. The author is grateful to the National Research Council of Canada (Grant # A-4065) for financial assistance.

of (**1**, §§ 1.3–1.6) goes through with only the most trivial modifications. Thus, it remains only to prove the following lemma.

LEMMA. *Let a_i, b_i ($i \in N$) be rational numbers such that*

$$\begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} \neq 0 \quad \textit{whenever } i \neq j.$$

Then there exist reals $s, t, x_0, x_1, \ldots$ such that (i)($x_i = a_i s + b_i t$) and such that $\langle D(x_i), D(x_j)\rangle$ is a generic pair whenever $i < j$.

Proof. Let $S_0 = T_0 = [0, 1]$. Given closed rational intervals S_n and T_n, let $\mathbf{F}$ be the $(n)_0$th sentence of $\mathscr{M}^*$, let $i = (n)_1$, and let $j = i + 1 + (n)_2$. Let P be a set of conditions such that $|P| \subseteq (a_i S_n + b_i T_n) \times (a_j S_n + b_j T_n)$. Let $Q_0 = (\mu Q)(\text{Ext}(Q, P, \mathbf{F}))$ (cf. (**1**, proof of Theorem 6); in particular, Q_0 extends P, and $Q_0 \Vdash \mathbf{F}$ or $Q_0 \Vdash \sim\mathbf{F}$). Let S_{n+1} and T_{n+1} be closed rational intervals of length less than $1/n$ with $S_{n+1} \subseteq S_n$, $T_{n+1} \subseteq T_n$, and

$$(a_i S_{n+1} + b_i T_{n+1}) \times (a_j S_{n+1} + b_j T_{n+1}) \subseteq |Q_0|.$$

Let $\langle s, t\rangle$ be the unique element of $\bigcap_n (S_n \times T_n)$ and $x_i = a_i s + b_i t$. That $\langle D(x_i), D(x_j)\rangle$ is a generic pair is now evident.

Note. The sets A_i constructed are all hyperarithmetic in Kleene's O. If one desires only a lattice-embedding of $\mathscr{L}$ in the degrees, a much more effective construction is possible: one approximates $\langle s, t\rangle$ by rational rectangles, but the step from $S_n \times T_n$ to $S_{n+1} \times T_{n+1}$ is suggested by the ordinary construction of two incomparable degrees with greatest lower bound zero. It may be possible to improve this method so as to embed $\mathscr{L}$ as an initial segment of degrees. It is certainly possible to use generalizations of the present methods to embed more complicated modular lattices in the degrees and hyperdegrees.

REFERENCE

1. S. K. Thomason, *The forcing method and the upper semi-lattice of hyperdegrees*, Trans. Amer. Math. Soc. *129* (1967), 38–57.

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